Example 2 – Positive definiteness of a Hessian Matrix

Consider
$$\begin{aligned} &Gx^* = -c \\ &with \\ &G = \begin{pmatrix} 5 & 3 \\ 3 & 2 \end{pmatrix} \\ &c = \begin{pmatrix} -5.5 \\ -3.5 \end{pmatrix} \end{aligned}$$

Using the eigenvalue solution method:

We search the eigenvalues of G which are solutions of the equation:

$$|G - \lambda I| = 0$$
$$\begin{vmatrix} 5-\lambda & 3 \\ 3 & 2-\lambda \end{vmatrix} = 0 \Rightarrow 10 - 7\lambda + \lambda^2 - 9 = 0$$

Hence the roots of the quadratic are the 2 eigenvalues:

$$\lambda_1 = \frac{7-3\sqrt{5}}{2} > 0, \lambda_2 = \frac{7+3\sqrt{5}}{2} > 0$$

Hence G is a positive definite matrix.

Method 2 using stationary points:

At a stationary point the condition $Gx^* = -c$ is satisfied or

$$\begin{pmatrix} 5 & 3 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \frac{11}{2} \\ \frac{7}{2} \end{pmatrix}$$

Using Cramer we get :

$$x^* = (1/2, 1)$$

The quadratic form is:

$$Q(x_1, x_2) = c^T x + \frac{1}{2} x^T G x = -\frac{11}{2} x_1 - 7/2\, x_2 + \frac{5}{2} x_1^2 + 3x_1 x_2$$
$$+x_2^2$$

Let us denote by $u = (u_1, u_2)$ the eigenvector corresponding to the eigenvalue λ

We know that $Q(x^* + \alpha u) = Q(x^*) + \frac{1}{2}\alpha^2 \lambda$

Let us compute $Q(x^* + \alpha u) =$

$$Q(x^*) + \frac{1}{2}\alpha^2 (5u_1^2 + 6u_1 u_2 + 2u_2^2)$$

i.e.

$$\lambda = 5u_1^2 + 6u_1 u_2 + 2u_2^2 = (\sqrt{5}u_1 + 3/\sqrt{5}\, u_2)^2 + (\frac{1}{\sqrt{5}} u_2)^2 > 0$$

So each eigenvalue is positive so G is positive definite.