

BASIC RESULTS

Kuhn-Tucker Necessary Conditions: Let x^* be a local minimum and a regular point. Then there exist unique Lagrange mult. vectors $\lambda^* = (\lambda_1^*, \dots, \lambda_m^*)$, $\mu^* = (\mu_1^*, \dots, \mu_r^*)$, such that

$$\nabla_x L(x^*, \lambda^*, \mu^*) = 0,$$

$$\mu_j^* \geq 0, \quad j = 1, \dots, r,$$

$$\mu_j^* = 0, \quad \forall j \notin A(x^*).$$

If f , h , and g are twice cont. differentiable,

$$y' \nabla_{xx}^2 L(x^*, \lambda^*, \mu^*) y \geq 0, \quad \text{for all } y \in V(x^*),$$

where

$$V(x^*) = \{y \mid \nabla h(x^*)' y = 0, \nabla g_j(x^*)' y = 0, j \in A(x^*)\}.$$

• Similar sufficiency conditions and sensitivity results. They require strict complementarity, i.e.,

$$\mu_j^* > 0, \quad \forall j \in A(x^*).$$

PROOF OF KUHN-TUCKER CONDITIONS

Use equality-constraints result to obtain all the conditions except for $\mu_j^* \geq 0$ for $j \in A(x^*)$. Introduce the penalty functions

$$g_j^+(x) = \max\{0, g_j(x)\}, \quad j = 1, \dots, r,$$

and for $k = 1, 2, \dots$, let x^k minimize

$$f(x) + \frac{k}{2} \|h(x)\|^2 + \frac{k}{2} \sum_{j=1}^r (g_j^+(x))^2 + \frac{1}{2} \|x - x^*\|^2$$

over a closed sphere of x such that $f(x^*) \leq f(x)$. Using the same argument as for equality constraints,

$$\lambda_i^* = \lim_{k \rightarrow \infty} kh_i(x^k), \quad i = 1, \dots, m,$$

$$\mu_j^* = \lim_{k \rightarrow \infty} kg_j^+(x^k), \quad j = 1, \dots, r.$$

Since $g_j^+(x^k) \geq 0$, we obtain $\mu_j^* \geq 0$ for all j .

GENERAL SUFFICIENCY CONDITION

Consider the problem

minimize $f(x)$

subject to $x \in X$, $g_j(x) \leq 0$, $j = 1, \dots, r$.

Let x^* be feasible and μ^* satisfy

$$\mu_j^* \geq 0, \quad j = 1, \dots, r, \quad \mu_j^* = 0, \quad \forall j \notin A(x^*),$$

$$x^* = \arg \min_{x \in X} L(x, \mu^*).$$

Then x^* is a global minimum of the problem.

Proof: We have

$$\begin{aligned} f(x^*) &= f(x^*) + \mu^{*\prime} g(x^*) = \min_{x \in X} \{ f(x) + \mu^{*\prime} g(x) \} \\ &\leq \min_{x \in X, g(x) \leq 0} \{ f(x) + \mu^{*\prime} g(x) \} \leq \min_{x \in X, g(x) \leq 0} f(x), \end{aligned}$$

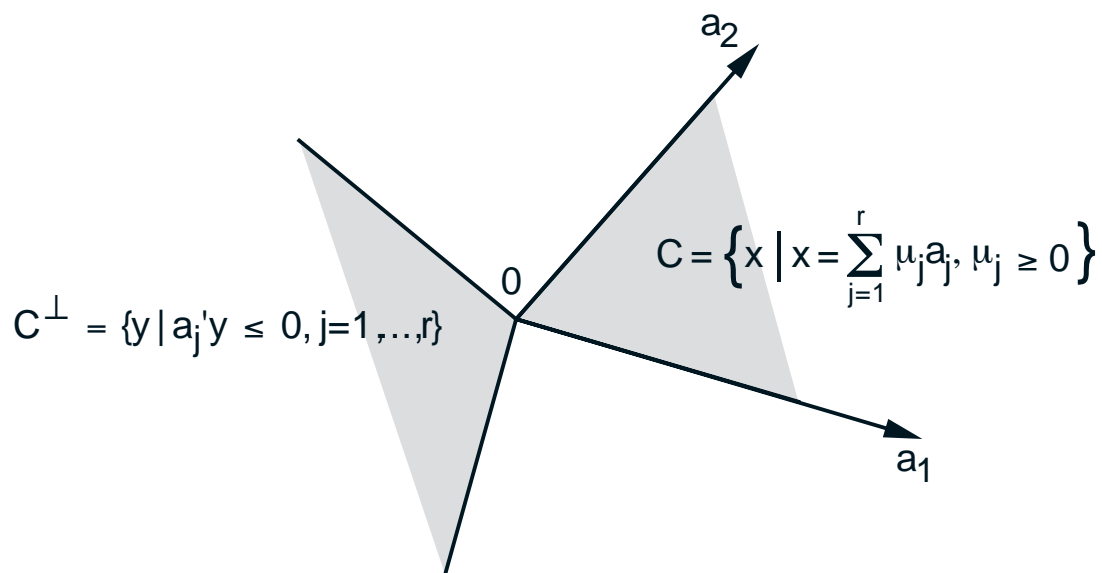
where the first equality follows from the hypothesis, which implies that $\mu^{*\prime} g(x^*) = 0$, and the last inequality follows from the nonnegativity of μ^* . **Q.E.D.**

LINEAR CONSTRAINTS

- Consider the problem $\min_{a'_j x \leq b_j, j=1, \dots, r} f(x)$.
- Remarkable property: No need for regularity.
- Proposition: If x^* is a local minimum, there exist μ_1^*, \dots, μ_r^* with $\mu_j^* \geq 0, j = 1, \dots, r$, such that

$$\nabla f(x^*) + \sum_{j=1}^r \mu_j^* a_j = 0, \quad \mu_j^* = 0, \quad \forall j \notin A(x^*).$$

- Proof uses Farkas Lemma: Consider the cones C and C^\perp



$$x \in C \quad \text{iff} \quad x' y \leq 0, \quad \forall y \in C^\perp.$$