A BASIC LEMMA

Lemma: Let P and Q be two symmetric matrices. Assume that $Q \geq 0$ and $P > 0$ on the nullspace of Q, i.e., $x'Px > 0$ for all $x \neq 0$ with $x'Qx = 0$. Then there exists a scalar $\overline{c}$ such that

$$P + cQ : \text{positive definite}, \qquad \forall\ c > \overline{c}.$$

Proof: Assume the contrary. Then for every k, there exists a vector x^k with $\|x^k\| = 1$ such that

$$x^{k'}Px^k + kx^{k'}Qx^k \leq 0.$$

Consider a subsequence $\{x^k\}_{k \in K}$ converging to some $\overline{x}$ with $\|\overline{x}\| = 1$. Taking the limit superior,

$$\overline{x}'P\overline{x} + \limsup_{k \to \infty,\, k \in K} (kx^{k'}Qx^k) \leq 0. \tag{*}$$

We have $x^{k'}Qx^k \geq 0$ (since $Q \geq 0$), so $\{x^{k'}Qx^k\}_{k \in K} \to 0$. Therefore, $\overline{x}'Q\overline{x} = 0$ and using the hypothesis, $\overline{x}'P\overline{x} > 0$. This contradicts (*).

PROOF OF SUFFICIENCY CONDITIONS

Consider the $augmented\ Lagrangian$ function

$$L_c(x,\lambda) = f(x) + \lambda' h(x) + \frac{c}{2}\|h(x)\|^2,$$

where c is a scalar. We have

$$\nabla_x L_c(x,\lambda) = \nabla_x L(x,\tilde{\lambda}),$$

$$\nabla^2_{xx} L_c(x,\lambda) = \nabla^2_{xx} L(x,\tilde{\lambda}) + c\nabla h(x)\nabla h(x)'$$

where $\tilde{\lambda} = \lambda + ch(x)$. If (x^*,λ^*) satisfy the suff. conditions, we have using the lemma,

$$\nabla_x L_c(x^*,\lambda^*) = 0, \qquad \nabla^2_{xx} L_c(x^*,\lambda^*) > 0,$$

for suff. large c. Hence for some $\gamma > 0$, $\epsilon > 0$,

$$L_c(x,\lambda^*) \geq L_c(x^*,\lambda^*) + \frac{\gamma}{2}\|x - x^*\|^2, \quad \text{if } \|x - x^*\| < \epsilon.$$

Since $L_c(x,\lambda^*) = f(x)$ when $h(x) = 0$,

$$f(x) \geq f(x^*) + \frac{\gamma}{2}\|x - x^*\|^2, \quad \text{if } h(x) = 0,\ \|x - x^*\| < \epsilon.$$