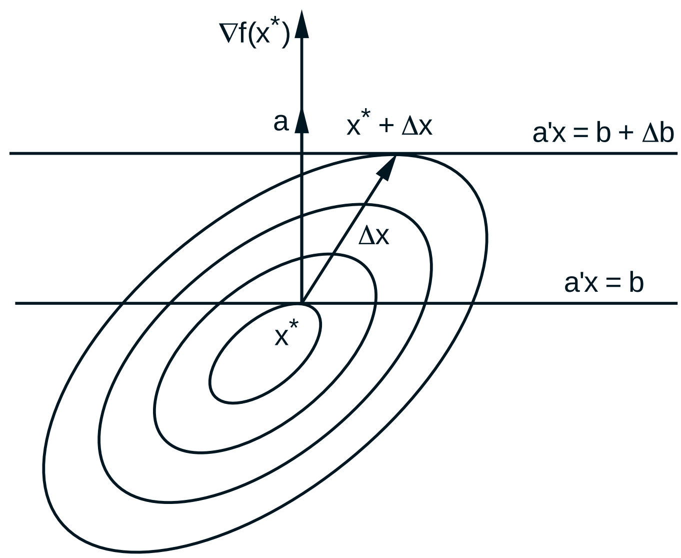
SENSITIVITY - GRAPHICAL DERIVATION



Sensitivity theorem for the problem $\min_{a'x=b} f(x)$. If b is changed to $b+\Delta b$, the minimum x^* will change to $x^*+\Delta x$. Since $b+\Delta b = a'(x^*+\Delta x) = a'x^* + a'\Delta x = b + a'\Delta x$, we have $a'\Delta x = \Delta b$. Using the condition $\nabla f(x^*) = -\lambda^* a$,

$$\begin{aligned}\Delta\text{cost} &= f(x^*+\Delta x) - f(x^*) = \nabla f(x^*)'\Delta x + o(\|\Delta x\|) \\ &= -\lambda^* a'\Delta x + o(\|\Delta x\|)\end{aligned}$$

Thus $\Delta\text{cost} = -\lambda^*\Delta b + o(\|\Delta x\|)$, so up to first order

$$\lambda^* = -\frac{\Delta\text{cost}}{\Delta b}.$$

For multiple constraints $a_i'x = b_i$, $i = 1, \ldots, n$, we have

$$\Delta\text{cost} = -\sum_{i=1}^{m} \lambda_i^* \Delta b_i + o(\|\Delta x\|).$$

SENSITIVITY THEOREM-LAGRANGE MU

Sensitivity Theorem: Consider the family of problems

$$\min_{h(x)=u} f(x) \qquad (*)$$

parameterized by $u \in \Re^m$. Assume that for $u = 0$, this problem has a local minimum x^*, which is regular and together with its unique Lagrange multiplier λ^* satisfies the sufficiency conditions.

Then there exists an open sphere S centered at $u = 0$ such that for every $u \in S$, there is an $x(u)$ and a $\lambda(u)$, which are a local minimum-Lagrange multiplier pair of problem (*). Furthermore, $x(\cdot)$ and $\lambda(\cdot)$ are continuously differentiable within S and we have $x(0) = x^*$, $\lambda(0) = \lambda^*$. In addition,

$$\nabla p(u) = -\lambda(u), \qquad \forall\ u \in S$$

where $p(u)$ is the *primal function*

$$p(u) = f\big(x(u)\big).$$

EXAMPLE

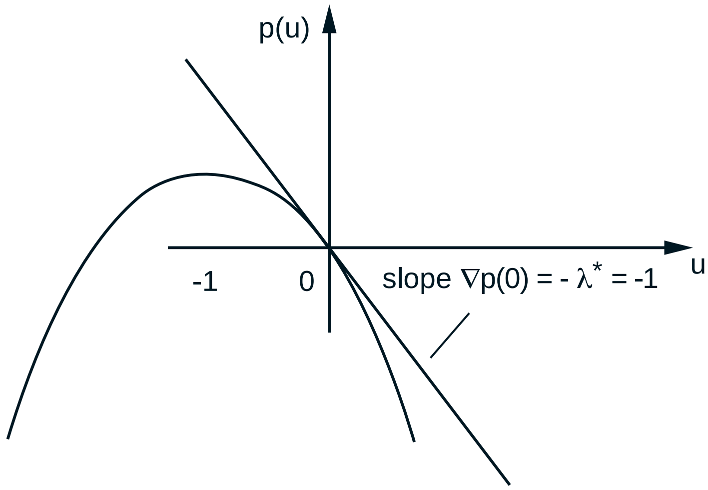


Illustration of the primal function $p(u) = f\big(x(u)\big)$ for the two-dimensional problem

$$\text{minimize} \quad f(x) = \tfrac{1}{2}\left(x_1^2 - x_2^2\right) - x_2$$
$$\text{subject to} \quad h(x) = x_2 = 0.$$

Here,

$$p(u) = \min_{h(x)=u} f(x) = -\tfrac{1}{2}u^2 - u$$

and $\lambda^* = -\nabla p(0) = 1$, consistently with the sensitivity theorem.

- Need for regularity of x^*: Change constraint to $h(x) = x_2^2 = 0$. Then $p(u) = -u/2 - \sqrt{u}$ for $u \geq 0$ and is undefined for $u < 0$.

PROOF OUTLINE OF SENSITIVITY THEOREM

Apply implicit function theorem to the system

$$\nabla f(x) + \nabla h(x)\lambda = 0, \qquad h(x) = u.$$

For $u = 0$ the system has the solution (x^*, λ^*), and the corresponding $(n+m) \times (n+m)$ Jacobian

$$J = \begin{pmatrix} \nabla^2 f(x^*) + \sum_{i=1}^m \lambda_i^* \nabla^2 h_i(x^*) & \nabla h(x^*) \\ \nabla h(x^*)' & 0 \end{pmatrix}$$

is shown nonsingular using the sufficiency conditions. Hence, for all u in some open sphere S centered at $u = 0$, there exist $x(u)$ and $\lambda(u)$ such that $x(0) = x^*$, $\lambda(0) = \lambda^*$, the functions $x(\cdot)$ and $\lambda(\cdot)$ are continuously differentiable, and

$$\nabla f\big(x(u)\big) + \nabla h\big(x(u)\big)\lambda(u) = 0, \quad h\big(x(u)\big) = u.$$

For u close to $u = 0$, using the sufficiency conditions, $x(u)$ and $\lambda(u)$ are a local minimum-Lagrange multiplier pair for the problem $\min_{h(x)=u} f(x)$.

To derive $\nabla p(u)$, differentiate $h\big(x(u)\big) = u$, to obtain $I = \nabla x(u)\nabla h\big(x(u)\big)$, and combine with the relations $\nabla x(u)\nabla f\big(x(u)\big) + \nabla x(u)\nabla h\big(x(u)\big)\lambda(u) = 0$ and $\nabla p(u) = \nabla_u \Big\{ f\big(x(u)\big)\Big\} = \nabla x(u)\nabla f\big(x(u)\big)$.