MODELING AND ANALYSIS OF A PERIODIC GINZBURG–LANDAU MODEL FOR TYPE-II SUPERCONDUCTORS*

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Abstract. A periodic Ginzburg–Landau model for superconductivity is considered. The model has two novel features compared to periodic problems arising in other settings. First, periodicity is defined with respect to a nonorthogonal lattice that is not necessarily aligned with the coordinate axes. Second, the periodicity of the physical variables implies nonstandard, in the context of periodic problems, relations for the primary dependent variables used in the model. Physical assumptions are introduced that form the basis for the model, and then the mathematical model is derived from these assumptions. The model discussed includes, as special cases, periodic Ginzburg–Landau models appearing in the literature. Then the model equations and its solutions are analyzed, addressing, among others, questions of existence and regularity. The paper closes with some remarks relevant to the use of the model in conjunction with analytic or numerical approximation methods.

Key words. superconductivity, Ginzburg–Landau equations, periodic solutions, type-II superconductors

AMS(MOS) subject classifications. 81J05, 35J60

1. Introduction. Following the discovery of materials that retain superconductive properties at “high” temperatures, there has been a renewal of interest in superconductivity among the physics, material science, engineering, and mathematics communities. A perceived need of those who wish to design superconducting devices and those who wish to study the physics of superconductivity is for robust and efficient algorithms for the numerical simulation of superconducting phenomena. Such simulations must be based on mathematical models that are, on the one hand, physically well founded and, on the other hand, amenable to discretizations resulting in practical algorithms.

There are a variety of excellent sources that may be used as an introduction to the subject of superconductivity. For example, see [5], [9], [21], [26], and [27]. Also, one may consult recent articles [8] and [12]. Here we just touch upon various issues connected with models for superconductivity, mainly to motivate the use of the periodic model discussed in §2. Details may be found in the references just cited. Our discussion leads us to a series of forks at each of which a modeling decision must be made. Specific choices made at these forks result in the periodic model.

All our considerations are for macroscopic, as opposed to microscopic, models of superconductivity. Thus we do not delve into superconducting phenomena at the atomic or subatomic level, but rather look at superconductors on scales comparable to the size of devices such as wires, films, and computer chips that use such materials. The study of microscopic models is likely to yield information about the basic physics of

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superconductivity. However, if one is interested in designing superconducting devices, then macroscopic models are of greater interest.

Another fork has already been alluded to, namely, the one leading to either high- or low-temperature superconductors. All superconducting materials lose their superconductive properties above some critical temperature $T_c$; the value of $T_c$ differs from one material to another. Thus, when we talk of “high-” or “low-temperature” superconductors, we really mean materials with, relatively speaking, high or low values for $T_c$, respectively. For low-$T_c$ superconductors, we find ourselves in the ideal situation of having a macroscopic model in hand that can be derived from a well-accepted microscopic model. The macroscopic model has the names of Ginzburg and Landau attached to it [16], and the microscopic model from which it can be derived is known as the BCS model [6]; that the latter is well accepted is evidenced by the fact that it gained a Nobel prize for its discoverers, Bardeen, Cooper, and Schrieffer (BCS). The connection between the microscopic BCS model and the macroscopic Ginzburg–Landau model was first made by Gor’kov [17].

For high-$T_c$ superconductors, satisfactory microscopic models still await discovery. On the other hand, it is a reasonably simple matter to generalize the low-$T_c$ Ginzburg–Landau model to account for high-$T_c$ phenomena. For example, the low-$T_c$ Ginzburg–Landau model deals with materials that are, from an electromagnetic viewpoint, homogeneous and isotropic; high-$T_c$ Ginzburg–Landau models could involve inhomogeneities and anisotropies. Of course, since there is currently no rigorous justification, emanating from a well-accepted microscopic theory, for the use of Ginzburg–Landau models for high-$T_c$ superconductivity, such justification must be made through comparisons with observations. In this regard there is mounting evidence that, indeed, Ginzburg–Landau models may be used in this context. In this paper, we confine ourselves to Ginzburg–Landau models for low-$T_c$ superconductivity. Our plans for the near future are to adapt and apply generalizations of this model to various problems involving high-$T_c$ superconductors.

Superconductors are divided into two types, known as type-I and type-II superconductors. From a technological point of view, type-II superconductors are of greater interest, mainly because they can maintain superconducting properties in the presence of higher external magnetic fields than can type-I superconductors. Of relevance to the present work are the observations that electromagnetic phenomena in type-I superconductors may be described by relatively smooth functions that can be well approximated through the use of standard discretization techniques, e.g., finite element methods, while on the other hand, such phenomena in type-II superconductors exhibit significant variations over length scales of the order of 1000 or fewer Ångstroms.

Suppose now that we wish to effect a numerical simulation of electromagnetic phenomena in a sample of superconducting material that is part of some device; typically, the size of this sample would be of the order of a centimeter or more. For a type-I superconductor, this poses little problem since, except for possible edge effects, all variables to be approximated are smooth with respect to the scale of the sample size. Finite element methods for just this situation are considered in [12]. However, for a type-II superconductor, it is not possible, using currently available computers, to simulate phenomena exhibiting spatial variations having length scales of the order of 1000 Ångstroms. For example, if we were to use a finite element method, then the number of degrees of freedom necessary to well approximate such functions would be prohibitively large.

The inability to perform simulations for typical material samples of type-II superconductors gives rise to the use of the common practice of neglecting the effects
due to the fact that the sample has boundaries. Thus we assume that we are far removed from the boundary of the superconducting sample and that, in such regions, the physically relevant variables, e.g., the magnetic field, the current, and so on, are, in some sense, periodic. The exact nature of the topology of the periodic behavior is well known for low-$T_c$ superconductors. The use of a periodic model allows us to focus on a piece of the sample that is of roughly the same size as that of the scale of variations in the interesting phenomena. In this case, it is possible to resolve these phenomena on currently available computers.

As a result of the above observations, we are led, in this paper, to study a periodic Ginzburg–Landau model for type-II superconductivity. Type-II superconductors are chosen because of their greater technological applicability; a Ginzburg–Landau model is chosen since we are interested in the macroscopic behavior of such materials, again for technological reasons; a periodic model is chosen as a result of the impossibility of resolving interesting phenomena occurring in material samples of realistic size and which have boundaries.

From the viewpoint of periodic models encountered in other applications, the model considered in this paper exhibits two novel features. In the first place, the periodicity is not necessarily with respect to a rectangular lattice, but may involve, e.g., triangular lattices. In fact, we consider general lattices so that all possible topologies for periodicity in the plane can be treated. Perhaps of even greater interest is the fact that the primary variables used in the model are not themselves periodic. One variable is scalar, complex-valued, and its phase suffers a jump across the lattice defining the periodicity, while another is vector-valued and itself suffers such a jump.

Periodic Ginzburg–Landau models have been used in the past as a setting for analyzing and approximating phenomena in type-II superconductors; see, e.g., [1], [7], [11], [14], [19], [20], [22], and [25]. Most of these deal with some sort of series solution of a periodic Ginzburg–Landau model. One notable exception is [11], which uses a Monte Carlo/simulated annealing approach, and represents the currently best-known effort for simulating type-II superconductors.

The plan of the paper is as follows. In the remainder of this section, we introduce the notation that is to be used throughout the rest of the paper. In §2 we describe the periodic Ginzburg–Landau model and give some new results concerning the model. Then, in §3 we consider the Ginzburg–Landau equations; these are a system of partial differential equations that are derived from the model. We also give some new results concerning solutions of these equations. In §4 we discuss the regularity of solutions of the periodic Ginzburg–Landau equations. In §5 we briefly consider some issues relevant to obtaining, by either numerical or analytic means, approximate solutions of the Ginzburg–Landau equations. The main existence proof is given in the Appendix. Finite element algorithms based on the model discussed in this paper are considered in [13].

1.1. Preliminaries. Throughout, we only deal with $\mathbb{R}^2$. Thus all vectors have only two components, say in the $x_1$ and $x_2$ directions, and these, as well as all scalar-valued functions, depend only on the two variables $x_1$ and $x_2$. We adopt the convention that the vector product of the two vectors $\mathbf{a} = (a_1, a_2)^T$ and $\mathbf{b} = (b_1, b_2)^T$ is the scalar $\mathbf{a} \times \mathbf{b} = a_1b_2 - a_2b_1$. Since we are dealing with the plane, it is convenient to introduce the following two curl operators:
\[
\text{curl } \mathbf{A} = \frac{\partial A_2}{\partial x_1} - \frac{\partial A_1}{\partial x_2} \quad \text{and} \quad \text{curl } \psi = \left( \begin{array}{c} \frac{\partial \psi}{\partial x_2} \\ \frac{\partial \psi}{\partial x_1} \end{array} \right).
\]

The first takes the two-component vector \( \mathbf{A} = (A_1, A_2)^T \) into the scalar \( \text{curl } \mathbf{A} \), and the second takes the scalar \( \psi \) into the two-component vector \( \text{curl } \psi \). We also have the divergence and gradient operators defined by

\[
\text{div } \mathbf{A} = \frac{\partial A_1}{\partial x_1} + \frac{\partial A_2}{\partial x_2} \quad \text{and} \quad \text{grad } \psi = \left( \begin{array}{c} \frac{\partial \psi}{\partial x_1} \\ \frac{\partial \psi}{\partial x_2} \end{array} \right),
\]

respectively. These various operators satisfy the usual relations, i.e., \( \text{div } \text{curl } \psi = 0 \) and \( \text{curl } \text{grad } \psi = 0 \) for any sufficiently smooth \( \psi \), as well as

\[
\Delta \psi = \frac{\partial^2 \psi}{\partial x_1^2} + \frac{\partial^2 \psi}{\partial x_2^2} = \text{div } \text{grad } \psi = -\text{curl } \text{curl } \psi.
\]

Also, we have the integration by parts formulas

\[
\int_D \psi \text{div } \mathbf{A} \, dD + \int_D \mathbf{A} \cdot \text{grad } \psi \, dD = \int_{\partial D} \psi \mathbf{A} \cdot \mathbf{n} \, d(\partial D)
\]

and

\[
\int_D \psi \text{curl } \mathbf{A} \, dD = \int_D \mathbf{A} \cdot \text{curl } \psi \, dD = \int_{\partial D} \psi \mathbf{A} \cdot \mathbf{\tau} \, d(\partial D),
\]

where \( D \) denotes a bounded subset of \( \mathbb{R}^2 \), \( \partial D \) its boundary, \( \mathbf{n} \) the unit outer normal vector to \( \partial D \), and \( \mathbf{\tau} \) the unit counterclockwise tangent vector to \( \partial D \).

Given two arbitrary vectors \( \mathbf{t}_1 \) and \( \mathbf{t}_2 \) that span \( \mathbb{R}^2 \), we say that a function \( f(\mathbf{x}) \) is periodic with respect to the lattice determined by \( \mathbf{t}_1 \) and \( \mathbf{t}_2 \) if

\[
f(\mathbf{x} + \mathbf{t}_k) = f(\mathbf{x}) \quad \forall \mathbf{x} \in \mathbb{R}^2.
\]

Here, \( f \) may be scalar or vector-valued and may be real or complex-valued. The vectors \( \mathbf{t}_k, k = 1, 2 \), are referred to as lattice vectors; without loss of generality, we assume that the counterclockwise angle between \( \mathbf{t}_1 \) and \( \mathbf{t}_2 \) is less than \( \pi \). For the sake of brevity, we often refer to functions satisfying (1.1) merely as being periodic. Note that (1.1) implies, for differentiable \( f \), that

\[
\left. \frac{\partial f}{\partial x_j}(\mathbf{x} + \mathbf{t}_k) = \frac{\partial f}{\partial x_j}(\mathbf{x}) \right|_{k = 1, 2, \ j = 1, 2} \quad \forall \mathbf{x} \in \mathbb{R}^2.
\]

We note that the only harmonic, periodic function is the constant function.

Given any point \( \mathbf{P} \in \mathbb{R}^2 \), a cell of the lattice with respect to the point \( \mathbf{P} \) is the open parallelogram \( \Omega_P \subset \mathbb{R}^2 \) depicted in Fig. 1.1. (There are numerous simple characterizations of this parallelogram, including the following one. Given the vectors \( \mathbf{t}_1 \) and \( \mathbf{t}_2 \), let \( A \) be the \( 2 \times 2 \) matrix whose columns are the vectors \( \mathbf{t}_1 \) and \( \mathbf{t}_2 \); then, a point \( \mathbf{X} \) is in the open parallelogram \( \Omega_P \) of Fig. 1.1 if and only if both components of the vector \( A^{-1}(\mathbf{X} - \mathbf{P}) \) are greater than zero and less than unity.) The boundary
of the cell $\Omega_P$ is denoted by $\Gamma_P$. When $P$ corresponds to the origin, we denote the corresponding cell by $\Omega$ and its boundary by $\Gamma$.

![Diagram](image)

**Fig. 1.1.** The cell $\Omega_P$ determined by the lattice vectors $t_1$ and $t_2$ and the point $P$.

We use the following function spaces. First, for any bounded set $D \subset \mathbb{R}^2$, we have, for any nonnegative integer $m$, the Sobolev space $H^m(D)$ consisting of square integrable functions over $D$ such that all derivatives of order up to and including $m$ are also square integrable. Norms for functions belonging to $H^m(D)$ are denoted by $\| \cdot \|_{m,D}$; when there is no chance of confusion, we omit the subscript $D$ from this notation for norms. Then we may define the spaces

$$H^m_{\text{loc}}(\mathbb{R}^2) = \left\{ \phi : \mathbb{R}^2 \to \mathbb{R} \mid \phi \in H^m(D) \quad \forall \text{ bounded } D \subset \mathbb{R}^2 \right\},$$

$$\mathcal{H}^m_{\text{loc}}(\mathbb{R}^2) = \left\{ \psi : \mathbb{R}^2 \to \mathbb{C} \mid \Re(\psi), \Im(\psi) \in H^m(D) \quad \forall \text{ bounded } D \subset \mathbb{R}^2 \right\},$$

where $\Re(\cdot)$ and $\Im(\cdot)$ denote the real and imaginary parts, respectively, and

$$H^m_{\text{loc}}(\mathbb{R}^2) = \left\{ \mathbf{A} = (A_1, A_2)^T : \mathbb{R}^2 \to \mathbb{R}^2 \mid A_1, A_2 \in H^m(D) \quad \forall \text{ bounded } D \subset \mathbb{R}^2 \right\}.$$

Given the lattice vectors $t_k, k = 1, 2$, we can also define, for $m \geq 1$, the space

$$H^m_{\text{per}}(\mathbb{R}^2) = \left\{ \phi \in H^m_{\text{loc}}(\mathbb{R}^2) \mid \phi(x + t_k) = \phi(x) \quad \text{for } k = 1, 2 \text{ and } \forall \mathbf{x} \in \mathbb{R}^2 \right\},$$

and the analogous spaces $\mathcal{H}^m_{\text{per}}(\mathbb{R}^2)$ and $H^1_{\text{per}}(\mathbb{R}^2)$ for complex- and vector-valued functions. Note that in the last definition the symbol $\forall$ should be interpreted in a weak sense. Details concerning these spaces may be found in, e.g., [2].

We now give a precise definition of gauge invariance. For any $\chi \in H^2_{\text{loc}}(\mathbb{R}^2)$, let the linear transformation $G_\chi$ from $\mathcal{H}^1_{\text{loc}}(\mathbb{R}^2) \times H^1_{\text{loc}}(\mathbb{R}^2)$ into itself be defined by

$$G_\chi(\psi, \mathbf{A}) = (\zeta, \mathbf{Q}) \in \mathcal{H}^1_{\text{loc}}(\mathbb{R}^2) \times H^1_{\text{loc}}(\mathbb{R}^2) \quad \forall \psi, \mathbf{A} \in \mathcal{H}^1_{\text{loc}}(\mathbb{R}^2) \times H^1_{\text{loc}}(\mathbb{R}^2)$$

if

$$\zeta = \psi e^{i\chi} \quad \text{and} \quad \mathbf{Q} = \mathbf{A} + \frac{1}{\kappa} \text{grad } \chi,$$
where \( \kappa \), a parameter appearing in the model, is defined below. Note that, if \((\zeta, \mathbf{Q}) = \mathbf{G}_\chi(\psi, \mathbf{A})\), then \((\psi, \mathbf{A}) = \mathbf{G}_{-\chi}(\zeta, \mathbf{Q})\).

**Definition.** \((\psi, \mathbf{A})\) and \((\zeta, \mathbf{Q})\) are said to be **gauge equivalent** if and only if there exists a \( \chi \in H^2_{\text{loc}}(\mathbb{R}^2) \) such that \((\psi, \mathbf{A}) = \mathbf{G}_{\chi}(\zeta, \mathbf{Q})\).

**2. The periodic Ginzburg–Landau model for superconductivity.** The variables used in Ginzburg–Landau models for superconductivity are the real, vector-valued **magnetic potential** \( \mathbf{A} \) and the complex, scalar-valued **order parameter** \( \psi \). These are related to (appropriately nondimensionalized) physical variables as follows:

\[
\begin{align*}
\text{magnetic field} & \quad h = \text{curl} \mathbf{A}, \\
\text{current} & \quad \mathbf{j} = \text{curl} h = \text{curl} \text{curl} \mathbf{A}, \\
\text{density of superconducting charge carriers} & \quad N_s = |\psi|^2.
\end{align*}
\]

Later, we see that the phase of the order parameter can be related to the current. Note that, since we are dealing with \( \mathbb{R}^2 \), the magnetic field may be defined as a scalar-valued function; viewed as a vector, the magnetic field points out of the \((x_1, x_2)\)-plane.

Ginzburg and Landau postulated, based on the Landau theory of second-order phase transitions, that the Gibbs free energy density of a superconducting material has the form

\[
f_0 - |\psi|^2 + \frac{1}{2}|\psi|^4 + \left| \left( \frac{i}{\kappa} \text{grad} + \mathbf{A} \right) \psi \right|^2 + |\text{curl} \mathbf{A}|^2,
\]

where \( \kappa \) denotes the Ginzburg–Landau parameter, a real constant whose value depends on the material properties and the temperature of the medium, and where \( f_0 \), the free energy density of the material in the nonsuperconducting state, is independent of \( \psi \) and \( \mathbf{A} \). See, e.g., [12], [16], [21], [26], or [27] for details. Then, the Gibbs free energy of the material occupying a lattice cell \( \Omega_P \) is given by

\[
(2.2) \quad \mathcal{G}(\psi, \mathbf{A}) = \int_{\Omega_P} \left( f_0 - |\psi|^2 + \frac{1}{2}|\psi|^4 + \left| \left( \frac{i}{\kappa} \text{grad} + \mathbf{A} \right) \psi \right|^2 + |\text{curl} \mathbf{A}|^2 \right) d\Omega.
\]

**2.1. Periodicity assumptions.** The first of the basic assumptions of the periodic Ginzburg–Landau model for superconductivity concerns the periodic nature of the physical attributes of the superconductor, i.e., the density of superconducting charge carriers \( N_s \), the magnetic field \( h \), and the free energy density are periodic with respect to the lattice vectors \( \mathbf{t}_1 \) and \( \mathbf{t}_2 \). In terms of the magnetic potential \( \mathbf{A} \) and order parameter \( \psi \), these infer that

\[
(2.3) \quad \text{curl} \mathbf{A}, \quad |\psi|, \quad \text{and} \quad \left( \frac{1}{\kappa} \text{grad} \phi - \mathbf{A} \right) \quad \text{are periodic},
\]

where \( \phi \) denotes the phase of the order parameter, i.e., \( \psi = |\psi|e^{\imath \phi} \). With (2.1) and (1.2), we have that the current \( \mathbf{j} \) is also periodic. Note that the magnetic potential and order parameter themselves are not assumed to be periodic. However, (2.2) and (2.3) imply that the value of the Gibbs free energy in a lattice cell is the same for any cell; i.e., it is independent of the choice of the point \( \mathbf{P} \).

The lattice vectors \( \mathbf{t}_1 \) and \( \mathbf{t}_2 \) are part of the problem specification. However, at this point, there is no advantage to making specific choices for these vectors. In §6 we discuss how to make physically motivated choices for the lattice vectors.
Another assumption concerns the notion of fluxoid quantization, i.e., in the present context, the fluxoid \( \Phi' \) satisfies
\[
\Phi' = \int_{\Omega_P} h \, d\Omega + \int_{\Gamma_P} \frac{1}{|\psi|^2} \cdot \tau \, d\Gamma = \frac{2\pi n}{\kappa},
\]
where \( n \) is an integer and where \( \Omega_P \) is any cell of the lattice and \( \Gamma_P \) its boundary. Again, for details, see the references cited above. The periodicity of \( j, |\psi|, \) and \( (\kappa A - \text{grad } \phi) \) with respect to any lattice cell \( \Omega_P \) and the relation \( h = \text{curl } A \) between the magnetic field and magnetic potential then yield that
\[
(2.4) \quad \kappa \int_{\Omega_P} \text{curl } A \, d\Omega = \kappa \int_{\Gamma_P} A \cdot \tau \, d\Gamma = \int_{\Gamma_P} \text{grad } \phi \cdot \tau \, d\Gamma = 2\pi n.
\]

Thus the fluxoid quantization requirement simply states that \( \phi \), the phase of the order parameter, must change by an integer multiple of \( 2\pi \) during one circuit of the boundary \( \Gamma_P \) of a typical lattice cell \( \Omega_P \). The average magnetic field \( \bar{B} \) over a cell \( \Omega_P \) is defined by
\[
(2.5) \quad \bar{B} = \frac{1}{|\Omega|} \int_{\Omega_P} h \, d\Omega = \frac{1}{|\Omega|} \int_{\Omega_P} \text{curl } A \, d\Omega,
\]
where \( |\Omega| \) denotes the area of the cell \( \Omega_P \). Note that \( \bar{B} \) and \( |\Omega| \) are independent of the choice of cell, i.e., of the choice of the point \( P \). The fluxoid quantization condition (2.4) can now be expressed as
\[
(2.6) \quad \kappa \bar{B} |\Omega| = 2\pi n,
\]
which relates the average magnetic field to the area of a lattice cell and the number of fluxoids it carries.

### 2.2. Choice of gauge. Choose \( Q \) to satisfy
\[
\Delta Q = -\text{curl curl } A \quad \forall \ x \in \mathbb{R}^3 \quad \text{and} \quad Q \ \text{periodic}.
\]

Then, concerning such \( Q \), we have the following results.

**Lemma 2.1.** Let \( A \in H^1_\text{loc}(\mathbb{R}^2) \) and \( \text{curl } A \) be periodic. Then (2.7) has a solution \( Q \) belonging to \( H^1_\text{loc}(\mathbb{R}^2) \). Moreover, solutions are unique up to an additive constant vector.

**Proof.** The proof is straightforward. For example, we may use a variational approach to show existence. For the uniqueness result, we may use the fact that, if \( Q^{(j)}, j = 1, 2 \), are two solutions of (2.7) with components \( q_1^{(j)} \) and \( q_2^{(j)} \), then
\[
\int_{\Omega_P} |\text{grad } (q_1^{(1)} - q_1^{(2)})|^2 \, d\Omega + \int_{\Omega_P} |\text{grad } (q_2^{(1)} - q_2^{(2)})|^2 \, d\Omega = 0
\]
for any lattice cell \( \Omega_P \). \( \square \)

**Lemma 2.2.** If \( Q \) is a solution of (2.7), then
\[
\text{div } Q = 0 \quad \text{a.e.}
\]
Proof. Let \( w = \text{div} \, Q \). Then (2.7) implies that \( \Delta w = 0 \) in the sense of distributions, and \( w \) is periodic. It follows that \( w \) is a constant almost everywhere Let \( \Omega_P \) be a lattice cell. Then

\[
 w|\Omega| = \int_{\Omega_P} \text{div} \, Q \, d\Omega = \int_{\Gamma_P} Q \cdot n \, d\Gamma = 0,
\]

where the last equality follows from the periodicity of \( Q \). Thus \( w = 0 \) almost everywhere.

\[\square\]

**Lemma 2.3.** If \( Q \) is a solution of (2.7), then \( \text{curl} \, Q \) is periodic and

\[
(2.8) \quad \text{curl} \, Q = \text{curl} \, A - \bar{B} = \text{curl} \, A - \frac{2\pi n}{\kappa|\Omega|} \quad \text{a.e.}
\]

**Proof.** The periodicity of \( \text{curl} \, Q \) follows from (1.2), applied to the components of \( Q \), and the periodicity of \( Q \). Let \( v = (\text{curl} \, Q - \text{curl} \, A) \). Then (2.7) and the periodicity of \( \text{curl} \, A \) imply that \( \Delta v = 0 \), in the sense of distributions, and \( v \) is periodic. It follows that \( v \) is a constant almost everywhere. Let \( \Omega_P \) be a lattice cell. Then (2.4), (2.5), and the periodicity of \( Q \) imply that

\[
v|\Omega| = \int_{\Omega_P} \text{curl} \, Q \, d\Omega - \int_{\Omega_P} \text{curl} \, A \, d\Omega = -\int_{\Omega_P} h \, d\Omega = -|\Omega|\bar{B}.
\]

Thus \( \text{curl} \, Q - \text{curl} \, A = v = -\bar{B} \) almost everywhere.

\[\square\]

**Theorem 2.4.** Let \( A \) and \( \psi = |\psi|e^{i\phi} \) satisfy (2.3). Let \( Q \) be defined by (2.7). Let

\[
(2.9) \quad A_\theta = \frac{\bar{B}}{2} \left\{ \left( \begin{array}{c} x_2 \\ -x_1 \end{array} \right) + \theta \left( \begin{array}{c} x_2 \\ x_1 \end{array} \right) \right\},
\]

where \( \theta \) is an arbitrary real number, and let

\[
(2.10) \quad \zeta = \psi e^{i\chi} = |\psi|e^{i(\phi+\chi)} = |\psi|e^{i\omega},
\]

where \( \omega = \phi + \chi \) and \( \chi \in H^2_{\text{loc}}(\mathbb{R}^2) \) satisfies

\[
(2.11) \quad \frac{1}{\kappa} \text{grad} \, \chi = Q - A - A_\theta.
\]

Then \((\psi, A)\) is gauge equivalent to \((\zeta, Q - A_\theta)\). Moreover, \( Q \) and \( \text{curl} \, Q \) are periodic, \( Q \) is uniquely determined up to an additive constant vector, \( \text{div} \, Q = 0 \), and

\[
(2.12) \quad A_\theta + \frac{1}{\kappa} \text{grad} \, (\phi + \chi) = A_\theta + \frac{1}{\kappa} \text{grad} \, \omega \quad \text{is periodic.}
\]

**Proof.** From (2.9), it follows that

\[
(2.13) \quad \text{div} \, A_\theta = 0 \quad \text{and} \quad \text{curl} \, A_\theta = -\bar{B}.
\]

As a result of (2.8) and (2.13), we have that

\[
\text{curl} \, (Q - A - A_\theta) = 0 \quad \text{a.e.,}
\]
which implies that there exists a function $\chi \in H^2_{\text{loc}}(\mathbb{R}^2)$ such that (2.11) is satisfied. Clearly, $(\psi, A)$ and $(\zeta, Q - A_\theta)$ are gauge equivalent; indeed, $(\zeta, Q - A_\theta) = G_\chi(\psi, A)$. The results concerning $Q$ follow from (2.7) and Lemmas 2.1–2.3. Finally, note that, from (2.11), we have that

$$A - \frac{1}{\kappa} \text{grad} \phi = Q - A_\theta - \frac{1}{\kappa} \text{grad} \phi - \frac{1}{\kappa} \text{grad} \chi = Q - A_\theta - \frac{1}{\kappa} \text{grad} \omega.$$ 

Since $Q$ and $[A - (1/\kappa)\text{grad} \phi]$ are periodic, (2.12) follows.

**Remark.** In the various papers concerning periodic Ginzburg–Landau models, different choices have been made for $\theta$. We further discuss these choices below.

**Remark.** If $\vec{B} \neq 0$, the magnetic field $h = \text{curl} \, A \neq \text{curl} \, Q$; rather, by (2.8) and (2.13), $h = \text{curl} \, Q + \vec{B} = \text{curl} \, Q - \text{curl} \, A_\theta$. This is also to be expected, since $A$ is related to $Q - A_\theta$ and not to just $Q$ through a gauge transformation.

### 2.3. Consequences of periodicity to the magnetic potential and order parameter

As was noted above, the order parameter, or more precisely, the phase of the order parameter, and the magnetic potential are not periodic. We now examine what can be said about these variables as a result of the periodicity assumptions contained in (2.3).

Let $t_k = (t_{k1}, t_{k2})^T$, $k = 1, 2$, denote the lattice vectors, and let $T_k = (t_{k2}, t_{k1})^T$. Let

$$g_{\theta k}(x) = \frac{-B}{2} \left\{ (x \times t_k) + \theta(x \cdot T_k) \right\}, \quad k = 1, 2.$$ 

Then it is easily verified that

$$A_\theta(x + t_k) - A_\theta(x) = -\text{grad} \, g_{\theta k}(x), \quad k = 1, 2,$$

and, as a result of (2.12),

$$\text{grad} \, \omega(x + t_k) - \text{grad} \, \omega(x) = \kappa \text{grad} \, g_{\theta k}(x), \quad k = 1, 2.$$ 

Then it follows that

$$\omega(x + t_k) - \omega(x) = \kappa g_{\theta k}(x) + \kappa C_k, \quad k = 1, 2,$$

for some constants $C_1$ and $C_2$.

We have seen, in Theorem 2.4, that $(\psi, A)$ is gauge equivalent to $(\zeta, Q - A_\theta)$, i.e., $(\zeta, Q - A_\theta) = G_\chi(\psi, A)$, where $Q$ is determined up to an arbitrary additive constant vector. The value we choose for this constant vector is irrelevant insofar as the results obtained so far. To see this, let $K$ denote an arbitrary constant vector,

$$\vec{Q} = Q + K \quad \text{and} \quad \tilde{\zeta} = \zeta e^{iK \cdot x} = |\psi| e^{i(\omega + \kappa K \cdot x)} = |\psi| e^{i\omega}.$$ 

Clearly, $(\zeta, Q - A_\theta)$ and $(\tilde{\zeta}, \tilde{Q} - A_\theta)$ are gauge equivalent, i.e., $(\tilde{\zeta}, \tilde{Q} - A_\theta) = G_{\kappa K \cdot x}(\zeta, Q - A_\theta)$, as are $(\psi, A)$ and $(\tilde{\zeta}, \tilde{Q} - A_\theta)$, i.e., $(\tilde{\zeta}, \tilde{Q} - A_\theta) = G_{(x + \kappa K \cdot x)}(\psi, A)$. The results of Theorem 2.4 for $(\zeta, Q)$ also hold for $(\tilde{\zeta}, \tilde{Q})$. Moreover, by (2.17) and (2.18), we have that

$$\tilde{\omega}(x + t_k) - \tilde{\omega}(x) = \kappa g_{\theta k}(x) + \kappa \tilde{C}_k, \quad k = 1, 2,$$
where \( \hat{C}_k = (C_k + K \cdot t_k) \), \( k = 1, 2 \). Thus any change to \( Q \) by an arbitrary additive constant vector can be accounted for by a change in the gauge transformation and a redefinition of the constants in the shifts in the phase \( \omega \) of \( \zeta \) across a lattice cell.

At this point, we are free to choose the constant \( \theta \) that appears in the definition (2.9) for \( A_\theta \). We can also add an arbitrary constant vector \( K \) to \( Q \) (see (2.18)) or add arbitrary constants \( C_1 \) and \( C_2 \) to the shifts in the phase \( \omega \) of \( \zeta \) across a lattice cell (see (2.17)). Unfortunately, the literature does not present a consistent choice for \( \theta \) or for \( C_1 \) and \( C_2 \). For example, in [25], \( \theta = C_1 = C_2 = 0 \), while, in [14], \( \theta = 1 \), \( C_1 = 0 \), and \( C_2 = -Bt_{22}t_{21}/2 \). Here we choose \( \theta = C_1 = C_2 = 0 \).

As a result of these choices, we have the following result.

**Corollary 2.5.** Let \( A \) and \( \psi = |\psi|e^{i\phi} \) satisfy (2.3). Let \( Q \) and \( \zeta \) be defined by (2.7) and (2.10), respectively. Let

\[
A_0 = \frac{B}{2} \left( \begin{array}{c} x_2 \\ -x_1 \end{array} \right).
\]

Then \((\psi, A)\) is gauge equivalent to \((\zeta, Q - A_0)\). Moreover, \( Q \) and curl \( Q \) are periodic, \( Q \) is uniquely determined up to a constant vector, \( \text{div} Q = 0 \), and, if

\[
g_k(x) = -\frac{B}{2} (x \times t_k), \quad k = 1, 2,
\]

then

\[
A_0(x + t_k) - A_0(x) = -\text{grad} g_k(x), \quad k = 1, 2
\]

and

\[
\omega(x + t_k) - \omega(x) = \kappa g_k(x), \quad k = 1, 2.
\]

Furthermore, the magnetic field \( h \) and the density of superconducting charge carriers may be recovered from \( \zeta \) and \( Q \) through the relations

\[
h = \text{curl} A = \text{curl} Q - \vec{B} \quad \text{and} \quad N_s = |\psi|^2 = |\zeta|^2,
\]

where \( \vec{B} \) is given by (2.6).

**Proof.** The results are obvious consequences of (2.1), (2.8), (2.11), (2.15), (2.17), and Theorem 2.4 after setting \( \theta = C_1 = C_2 = 0 \). \( \square \)

2.4. **Minimization problems for the Ginzburg–Landau free energy.** The Gibbs free energy is invariant to gauge transformations, so that (2.2) may be written in the form

\[
G(\zeta, Q) = \int_{\Omega_F} \left( f_0 - |\zeta|^2 + \frac{1}{2} |\zeta|^4 + \left| \frac{i}{\kappa} \text{grad} Q + A_0 \right|^2 \right) d\Omega,
\]

where \( \zeta \), \( Q \), and \( A_0 \) satisfy the results of Corollary 2.5.

The fundamental hypothesis of the Landau theory of phase transitions, and thus of the Ginzburg–Landau theory of superconductivity, is that the medium is in a state
such that the Gibbs free energy is minimized. To make this statement precise, we must introduce some additional notation.

Denote the four sides of the parallelogram $\Omega_P$ by $\Gamma_{+1}$, $\Gamma_{-1}$, $\Gamma_{+2}$, and $\Gamma_{-2}$, using the convention of Fig. 2.1. Note that, for $k = 1$ or 2, $\Gamma_{+k}$ is the locus of points $y \in \mathbb{R}^2$ such that $y = x + t_k$ for $x \in \Gamma_{-k}$. We may define the function spaces

$$\mathcal{H}_P^1(\Omega_P) = \left\{ \zeta \in \mathcal{H}^1(\Omega_P) \mid \zeta(x + t_k) = \zeta(x)e^{ikg_k(x)} \quad \forall x \in \Gamma_{-k}, k = 1, 2 \right\},$$

$$\mathbf{H}_P^1(\Omega_P) = \left\{ \mathbf{Q} \in \mathbf{H}^1(\Omega_P) \mid \mathbf{Q}(x + t_k) = \mathbf{Q}(x) \quad \forall x \in \Gamma_{-k}, k = 1, 2 \right\},$$

and

$$\mathbf{H}_P^1(\text{div}; \Omega_P) = \left\{ \mathbf{Q} \in \mathbf{H}_P^1(\Omega_P) \mid \text{div} \mathbf{Q} = 0 \quad \text{in} \ \Omega_P \right\}.$$

![Fig. 2.1. Boundary segments and normal vectors of the cell $\Omega_P$.](image)

The problem at hand is then to

$$\begin{equation}
\text{minimize } \mathcal{G}(\zeta, \mathbf{Q}), \text{ given by (2.24), over all } \zeta \in \mathcal{H}_P^1(\Omega_P) \text{ and } \mathbf{Q} \in \mathbf{H}_P^1(\text{div}; \Omega_P). \tag{2.25}\end{equation}$$

The constraints placed on candidate minimizers are motivated by the conditions on $\zeta$ and $\mathbf{Q}$ found in Corollary 2.5.

In the minimization problem (2.25), one of the constraints placed on candidate minimizers is $\text{div} \mathbf{Q} = 0$ in $\Omega_P$. This constraint is an onerous one from the point of view of devising discretization schemes. To avoid this constraint, we introduce the functional

$$\begin{equation}
\mathcal{F}(\zeta, \mathbf{Q}) = \mathcal{G}(\zeta, \mathbf{Q}) + \int_{\Omega_P} (\text{div} \mathbf{Q})^2 \, d\Omega \tag{2.26}\end{equation}$$

$$\begin{align*}
= & \int_{\Omega_P} \left( f_0 - |\zeta|^2 + \frac{1}{2}|\zeta|^4 + \left( \frac{i}{\kappa \sigma} \text{grad} \mathbf{Q} - \mathbf{A}_0 \right) \zeta \right)^2 \, d\Omega \\
& + \int_{\Omega_P} \left( (\text{div} \mathbf{Q})^2 + |\text{curl} (\mathbf{Q} - \mathbf{A}_0)|^2 \right) \, d\Omega
\end{align*}$$
and the problem
\begin{align}
\text{minimize } & \mathcal{F} (\zeta, Q), \text{ given by } (2.26), \text{ over all } \zeta \in \mathcal{H}_p^1 (\Omega_P) \\
& \text{ and } Q \in H^1_p (\Omega_P). 
\end{align}

2.5. Existence of minimizers. We now consider the existence of minimizers of $\mathcal{F}$ and $\mathcal{G}$, i.e., of solutions of problems (2.25) and (2.27), and also relations between such minimizers.

**Theorem 2.6.** The functional $\mathcal{G}$, given by (2.24), has at least one minimizer belonging to $\mathcal{H}_p^1 (\Omega_P) \times H^1_p (\text{div;} \Omega_P)$.

**Proof.** The proof is given in the Appendix. \( \Box \)

As a consequence of the gauge invariance, we then have the following corollary.

**Corollary 2.7.** $\mathcal{G}$ has at least one minimizer belonging to $\mathcal{H}_p^1 (\Omega_P) \times H^1_p (\text{div;} \Omega_P)$. Moreover, any minimizer of $\mathcal{G}$ in $\mathcal{H}_p^1 (\Omega_P) \times H^1_p (\text{div;} \Omega_P)$ is also a minimizer of $\mathcal{G}$ in $\mathcal{H}_p^1 (\Omega_P) \times H^1_p (\Omega_P)$, and any minimizer of $\mathcal{G}$ in $\mathcal{H}_p^1 (\Omega_P) \times H^1_p (\text{div;} \Omega_P)$ is gauge equivalent to a minimizer of $\mathcal{G}$ in $\mathcal{H}_p^1 (\Omega_P) \times H^1_p (\text{div;} \Omega_P)$.

Now we observe the following relations between $\mathcal{F}$ and $\mathcal{G}$ and their minimizers.

**Lemma 2.8.** For any $(\zeta, Q) \in \mathcal{H}_p^1 (\Omega_P) \times H^1_p (\Omega_P)$, let $\chi \in H^2 (\mathbb{R}^2)$ satisfy
\[ \Delta \chi = -\kappa \text{div } Q \text{ in } \Omega_P. \]

Then
\[ \mathcal{F} (G_\chi (\zeta, Q)) = \mathcal{G} (\zeta, Q). \]

**Corollary 2.9.** $\mathcal{F}$ has at least one minimizer belonging to $\mathcal{H}_p^1 (\Omega_P) \times H^1_p (\text{div;} \Omega_P)$, and all minimizers $(\zeta, Q)$ of $\mathcal{F}$ satisfy $\text{div } Q = 0$. Moreover,
\[ \min_{\mathcal{H}_p^1 (\Omega_P) \times H^1_p (\text{div;} \Omega_P)} \mathcal{F} = \min_{\mathcal{H}_p^1 (\Omega_P) \times H^1_p (\text{div;} \Omega_P)} \mathcal{G} = \min_{\mathcal{H}_p^1 (\Omega_P) \times H^1_p (\text{div;} \Omega_P)} G. \]

The above results imply that we can locate a minimizer of $\mathcal{G}$ in $\mathcal{H}_p^1 (\Omega_P) \times H^1_p (\text{div;} \Omega_P)$ by seeking a minimizer of $\mathcal{F}$ in $\mathcal{H}_p^1 (\Omega_P) \times H^1_p (\Omega_P)$. The latter problem is more convenient from a computational point of view.

3. The Ginzburg–Landau equations. It is well known that necessarily the first variations of (2.24) with respect to both $\zeta$ and $Q$ must vanish at a minimizer. Candidate minimizers $(\hat{\zeta}, \hat{Q})$ satisfy $\hat{\zeta} \in \mathcal{H}_p^1 (\Omega_P)$ and $\hat{Q} \in H^1_p (\text{div;} \Omega_P)$. Admissible variations $(\zeta, Q)$ must therefore be such that $\hat{\zeta} + \zeta \in \mathcal{H}_p^1 (\Omega_P)$ and $\hat{Q} + Q \in H^1_p (\text{div;} \Omega_P)$. Then it is easily seen that admissible variations $(\zeta, Q)$ must themselves satisfy $\zeta \in \mathcal{H}_p^1 (\Omega_P)$ and $Q \in H^1_p (\text{div;} \Omega_P)$.

Standard techniques from the calculus of variations can then be used to derive the Euler–Lagrange equations corresponding to the vanishing of the first variations. Variations in $\mathcal{G} (\zeta, Q)$ with respect to $Q$ then yield that any minimizing pair $(\hat{\zeta}, \hat{Q}) \in \mathcal{H}_p^1 (\Omega_P) \times H^1_p (\text{div;} \Omega_P)$ satisfies
\begin{align}
\int_{\Omega_P} \left( \text{curl } \hat{Q} \cdot \text{curl } Q + |\hat{\zeta}|^2 \hat{Q} \cdot \hat{Q} + \Re \left\{ \hat{\zeta} \left( \frac{\iota}{\kappa} \text{grad } A_0 \right) \hat{\zeta} \right\} \cdot \hat{Q} \right) d\Omega \\
= \int_{\Omega_P} \text{curl } A_0 \cdot \text{curl } Q d\Omega = -\bar{B} \int_{\Omega_P} \text{curl } \hat{Q} d\Omega = 0 \quad \forall \hat{Q} \in H^1_p (\text{div;} \Omega_P),
\end{align}
where the last equality follows from the fact that \( \tilde{Q}(x + t_k) = \tilde{Q}(x) \) for all \( x \in \Gamma_{-k}, \quad k = 1, 2 \). Variations in \( G(\zeta, \tilde{Q}) \) with respect to \( \zeta \) then yield that any minimizing pair 
\((\tilde{\zeta}, \tilde{Q}) \in \mathcal{H}^1_p(\Omega_P) \times \mathcal{H}^1_p(\text{div}; \Omega_P)\) satisfies
\[
\int_{\Omega_P} \Re \left\{ \left( \frac{i}{\kappa} \text{grad} - A_0 \right) \tilde{\zeta} \cdot \left( -\frac{i}{\kappa} \text{grad} - A_0 \right) \tilde{\zeta}^* \right\} \, d\Omega
\]
\[
+ \int_{\Omega_P} \Re \left\{ \tilde{\zeta}^* \left( \frac{i}{\kappa} \text{grad} - A_0 \right) \tilde{\zeta} + \tilde{\zeta} \left( -\frac{i}{\kappa} \text{grad} - A_0 \right) \tilde{\zeta}^* \right\} \cdot \tilde{Q} \, d\Omega
\]
\[
+ \int_{\Omega_P} (|\tilde{Q}|^2 + |\tilde{\zeta}|^2 - 1) \Re \left\{ \tilde{\zeta} \tilde{\zeta}^* \right\} \, d\Omega = 0 \quad \forall \tilde{\zeta} \in \mathcal{H}^1_p(\Omega_P).
\]

It can be shown that (3.1) also holds for test functions \( \tilde{Q} \in \mathcal{H}^1_p(\Omega_P) \), i.e., for test functions that are not necessarily solenoidal. To see this, we merely must show that (3.1) holds for test functions that are gradients, i.e., for \( Q = \text{grad} \xi \) for some \( \xi \in \mathcal{H}^1_{\text{per}}(\mathbb{R}^2) \). That this is the case follows by setting \( \zeta = \frac{1}{i\kappa} \tilde{\zeta} \tilde{\zeta} \) in (3.2).

In the case where \( \tilde{\zeta} \) and \( \tilde{Q} \) are sufficiently smooth, (3.1) and (3.2) may be integrated by parts so that no derivatives of the test functions \( \tilde{\zeta} \) and \( \tilde{Q} \) appear in the result. For example, after appropriate integration by parts in (3.1), we obtain that
\[
\int_{\Omega_P} \left( \text{curl} \text{curl} \tilde{Q} + |\tilde{\zeta}|^2 \tilde{Q} + \Re \left\{ \tilde{\zeta}^* \left( \frac{i}{\kappa} \text{grad} - A_0 \right) \tilde{\zeta} \right\} \right) \cdot \tilde{Q} \, d\Omega
\]
\[
= \int_{\Gamma_P} \text{curl} (\tilde{Q} - A_0)(\tau \cdot \tilde{Q}) \, d\Gamma,
\]
where \( \text{curl} \text{curl} A_0 = -\text{curl} B = 0 \) has been used. Due to the arbitrariness of \( \tilde{Q} \), we deduce from (3.3) that
\[
\text{curl} \text{curl} \tilde{Q} + |\tilde{\zeta}|^2 \tilde{Q} + \Re \left\{ \tilde{\zeta}^* \left( \frac{i}{\kappa} \text{grad} - A_0 \right) \tilde{\zeta} \right\} = 0 \quad \text{in} \ \Omega_P
\]
and, since \( \tilde{Q} \) is periodic,
\[
\text{curl} (\tilde{Q} - A_0)|_{x + t_k} = \text{curl} (\tilde{Q} - A_0)|_x \quad \text{for} \ x \in \Gamma_{-k}, \ k = 1, 2.
\]

Thus (3.1) has been shown to be a weak formulation of (3.4), the first of the Ginzburg–Landau equations and has, as a natural boundary condition, the periodicity of \( \text{curl} (\tilde{Q} - A_0) \) across a lattice cell. The latter, due to (2.23), implies the like periodicity of the magnetic field. Of course, the periodicity across a lattice cell of \( \tilde{Q} \) is an essential boundary condition. Also, \( \tilde{Q} \in \mathcal{H}^1_p(\text{div}; \Omega_P) \) implies that \( \text{div} \tilde{Q} = 0 \) almost everywhere in \( \Omega_P \).

Analogously, integrating the appropriate terms in (3.2) by parts yields that
\[
\Re \int_{\Omega_P} \left( \left( \frac{i}{\kappa} \text{grad} - A_0 \right) \cdot \left( \frac{i}{\kappa} \text{grad} - A_0 \right) \tilde{\zeta} + (|\tilde{Q}|^2 + |\tilde{\zeta}|^2 - 1) \tilde{\zeta} \right)
\]
\[
+ Q \cdot \left( \frac{i}{\kappa} \text{grad} - A_0 \right) \tilde{\zeta} + \left( \frac{i}{\kappa} \text{grad} - A_0 \right) \cdot (\tilde{\zeta} \tilde{Q}) \tilde{\zeta}^* \, d\Omega
\]
\[
= \Re \int_{\Gamma_P} \left( \frac{i}{\kappa} \text{grad} - A_0 \right) \tilde{\zeta} + \frac{i}{\kappa} \tilde{\zeta} \tilde{Q} \cdot n \tilde{\zeta}^* \, d\Gamma.
\]
Due to the arbitrariness of $\tilde{\zeta}$ and since $\text{div} \, \tilde{\mathbf{Q}} = 0$, we deduce from (3.6) that

$$
\left( \frac{i}{\kappa} \text{grad} - A_0 \right) \cdot \left( \frac{i}{\kappa} \text{grad} - A_0 \right) \tilde{\zeta} + (|\tilde{\mathbf{Q}}|^2 + |\tilde{\zeta}|^2 - 1)\tilde{\zeta} + 2\tilde{\mathbf{Q}} \cdot \left( \frac{i}{\kappa} \text{grad} - A_0 \right) \tilde{\zeta} = 0 \quad \text{in } \Omega_P
$$

(3.7)

and, since $\tilde{\zeta}(x + t_k) = e^{i\kappa g_k(x)} \tilde{\zeta}(x)$ for all $x \in \Gamma_{-k}$, $k = 1, 2$,

$$
e^{-i\kappa g_k(x)} \left( \left( \frac{i}{\kappa} \text{grad} - A_0 \right) \tilde{\zeta} + \tilde{\mathbf{Q}} \tilde{\zeta} \right) \bigg|_{x + t_k} \cdot n_{+k} + \left( \left( \frac{i}{\kappa} \text{grad} - A_0 \right) \tilde{\zeta} + \tilde{\mathbf{Q}} \tilde{\zeta} \right) \bigg|_x \cdot n_{-k} = 0 \quad \text{for } x \in \Gamma_{-k}, k = 1, 2.
$$

(3.8)

Thus (3.2) has been shown to be a weak formulation of (3.7), the second Ginzburg–Landau equation, and has as a natural boundary condition (3.8). We also have the essential boundary condition $\tilde{\zeta}(x + t_k) = e^{i\kappa g_k(x)} \tilde{\zeta}(x)$ for all $x \in \Gamma_{-k}$, $k = 1, 2$, which, with $\tilde{\zeta} = |\tilde{\zeta}| e^{i\tilde{\omega}}$, implies that

$$
|\tilde{\zeta}(x + t_k)| = |\tilde{\zeta}(x)| \quad \forall \, x \in \Gamma_{-k}, \, k = 1, 2
$$

(3.9)

and

$$
\tilde{\omega}(x + t_k) = \tilde{\omega}(x) + \kappa g_k(x) \quad \forall \, x \in \Gamma_{-k}, \, k = 1, 2.
$$

(3.10)

Note that using (2.23), we see that the first of these implies the periodicity across a lattice cell of the density of superconducting charge carriers.

Let us now examine some implications of (3.8)–(3.10). Since $n_{-k} = -n_{+k}$, these imply that

$$
i \text{grad} |\tilde{\zeta}| - |\tilde{\zeta}| \left( \text{grad} \tilde{\omega} - \kappa (\tilde{\mathbf{Q}} - A_0) \right) \bigg|_{x + t_k} \cdot n_{+k}
$$

$$= i \text{grad} |\tilde{\zeta}| - |\tilde{\zeta}| \left( \text{grad} \tilde{\omega} - \kappa (\tilde{\mathbf{Q}} - A_0) \right) \bigg|_x \cdot n_{+k} \quad \forall \, x \in \Gamma_{-k}, \, k = 1, 2.
$$

(3.11)

On the other hand, (2.21), (3.10), and the periodicity of $\tilde{\mathbf{Q}}$ imply that

$$
\left\{ i \text{grad} |\tilde{\zeta}| - |\tilde{\zeta}| \left( \text{grad} \tilde{\omega} - \kappa (\tilde{\mathbf{Q}} - A_0) \right) \right\} \bigg|_{x + t_k} \times n_{+k}
$$

$$= \left\{ i \text{grad} |\tilde{\zeta}| - |\tilde{\zeta}| \left( \text{grad} \tilde{\omega} - \kappa (\tilde{\mathbf{Q}} - A_0) \right) \right\} \bigg|_x \times n_{+k} \quad \forall \, x \in \Gamma_{-k}, \, k = 1, 2.
$$

(3.12)

Together, (3.11) and (3.12) result in

$$
\left( \text{grad} |\tilde{\zeta}| \right) \bigg|_{x + t_k} = \left( \text{grad} |\tilde{\zeta}| \right) \bigg|_x \quad \forall \, x \in \Gamma_{-k}, \, k = 1, 2
$$

(3.13)

and

$$
|\tilde{\zeta}| \left( \text{grad} \tilde{\omega} - \kappa (\tilde{\mathbf{Q}} - A_0) \right) \bigg|_{x + t_k}
$$

$$= |\tilde{\zeta}| \left( \text{grad} \tilde{\omega} - \kappa (\tilde{\mathbf{Q}} - A_0) \right) \bigg|_x \quad \forall \, x \in \Gamma_{-k}, \, k = 1, 2.
$$

(3.14)
As a result of (3.4), the current \( j \) satisfies

\[
(3.15) \quad j = \nabla \times h = \nabla \times \nabla \times \hat{Q} = |\hat{\zeta}|^2 \left( \frac{i}{\kappa} \nabla \hat{\omega} - \hat{A}_0 \right).
\]

### 3.1. Recapitulation of the Ginzburg–Landau equations and boundary conditions for the periodic model.

We collect the above results. The minimization problem (2.25) leads to the Ginzburg–Landau equations

\[
(3.16) \quad \nabla \times \nabla \times \hat{Q} + |\hat{\zeta}|^2 \hat{Q} + \Re \left\{ \hat{\zeta} \cdot \left( \frac{i}{\kappa} \nabla \hat{\omega} - \hat{A}_0 \right) \right\} = 0 \quad \text{in } \Omega_P
\]

and

\[
(3.17) \quad \left( \frac{i}{\kappa} \nabla \hat{\omega} - \hat{A}_0 \right) \cdot \left( \frac{i}{\kappa} \nabla \hat{\omega} - \hat{A}_0 \right) \hat{\zeta} + (|\hat{Q}|^2 + |\hat{\zeta}|^2 - 1) \hat{\zeta} \\
+ 2 \hat{Q} \cdot \left( \frac{i}{\kappa} \nabla \hat{\omega} - \hat{A}_0 \right) \hat{\zeta} = 0 \quad \text{in } \Omega_P.
\]

The constraints in (2.25) placed on candidate minimizers imply the essential conditions

\[
(3.18) \quad \text{div } \hat{Q} = 0 \quad \text{in } \Omega_P,
\]

\[
(3.19) \quad \hat{Q}(x + t_k) = \hat{Q}(x) \quad \forall \, x \in \Gamma_{-k}, \, k = 1, 2,
\]

and

\[
(3.20) \quad \hat{\zeta}(x + t_k) = \hat{\zeta}(x)e^{i\kappa g_k(x)} \quad \forall \, x \in \Gamma_{-k}, \, k = 1, 2.
\]

The natural constraints associated with the minimization problem (2.25), together with the essential constraints (3.18)–(3.20), imply that

\[
(3.21) \quad \left( \nabla |\hat{\zeta}| \right)|_{x+t_k} = \left( \nabla |\hat{\zeta}| \right)|_{x} \quad \forall \, x \in \Gamma_{-k}, \, k = 1, 2,
\]

\[
(3.22) \quad \text{curl } (\hat{Q} - \hat{A}_0)|_{x+t_k} = \text{curl } (\hat{Q} - \hat{A}_0)|_{x} \quad \forall \, x \in \Gamma_{-k}, \, k = 1, 2,
\]

and

\[
(3.23) \quad |\hat{\zeta}| \left( \nabla \hat{\omega} - \kappa (\hat{Q} - \hat{A}_0) \right)|_{x+t_k} \\
= |\hat{\zeta}| \left( \nabla \hat{\omega} - \kappa (\hat{Q} - \hat{A}_0) \right)|_{x} \quad \forall \, x \in \Gamma_{-k}, \, k = 1, 2,
\]

where \( \hat{\omega} \) denotes the phase of \( \hat{\zeta} \). We have also seen how (3.20)–(3.23) imply that the magnetic field, the current, and the density of superconducting charge carriers are periodic across a lattice cell.

### 3.2. The weak enforcement of the divQ = 0 gauge condition.

It is often not practical to require the test and trial functions to satisfy (3.18). In §2.5 it was seen that, instead of minimizing \( \mathcal{G} \) given by (2.24), i.e., solving problem (2.25), we may
equivalently minimize $\mathcal{F}$ given by (2.26), i.e., solve problem (2.27). The advantage of
the latter is that candidate minimizers and test functions need not be solenoidal, as is
the case for the former. Setting the first variation of (2.26) to zero then implies that
a minimizing pair $(\hat{\zeta}, \hat{Q}) \in \mathcal{H}^1_p(\Omega_P) \times \mathbf{H}^1_p(\Omega_P)$ must satisfy, instead of (3.1),
\[ \int_{\Omega_P} \left( \text{curl} \hat{Q} \cdot \text{curl} \hat{Q} + \text{div} \hat{Q} \cdot \text{div} \hat{Q} + |\hat{c}|^2 \hat{Q} \cdot \hat{Q} \right) \, d\Omega + \text{Re} \left\{ \int_{\Omega_P} \hat{c}^* \left( \frac{i}{k} \text{grad} - A_0 \right) \hat{c} \, d\Omega \right\} = 0 \quad \forall \hat{Q} \in \mathbf{H}^1_p(\Omega_P). \]

Note that functions belonging to $\mathbf{H}^1_p(\Omega_P)$ are not necessarily solenoidal.

### 3.3. Properties of solutions of the Ginzburg–Landau equations

Various properties of the solutions of the Ginzburg–Landau equations (3.16)–(3.23) can be
readily obtained. We omit many of the proofs since they are similar to those in [12]
for the analogous results on bounded domains.

First, we have that the order parameter is bounded.

**Proposition 3.1.** If $(\hat{\zeta}, \hat{Q})$ is a solution of the Ginzburg–Landau equations (3.16)–(3.23), then $|\hat{\zeta}| \leq 1$ almost everywhere.

With regard to the Ginzburg–Landau functionals $\mathcal{G}$ and $\mathcal{F}$, we have the following
proposition.

**Proposition 3.2.** A solution of the Ginzburg–Landau equations (3.16)–(3.23)
cannot be a local maximum of the functionals $\mathcal{G}$ or $\mathcal{F}$ defined by (2.24) and (2.26).

Next, we look at constant solutions of the Ginzburg–Landau equation; in particular,
we look at the normal state $(\hat{\zeta} = 0)$ and the perfect superconducting state
$(|\hat{\zeta}| = 1)$.

**Proposition 3.3.** $(\hat{\zeta} = 0, \hat{Q} = \text{constant})$ is always a trivial solution of the
Ginzburg–Landau equations (3.16)–(3.23). If $\hat{B} = 0$, then $\hat{\zeta} = \exp(i\theta)$ for some constant
$\theta \in [0, 2\pi)$, and $\hat{Q} = 0$ are the only other solutions belonging to $\mathcal{H}^1_p(\Omega_P) \times
\mathbf{H}^1_p(\text{div} ; \Omega_P)$ with $\hat{\zeta} = \text{constant almost everywhere}$. Moreover, these are the only
global minimizers of $\mathcal{G}$ when $\hat{B} = 0$.

**Proof.** That $(\hat{\zeta} = 0, \hat{Q} = 0)$ is always a solution of the Ginzburg–Landau
equations is obvious.

Now, if $\hat{B} = 0$, then $\hat{A}_0 = 0$ and $g_k = 0$, and then it can be shown that $(1, 0) \in
\mathcal{H}^1_p(\Omega_P) \times \mathbf{H}^1_p(\text{div} ; \Omega_P)$ satisfies the Ginzburg–Landau equations. Moreover,
\[ \mathcal{G}(\hat{\zeta}, \hat{Q}) \geq \mathcal{G}(1, 0) \quad \forall (\hat{\zeta}, \hat{Q}) \in \mathcal{H}^1_p(\Omega_P) \times \mathbf{H}^1_p(\text{div} ; \Omega_P). \]

We see that, by gauge invariance, $\hat{\zeta} = \exp(i\theta)$, $\theta \in [0, 2\pi)$, with $\hat{Q} = 0$ are the only
global minimizers of $\mathcal{G}$. It remains to show that, if $\hat{\zeta} = c$ almost everywhere and $c \neq 0$,
then necessarily $|c| = 1$ and $\hat{Q} = 0$. Note that, in this case, we have the following
equations for $\hat{Q} \in \mathbf{H}^1_p(\text{div} ; \Omega_P)$:
\[ \text{curl} \text{curl} \hat{Q} + c^2 \hat{Q} = 0, \quad \text{div} \hat{Q} = 0, \]
and $|\hat{Q}|^2 = 1 - c^2$, almost everywhere. It is easy to see that any such $\hat{Q}$ is smooth.

Let $\beta = \text{curl} \hat{Q}$. A simple calculation shows that $|\text{grad} \beta|^2 = c^2(1 - c^2)$. On the other
hand, $\beta$ is periodic and smooth, so that $\text{grad} \beta$ must vanish at some points. This
implies that $c^2 = 1$. Thus $\hat{Q} = 0$ and $\hat{\zeta} = \exp(i\theta)$ for some $\theta \in [0, 2\pi)$. \[\Box\]
Proposition 3.4. If \( \tilde{B} \neq 0 \), then the only solution of the Ginzburg–Landau equations (3.16)–(3.23) such that \( (\zeta, \mathbf{Q}) \in \mathcal{H}^1_p(\Omega_P) \times \mathbf{H}^1_p(\text{div}; \Omega_P) \) and \( \zeta = \text{constant almost everywhere} \) is \( \zeta = 0 \) and \( \mathbf{Q} = \text{constant} \).

Proof. When \( \tilde{B} \neq 0 \), \( |A_0| \) is not a constant. The only constant function \( \zeta \in \mathcal{H}^1_p(\Omega_P) \) is \( \zeta = 0 \) and \( \mathbf{Q} \in \mathbf{H}^1_p(\text{div}; \Omega_P) \) satisfies

\[
\text{curl} \text{curl} \mathbf{Q} = 0 \quad \text{and} \quad \text{div} \mathbf{Q} = 0
\]

so that, as above, \( \mathbf{Q} \) must be a constant vector almost everywhere. \( \square \)

4. Regularity of solutions of the periodic Ginzburg–Landau model. We now examine the regularity of solutions of the Ginzburg–Landau equations (3.16)–(3.23). In fact, we show that these solutions are infinitely differentiable. Details connected with the development given below may be found in, e.g., [3], [4], [15], [18], [23], or [24].

We use the following function spaces. First, for any bounded set \( \mathcal{D} \subset \mathbb{R}^2 \), we have the space \( \mathcal{L}^p(\mathcal{D}) \) consisting of complex-valued functions having real and imaginary parts belonging to \( L^p(\mathcal{D}) \). Analogously, we have the space \( \mathcal{L}^p(\mathcal{D}) \) of vector-valued functions. Also, for any nonnegative integer \( m \) and any real number \( p \) such that \( 1 < p < \infty \), we have the Sobolev space \( W^{m,p}(\mathcal{D}) \) consisting of \( L^p \) integrable functions over \( \mathcal{D} \) such that all derivatives of order up to \( m \) are also \( L^p \) integrable. In particular, if \( p = 2 \), we use the convention \( W^{m,2}(\mathcal{D}) = H^m(\mathcal{D}) \). We may then define the spaces

\[
W^{m,p}_{\text{loc}}(\mathbb{R}^2) = \left\{ \psi : \mathbb{R}^2 \to \mathbb{C} \mid \Re(\psi), \Im(\psi) \in W^{m,p}(\mathcal{D}) \quad \forall \text{ bounded } \mathcal{D} \subset \mathbb{R}^2 \right\}
\]

and

\[
W^{m,p}_{\text{loc}}(\mathbb{R}^2) = \left\{ \mathbf{A} = (A_1, A_2)^T : \mathbb{R}^2 \to \mathbb{R}^2 \mid A_1, A_2 \in W^{m,p}(\mathcal{D}) \quad \forall \text{ bounded } \mathcal{D} \subset \mathbb{R}^2 \right\}.
\]

Details concerning these spaces may be found in, e.g., [2]. Given the lattice vectors \( \mathbf{t}_k \) and the lattice functions \( g_k(\mathbf{x}) \), \( k = 1, 2 \), we define, for \( m \geq 0 \), the space of periodic functions

\[
W^{m,p}_{\text{per}}(\mathbb{R}^2) = \left\{ \mathbf{A} \in W^{m,p}_{\text{loc}}(\mathbb{R}^2) \mid \mathbf{A}(\mathbf{x} + \mathbf{t}_k) = \mathbf{A}(\mathbf{x}) \quad \text{for } k = 1, 2 \text{ and } \forall \mathbf{x} \in \mathbb{R}^2 \right\}
\]

and the space of “quasi”-periodic functions

\[
W^{m,p}_{\text{qp}}(\mathbb{R}^2) = \left\{ \psi \in W^{m,p}_{\text{loc}}(\mathbb{R}^2) \mid \psi(\mathbf{x} + \mathbf{t}_k) = \psi(\mathbf{x})e^{i\phi g_k(\mathbf{x})} \quad \text{for } k = 1, 2 \text{ and } \forall \mathbf{x} \in \mathbb{R}^2 \right\}.
\]

Note that here and throughout, the symbol \( \forall \) is to be interpreted in the sense of measures, or in the weak sense. From these, we define the spaces of functions restricted to \( \Omega_P \)

\[
W^{m,p}_{p}(\Omega_P) = \left\{ \mathbf{A}|_{\Omega_P} \mid \mathbf{A} \in W^{m,p}_{\text{per}}(\mathbb{R}^2) \right\}
\]

and

\[
W^{m,p}_{p}(\Omega_P) = \left\{ \psi|_{\Omega_P} \mid \psi \in W^{m,p}_{\text{qp}}(\mathbb{R}^2) \right\}.
\]
Whenever $m \geq 1$, we also define the spaces
\[ W^{m,p}_P(\Omega_P) = \left\{ A \in W^{m,p}(\Omega_P) \mid A(x + t_k) = A(x) \quad \forall x \in \Gamma_{-k}, \quad k = 1, 2 \right\} \]
and
\[ W^{m,p}_{P}(\Omega_P) = \left\{ \psi \in W^{m,p}(\Omega_P) \mid \psi(x + t_k) = \psi(x)e^{i\kappa g_k} \quad \forall x \in \Gamma_{-k} \quad k = 1, 2 \right\}. \]
Note that
\[ \tilde{W}^{m,p}_P(\Omega_P) \subset W^{m,p}_P(\Omega_P) \quad \text{and} \quad \bar{W}^{m,p}_P(\Omega_P) \subset W^{m,p}_P(\Omega_P). \]
Also, the following results are easily verified.

**Lemma 4.1.** It holds that
\[ \tilde{W}^{0,p}_P(\Omega_P) = L^p(\Omega_P), \quad \bar{W}^{0,p}_P(\Omega_P) = L^p(\Omega_P), \]
\[ W^{1,p}_P(\Omega_P) = \bar{W}^{1,p}_P(\Omega_P), \quad \text{and} \quad W^{1,p}_P(\Omega_P) = \bar{W}^{1,p}_P(\Omega_P). \]

**4.1. Regularity results for associated linear problems.** We first consider the regularity of solutions for a linear problem related to (3.17), one of the Ginzburg–Landau equations. For $f \in L^p(\Omega_P)$, consider the following problem: seek $\zeta \in H^1_P(\Omega_P) = W^{1,2}_P(\Omega_P)$ such that
\begin{equation}
(4.1) \quad a(\zeta, \tilde{\zeta}) = \Re\left[(f, \tilde{\zeta})\right] \quad \forall \tilde{\zeta} \in H^1_P(\Omega_P),
\end{equation}
where the bilinear form $a(\cdot, \cdot)$ is defined by
\[ a(\zeta, \tilde{\zeta}) = \int_{\Omega_P} \Re \left[ \left( \frac{i}{\kappa} \text{grad} \zeta - A_0 \zeta \right) \cdot \left( \frac{i}{\kappa} \text{grad} \tilde{\zeta}^* - A_0 \tilde{\zeta}^* \right) + \zeta \tilde{\zeta}^* \right] d\Omega \quad \forall \zeta, \tilde{\zeta} \in H^1_P(\Omega_P) \]
and where $(f, \tilde{\zeta}) = \int_{\Omega_P} f \tilde{\zeta}^* d\Omega$. Note that, on $H^1_P(\Omega_P)$, $\sqrt{a(\zeta, \tilde{\zeta})}$ defines a norm equivalent to the standard 1-norm. Obviously, problem (4.1) is equivalent to the minimization problem
\begin{equation}
(4.2) \quad \min_{\zeta \in H^1_P(\Omega_P)} \left\{ a(\zeta, \zeta) - 2\Re[(f, \zeta)] \right\}.
\end{equation}
Standard variational arguments may then be used to obtain the following result.

**Lemma 4.2.** For $f \in L^p(\Omega_P)$, the minimization problem (4.2) has a unique solution belonging to $H^1_P(\Omega_P)$. The minimizer satisfies the weak form (4.1).

Using integration by parts, we obtain the differential equation corresponding to the weak formulation (4.1), i.e.,
\begin{equation}
(4.3) \quad \left( \frac{i}{\kappa} \text{grad} - A_0 \right) \cdot \left( \frac{i}{\kappa} \text{grad} - A_0 \right) \zeta + \zeta = f \quad \text{in} \quad \Omega_P.
\end{equation}
From this equation, we obtain that, for $q = \min\{p, 2\}$, $\Delta \zeta \in L^q(\Omega_P)$; then, using a trace theorem, we have that $\partial \zeta/\partial n \in L^2(\Gamma)$. We may then derive the natural boundary conditions corresponding to the weak formulation (4.1), i.e.,
\begin{equation}
(4.4) \quad \left[ \left( \frac{i}{\kappa} \text{grad} - A_0 \right) \zeta \right]_{x + t_k \cdot n_{+k}} = \left[ \left( \frac{i}{\kappa} \text{grad} - A_0 \right) \zeta e^{i\kappa g_k} \right]_{x \cdot n_{+k}} \quad \forall x \in \Gamma_{-k}, \quad k = 1, 2,
\end{equation}
where again, the symbol $\forall$ is to be interpreted in the sense of measures.

From (4.3) we obtain, for any test function $\zeta \in C^\infty(\Omega_P) = \{ \zeta \mid \Re(\zeta), \Im(\zeta) \in C^\infty(\Omega_P) \},$

$$a(\zeta, \tilde{\zeta}) = \Re(\langle f, \tilde{\zeta} \rangle) + b(\zeta, \tilde{\zeta}),$$

where

$$b(\zeta, \tilde{\zeta}) = \int_{\Gamma_P} \Re \left[ \left( \frac{i}{\kappa} \text{grad} \zeta - A_0 \zeta \right) \cdot \mathbf{n} \frac{i}{\kappa} \tilde{\zeta}^* \right] d\Gamma.$$  

Now, if we extend the solution $\zeta$ to $\mathbb{R}^2$ by

$$\zeta(x + t_k) = \zeta(x) e^{i\kappa x_k} \quad \forall \ x \in \mathbb{R}^2$$

and extend $f$ in a similar manner, then, by (4.5), we conclude that

$$a(\zeta, \tilde{\zeta}) = \Re(\langle f, \tilde{\zeta} \rangle) \quad \forall \ \tilde{\zeta} \in C^\infty_0(\mathbb{R}^2),$$

i.e., for all infinitely differentiable complex-valued test functions having compact support in $\mathbb{R}^2$.

From the standard elliptic regularity theory, we have $\zeta \in H^{2,p}_{\text{loc}}(\mathbb{R}^2)$. In particular, we obtain the following result. (Here and throughout, all norms are with respect to a single cell $\Omega_P$.)

**Lemma 4.3.** There exists a constant $C_{p, \Omega_P} > 0$ such that, for any $f \in L^p(\Omega_P)$, the solution $\zeta$ of (4.1) belongs to $\tilde{W}^{2,p}_{\text{loc}}(\Omega_P)$ and satisfies $\|\zeta\|_{2,p,\Omega_P} \leq C_{p, \Omega_P} \|f\|_{0,p,\Omega_P}.$

This result may be extended to higher derivatives.

**Lemma 4.4.** For $m \geq 0$ and $1 < p < \infty$, there exists a constant $C_{m,p,\omega_P} > 0$ such that for any $f \in \tilde{W}^{m,p}_{\text{loc}}(\Omega_P)$ the solution $\zeta$ of (4.1) belongs to $\tilde{W}^{m+2,p}(\Omega_P)$ and satisfies $\|\zeta\|_{m+2,p,\Omega_P} \leq C_{m,p,\omega_P} \|f\|_{m,p,\Omega_P}.$

In a similar manner, we may obtain regularity results for a linear problem associated with (3.16), i.e., the other Ginzburg-Landau equation. Consider the system

$$\text{curl} \text{curl} \mathbf{Q} + \mathbf{Q} = \mathbf{F} \quad \text{in} \ \Omega_P$$

for $\mathbf{F} \in L^p(\Omega_P)$ with $(\mathbf{F}, \nabla q) = 0$ for all $q \in H^2_{\text{per}}(\mathbb{R}^2)$, and

$$\text{div} \mathbf{Q} = 0 \quad \text{in} \ \Omega_P$$

subject to the periodic boundary conditions

$$\mathbf{Q}(x + t_k) = \mathbf{Q}(x) \quad \forall \ x \in \Gamma_{\omega_k}, \ k = 1, 2$$

and

$$\text{curl} \mathbf{Q}(x + t_k) = \text{curl} \mathbf{Q}(x) \quad \forall \ x \in \Gamma_{\omega_k}, \ k = 1, 2.$$  

This system has a corresponding weak formulation

$$\int_{\Omega_P} \left( \text{div} \mathbf{Q} \text{div} \tilde{\mathbf{Q}} + \text{curl} \mathbf{Q} \text{curl} \tilde{\mathbf{Q}} + \mathbf{Q} \cdot \tilde{\mathbf{Q}} \right) d\Omega = \int_{\Omega_P} \mathbf{F} \cdot \tilde{\mathbf{Q}} d\Omega \quad \forall \ \tilde{\mathbf{Q}} \in H^1_p(\Omega_P).$$
In fact, the solution also corresponds to the solution of the minimization problem

\begin{equation}
\min_{Q \in H^1_p(\Omega_P)} \left[ \int_{\Omega_P} \left( |\text{div} \, Q|^2 + |\text{curl} \, Q|^2 + |Q|^2 - 2Q \cdot F \right) d\Omega \right].
\end{equation}

We may again extend the solution, the equation, and the weak form to \( \mathbb{R}^2 \) using the periodicity to obtain the following result.

**Lemma 4.5.** For \( 1 < p < \infty \), there exists a constant \( C_{p, \Omega_P} > 0 \) such that for any \( F \in L^p(\Omega_P) \) with \( (F, \nabla q) = 0 \) for all \( q \in H^2_{\text{per}}(\mathbb{R}^2) \), the solution \( Q \) of (4.12) belongs to \( \tilde{W}^{2,p}_P(\Omega_P) \) and satisfies \( \|Q\|_{2,p,\Omega_P} \leq C_{p,\Omega_P} \|F\|_{p,\Omega_P} \).

The higher-order regularity also follows.

**Lemma 4.6.** For \( m \geq 1 \) and \( 1 < p < \infty \), there exists a constant \( C_{m,p,\Omega_P} > 0 \) such that for any \( F \in \tilde{W}^{m+2,p}_P(\Omega_P) \) such that \( \text{div} \, F = 0 \) in \( \Omega_P \), the solution \( Q \) of (4.13) belongs to \( \tilde{W}^{m+2,p}_P(\Omega_P) \) and satisfies \( \|Q\|_{m+2,p,\Omega_P} \leq C_{m,p,\Omega_P} \|F\|_{m,p,\Omega_P} \).

### 4.2. Regularity results for the nonlinear problem

Now let us consider the nonlinear Ginzburg–Landau equations (3.16)–(3.23). Note that the boundary conditions (3.19)–(3.23) imply the boundary conditions (4.4), (4.10), and (4.11).

From Corollary 2.7, we have that \( (\hat{\zeta}, \hat{Q}) \in H^1_p(\Omega_P) \times H^1_p(\Omega_P) \) satisfies, in a weak sense,

\begin{equation}
\left( \frac{i}{\kappa} \text{grad} - A_0 \right) \hat{\zeta} + \hat{Q} = \hat{\zeta} - (|\hat{Q}|^2 + |\hat{\zeta}|^2 - 1) \hat{\zeta} - 2\hat{Q} \cdot \left( \frac{i}{\kappa} \text{grad} - A_0 \right) \hat{\zeta} \quad \text{in } \Omega_P,
\end{equation}

(4.14)

\begin{equation}
\text{curl} \, \text{curl} \, \hat{Q} + \hat{Q} = \hat{Q} - |\hat{\zeta}|^2 \hat{Q} - \Re \left\{ \hat{\zeta}^* \left( \frac{i}{\kappa} \text{grad} - A_0 \right) \hat{\zeta} \right\} \quad \text{in } \Omega_P,
\end{equation}

(4.15)

and

\begin{equation}
\text{div} \, \hat{Q} = 0 \quad \text{in } \Omega_P,
\end{equation}

(4.16)

along with the already quoted boundary conditions for \( \hat{\zeta} \) and \( \hat{Q} \). Note that we have previously shown that the right-hand side of (4.15) satisfies the hypotheses on \( F \) in Lemmas 4.5 and 4.6; i.e., the right-hand side is solenoidal.

Using well-known imbedding theorems, the right-hand side terms of (4.14) and (4.15) belong to \( L^p(\Omega_P) \) and \( L^p(\Omega_P) \), respectively, where \( 1 < p < 2 \). Based on the regularity results for the linear problems, if we extend \( \hat{\zeta} \) according to (4.6) and extend \( \hat{Q} \) periodically to \( \mathbb{R}^2 \), the right-hand side terms are extended accordingly, and we obtain that (4.14)–(4.16) are valid almost everywhere in \( \mathbb{R}^2 \) and the following result.

**Lemma 4.7.** For \( 1 < p < 2 \), solutions \( (\hat{\zeta}, \hat{Q}) \) of the nonlinear Ginzburg–Landau equations (3.16)–(3.23) belong to \( \tilde{W}^{2,p}_P(\Omega_P) \times \tilde{W}^{2,p}_P(\Omega_P) \).

The regularity results of the above lemma imply better regularity of the terms on the right-hand sides of (4.14) and (4.15). Thus a standard bootstrapping argument results in the following theorem.

**Theorem 4.8.** Let \( (\hat{\zeta}, \hat{Q}) \) denote a solution of the nonlinear Ginzburg–Landau equations (3.16)–(3.23). Then \( \hat{\zeta} \) can be extended to \( \mathbb{R}^2 \) by (4.6) and \( \hat{Q} \) can be extended periodically to \( \mathbb{R}^2 \) as infinitely differentiable functions, and they satisfy the Ginzburg–Landau equations in the classical sense.
5. Concluding remarks. We conclude our study by briefly considering various issues of importance to the approximate solution, be it by analytical or numerical techniques, of the periodic Ginzburg–Landau model.

5.1. Specification of data for the periodic Ginzburg–Landau model. An examination of the periodic Ginzburg–Landau model, either in its weak form (3.1) and (3.2) or strong form (3.16)–(3.23), indicates that the model is uniquely specified once \( \kappa, A_0, g_1(x), g_2(x), \) and \( \Omega_P \) are chosen. Note that (2.19) and (2.20) yield that \( A_0 \) is determined once the average magnetic field \( \bar{B} \) is specified, and \( g_k(x), k = 1, 2, \) are determined once \( \bar{B} \) and the lattice vectors \( t_k, k = 1, 2, \) are specified. Of course, these lattice vectors and a specification of the point \( P \) completely determine the lattice cell \( \Omega_P \). Thus choosing particular values for \( \kappa, \bar{B}, P, t_1, \) and \( t_2 \) suffices to uniquely specify the model. However, not all of these may be chosen independently; they are related through the fluxoid quantization condition (2.6), which contains the additional parameter \( n \), the number of fluxoids associated with the lattice cell \( \Omega_P \).

The following is a list of the parameters, some chosen, some derived, that uniquely specify the periodic Ginzburg–Landau model and are such that the fluxoid quantization condition is satisfied; although this is the customary way the model is specified, it certainly is not the only way.

1. The Ginzburg–Landau parameter \( \kappa \), a material constant whose value depends on the temperature and which, for type-II superconductors, satisfies \( \kappa \in (1/\sqrt{2}, \infty) \);
2. The positive integer \( n \), the number of fluxoids associated with a lattice cell;
3. The position of the point \( P \) that determines the position of the lattice cell with respect to the origin; the usual choice for \( P \) is the origin itself;
4. The average magnetic field \( \bar{B} \); for a given value of \( \kappa, \bar{B} \) may be chosen in the interval \((0, \kappa)\). For \( \bar{B} > \kappa \), it is known that the material loses its superconducting properties; see, e.g., [9], [21], [26], and [27];
5. The directions and relative magnitudes of the lattice vectors \( t_1 \) and \( t_2 \). Without loss of generality, we may choose \( t_1 \) to be aligned with the \( x_1 \)-axis. We also choose \( \vartheta \neq 0 \), the angle between \( t_1 \) and \( t_2 \), and \( \gamma > 0 \), the ratio of the magnitudes of \( t_2 \) and \( t_1 \). Then \(|\Omega| = \gamma |t_1|^2 \sin \vartheta\). On the other hand, the fluxoid quantization condition (2.6) requires that \( |\bar{B}|/|\kappa = 2\pi n \) so that we must choose \(|t_1|^2 = (2\pi n)/(\gamma \kappa \bar{B} \sin \vartheta)\). As a result, we have that

\[
(5.1) \quad t_1 = \sqrt{\frac{2\pi n}{\gamma \kappa \bar{B} \sin \vartheta}} i_1 \quad \text{and} \quad t_2 = \sqrt{\frac{2\pi n \gamma}{\kappa \bar{B} \sin \vartheta}} \left((\cos \vartheta) i_1 + (\sin \vartheta) i_2\right),
\]

where \( i_k, k = 1, 2, \) denote the unit vectors in the directions of the \( x_k, k = 1, 2, \) axes, respectively.

Thus the parameters that determine the model are \( P, \kappa, n, \gamma, \vartheta, \) and \( \bar{B} \). Further discussions of the issues that enter into the choice of these parameters are given in §§5.2 and 5.3.

5.2. Choice of lattice vectors. So far, the lattice vectors \( t_1 \) and \( t_2 \) may be arbitrarily chosen, except for the constraint imposed by the fluxoid quantization condition, i.e., the parameters \( \gamma > 0 \) and \( \vartheta \neq 0 \) may be chosen at will. Specific choices for these parameters are governed by well-known physical considerations.

The two most interesting periodicity structures are those corresponding to equilateral triangular and to square lattices. For an equilateral triangular lattice, we must have that \( \gamma = 1 \) and \( \vartheta = \pi/3 \). In this case, the area \(|\Omega|\) of any lattice cell is given by
\(|\Omega| = \sqrt{3}|t_1|^2/2\) and, from (5.1),

\[
(5.2) \quad t_1 = \sqrt{\frac{4\pi n}{\sqrt{3}KB}}i_1 \quad \text{and} \quad t_2 = \sqrt{\frac{4\pi n}{\sqrt{3}KB}} \left( \frac{1}{2}i_1 + \frac{\sqrt{3}}{2}i_2 \right).
\]

For a square lattice, we find that \(\gamma = 1\) and \(\vartheta = \pi/2\) so that \(|\Omega| = |t_1|^2\) and, from (5.1),

\[
t_1 = \sqrt{\frac{2\pi n}{KB}}i_1 \quad \text{and} \quad t_2 = \sqrt{\frac{2\pi n}{KB}}i_2.
\]

It is known (see, e.g., [9], [21], [26], or [27]) that an equilateral triangular arrangement of vortex-like structures having one fluxoid associated with each vortex yields the smallest value for the Gibbs free energy. Thus the preponderance of calculations using periodic Ginzburg–Landau models have been carried out for such an arrangement. For example, we could choose \(n = 1\) and use the lattice vectors defined in (5.2) to treat this case, i.e.,

\[
(5.2) \quad t_1 = \sqrt{\frac{4\pi}{\sqrt{3}KB}}i_1 \quad \text{and} \quad t_2 = \sqrt{\frac{4\pi}{\sqrt{3}KB}} \left( \frac{1}{2}i_1 + \frac{\sqrt{3}}{2}i_2 \right).
\]

With these choices, there will be one vortex-like structure associated with each lattice cell; the cells themselves are rhombuses.

We may also treat the case of an equilateral triangular arrangement of vortex-like structures using rectangular cells having two vortices associated with each cell; see [11]. To make sure the vortex structure is equilateral triangular, we choose \(n = 2\), \(\vartheta = \pi/2\), and \(\gamma = \sqrt{3}\) so that, from (5.1),

\[
t_1 = \sqrt{\frac{4\pi}{\sqrt{3}KB}}i_1 \quad \text{and} \quad t_2 = \sqrt{\frac{4\sqrt{3}\pi}{KB}}i_2.
\]

5.3. Determination of the external magnetic field and other variables of interest. One of the parameters that specifies the periodic Ginzburg–Landau model is the average magnetic field \(\bar{B}\) defined by (2.5). Often, we would instead like to specify the external applied field \(H_e\). Note that the model has been formulated in such a way that \(H_e\) does not explicitly appear in its specification. However, to any given set of values of the parameters that do appear explicitly in the specification of the model, including that for \(\bar{B}\), there corresponds a unique value of \(H_e\). In fact (see, e.g., [10], [21], and [27]),

\[
H_e = \frac{1}{2|\Omega|} \frac{\partial G}{\partial \bar{B}} (\hat{\zeta}, \hat{Q}).
\]

The minimizing pair \((\hat{\zeta}, \hat{Q})\) is now viewed as a function of the parameters \(P, \kappa, n, \gamma, \vartheta, \) and \(\bar{B};\) i.e., we choose values for these parameters and then determine \((\hat{\zeta}, \hat{Q})\) by solving the problem (2.25).

One method for determining the value of \(H_e\) corresponding to a set of parameters is to use a finite difference quotient. For fixed values of \(P, \kappa, n, \gamma, \) and \(\vartheta\) and for some "small" value of \(\epsilon\), we first determine the values of the minimizing pair \((\hat{\zeta}, \hat{Q})\) for the average fields \(\bar{B} + \epsilon\) and \(\bar{B} - \epsilon\). Then we determine the values of the Gibbs free energy
for these values \((\hat{\zeta}, \hat{Q})\); denote these by \(G(\hat{B} + \epsilon)\) and \(G(\hat{B} - \epsilon)\). We then use the approximation

\[
H_e(\hat{B}) \approx \frac{1}{2|\Omega|} \left( \frac{G(\hat{B} + \epsilon) - G(\hat{B} - \epsilon)}{2\epsilon} \right)
\]

to estimate the value of \(H_e\) corresponding to the average magnetic field \(\hat{B}\).

A better method for determining \(H_e\) is developed in [10], where it is shown that

\[
(5.3) \quad \hat{B}H_e = \frac{1}{|\Omega|} \int_{\Omega_p} \left( \frac{1}{2} \left| \left( \frac{i}{\kappa} \text{grad} + \hat{Q} - A_0 \right) \hat{\zeta} + |\text{curl}(\hat{Q} - A_0)|^2 \right| \right) d\Omega.
\]

Thus, once \((\hat{\zeta}, \hat{Q})\) has been determined from a given set of parameters \(P, \kappa, n, \gamma, \vartheta, \) and \(\hat{B}, H_e\) can be determined by evaluating integrals.

The reduced magnetic potential \(\hat{Q}\) and order parameter \(\hat{\zeta}\) are the primary dependent variables used in the Ginzburg–Landau model. However, of more interest are the magnetic field \(h\), the current \(j\), and the density of superconducting charge carriers \(N_s\). All of the latter may be obtained from \(\hat{Q}\) and \(\hat{\zeta}\) through (2.23) and (3.15). The external applied magnetic field \(H_e\) may be determined from (5.3). Another quantity of interest is the magnetization (or magnetic moment per unit volume) \(M\) defined by \(-4\pi M = H_e - \hat{B}\).

### 5.4. A reformulated model having only exactly periodic boundary conditions.

Consider the Ginzburg–Landau equations and boundary conditions (3.16)–(3.23) that include the “quasi”-periodic condition (3.20). Let

\[
\hat{\zeta} = |\hat{\zeta}| e^{i\hat{\omega}} \quad \text{and} \quad \hat{Z} = \frac{1}{\kappa} \text{grad} \hat{\omega} + A_0.
\]

Then, system (3.16)–(3.23) is equivalent to the following system:

\[
\begin{align*}
\text{curl} \text{curl} \hat{Q} + |\hat{\zeta}|^2 (\hat{Q} - \hat{Z}) &= 0 \quad \text{in } \Omega_P, \\
\text{div} \hat{Q} &= 0 \quad \text{in } \Omega_P, \\
\hat{Q}(x + t_k) &= \hat{Q}(x) \quad \forall \ x \in \Gamma_{-k}, \ k = 1, 2, \\
\left( \text{curl} \hat{Q} \right)|_{x+t_k} &= \left( \text{curl} \hat{Q} \right)|_x \quad \forall \ x \in \Gamma_{-k}, \ k = 1, 2, \\
\frac{1}{\kappa^2} \Delta |\hat{\zeta}| + |\hat{\zeta}| |\hat{Z}|^2 &= 0 \quad \text{in } \Omega_P, \\
|\hat{\zeta}(x + t_k)| &= |\hat{\zeta}(x)| \quad \forall \ x \in \Gamma_{-k}, \ k = 1, 2, \\
\left( \text{grad} |\hat{\zeta}| \right)|_{x+t_k} &= \left( \text{grad} |\hat{\zeta}| \right)|_x \quad \forall \ x \in \Gamma_{-k}, \ k = 1, 2, \\
|\hat{\zeta}| \text{div} \hat{Z} - \frac{2}{\kappa} \hat{Z} \cdot \text{grad} |\hat{\zeta}| &= 0 \quad \text{in } \Omega_P,
\end{align*}
\]

and

\[
\hat{Z}(x + t_k) = \hat{Z}(x) \quad \forall \ x \in \Gamma_{-k}, \ k = 1, 2.
\]

Note that the boundary conditions on \(\hat{Q}, |\hat{\zeta}|,\) and \(\hat{Z}\) are all exactly periodic. This indicates that it may be advantageous to choose \(\hat{Q}, |\hat{\zeta}|,\) and \(\hat{Z}\) as the dependent variables. Indeed in [7], for example, such a choice is profitably used.
However, there is a price to be paid for using \((\mathbf{Q}, |\mathbf{\zeta}|, \tilde{\mathbf{Z}})\) as dependent variables, especially in numerical simulations. The problem is that \(\tilde{\mathbf{Z}}\) or \(\tilde{\omega}\) are singular at points where \(|\mathbf{\zeta}| = 0\), i.e., at the center of the vortex-like structures. In [7] this problem was handled by using singular basis functions in the approximation scheme, and such a methodology can also be used within the context of finite element methods for example. On the other hand, if we instead use \(\mathbf{Q}, \Re(\mathbf{\zeta}), \text{ and } \Im(\mathbf{\zeta})\) as dependent variables, then these are all nonsingular, and more standard finite element basis functions can be used. Thus we have the choice of having exactly periodic boundary conditions but having to deal with singular solutions, or having nonsingular solutions but having to deal with nonstandard “quasi”-periodic boundary conditions.

**Appendix. Proof of Theorem 2.6.** To achieve our goal of proving Theorem 2.6, we use the spaces

\[
\mathbf{H}_{P,0}^1(\Omega_P) = \left\{ \mathbf{Q} \in \mathbf{H}^1(\Omega_P) \mid \int_{\Omega_P} \mathbf{Q} \, d\Omega = 0, \quad \mathbf{Q}(\mathbf{x} + t_k) = \mathbf{Q}(\mathbf{x}) \quad \forall \mathbf{x} \in \Gamma_{-k}, \ k = 1, 2 \right\}
\]

and

\[
\mathbf{H}^1_p(\text{div} \ ; \Omega_P) = \left\{ \mathbf{Q} \in \mathbf{H}^1_{P,0}(\Omega_P) \mid \text{div} \mathbf{Q} = 0 \text{ in } \Omega_P \right\},
\]

and the auxiliary functional

\[
J(\zeta, \mathbf{Q}, \mathbf{\alpha}) = \int_{\Omega_P} \left( f_0 - |\zeta|^2 + \frac{1}{2}|\zeta|^4 + \left| \left( \frac{i}{\kappa} \text{grad} + \mathbf{Q} + \mathbf{\alpha} - \mathbf{A}_0 \right) \zeta \right|^2 + |\text{curl} (\mathbf{Q} - \mathbf{A}_0)|^2 \right) \, d\Omega.
\]

where \(\mathbf{A}_0\) is given by (2.19) and \(\zeta \in \mathcal{H}_P(\Omega_P), \mathbf{Q} \in \mathbf{H}^1_{P,0}(\Omega_P), \text{ and } \mathbf{\alpha} \in \mathbb{R}^2\).

It is easy to see that the minimization problem (2.25) is equivalent to

\[
\text{minimize } J(\zeta, \mathbf{Q}, \mathbf{\alpha}), \text{ given by (A.1), over all } \zeta \in \mathcal{H}_P(\Omega_P) \quad \mathbf{Q} \in \mathbf{H}^1_{P,0}(\text{div} \ ; \Omega_P) \quad \text{and } \mathbf{\alpha} \in \mathbb{R}^2.
\]

We also obtain the following correspondence between minimizers of the functional \(G\) given in (2.24) and those of the functional \(J\) given in (A.1).

**Lemma A.1.** \(J\) has a minimizer in \(\mathcal{H}_P(\Omega_P) \times \mathbf{H}^1_{P,0}(\text{div} \ ; \Omega_P) \times \mathbb{R}^2\) if and only if \(G\) has a minimizer in \(\mathcal{H}^1_p(\Omega_P) \times \mathbf{H}^1_p(\text{div} \ ; \Omega_P)\). Moreover, if the minimizers exists, then

\[
\min_{\mathcal{H}_P(\Omega_P) \times \mathbf{H}^1_{P,0}(\text{div} \ ; \Omega_P)} \quad \min_{\mathcal{H}^1_p(\Omega_P) \times \mathbf{H}^1_p(\text{div} \ ; \Omega_P) \times \mathbb{R}^2} \quad J.
\]

Thus, if problem (A.1) has a solution, then so does problem (2.25). Thus, to prove Theorem 2.6, we need only obtain the following result.

**Theorem A.2.** \(J\) has at least one minimizer in \(\mathcal{H}_P(\Omega_P) \times \mathbf{H}^1_{P,0}(\text{div} \ ; \Omega_P) \times \mathbb{R}^2\).

**Proof.** The proof of this result is given in §A.3. \(\square\)

To prove Theorem A.2, we need some preliminary results concerned with minimizers of \(J\) in the case of \(\mathbf{\alpha}\) being fixed. We collect these results in §§A.1 and A.2.
A.1. Minimizers with fixed $\alpha$. For any fixed value of $\alpha$, let

$$J_\alpha(\zeta, Q) = J(\zeta, Q, \alpha) \quad \forall \zeta \in H^1_\Omega(\Omega_P), \; Q \in H^1_{P,0}(\text{div}; \Omega_P).$$

As a consequence of the fact that, for $Q \in H^1_{P,0}(\text{div}; \Omega_P), \|\text{curl} \, Q\|_0$ is a norm equivalent to $\|Q\|_1$, we may use standard variational arguments to prove the following lemma.

**Lemma A.3.** For any given $\alpha \in \mathbb{R}^2$, $J_\alpha$ has at least one minimizer belonging to $H^1_{P}(\Omega_P) \times H^1_{P,0}(\text{div}; \Omega_P)$.

Then, as a consequence of gauge invariance, we have the following result.

**Corollary A.4.** For given $\alpha \in \mathbb{R}^2$, $J_\alpha$ has at least one minimizer belonging to $H^1_{P}(\Omega_P) \times H^1_{P,0}(\Omega_P)$. Moreover, any minimizer of $J_\alpha$ in $H^1_{P}(\Omega_P) \times H^1_{P,0}(\text{div}; \Omega_P)$ is also a minimizer of $J_\alpha$ in $H^1_{P}(\Omega_P) \times H^1_{P,0}(\Omega_P)$ and any minimizer of $J_\alpha$ in $H^1_{P}(\Omega_P) \times H^1_{P,0}(\text{div}; \Omega_P)$ is gauge equivalent to a minimizer of $J_\alpha$ in $H^1_{P}(\Omega_P) \times H^1_{P,0}(\text{div}; \Omega_P)$.

Now we examine some properties of minimizers $(\zeta(\alpha), Q(\alpha))$ of the functional $J_\alpha$ for a given, fixed value of $\alpha \in \mathbb{R}^2$.

In a manner analogous to the development in §3.2, we find that minimizers of $J_\alpha$ satisfy

$$\left(\frac{i}{\kappa} \text{grad} - A_0 \right) \cdot \left(\frac{i}{\kappa} \text{grad} - A_0 \right) \zeta(\alpha) + (|(Q(\alpha) + \alpha)|^2 + |\zeta(\alpha)|^2 - 1) \zeta(\alpha)$$

$$+ 2(Q(\alpha) + \alpha) \cdot \left(\frac{i}{\kappa} \text{grad} - A_0 \right) \zeta(\alpha) = 0 \quad \text{in } \Omega_P$$

and

$$\text{curl} \, \text{curl} \, Q(\alpha) + |\tilde{\zeta}|^2 (Q(\alpha) + \alpha)$$

$$+ \Re \left\{ \zeta(\alpha)^* \left(\frac{i}{\kappa} \text{grad} - A_0 \right) \zeta(\alpha) \right\} = C(\alpha) \quad \text{in } \Omega_P,$$

together with the divergence-free condition and boundary conditions that are the same as (3.19)–(3.23). In (A.5), $C(\alpha)$ is a constant vector that serves as the Lagrange multiplier for the mean-zero constraint on $Q(\alpha)$, i.e., to enforce the constraint

$$\int_{\Omega_P} Q(\alpha) \, d\Omega = 0.$$

By integrating (A.5) over $\Omega_P$, we have that

$$C(\alpha) = \frac{1}{|\Omega_P|} \int_{\Omega_P} \Re \left\{ \zeta(\alpha)^* \left(\frac{1}{\kappa} \text{grad} + Q(\alpha) + \alpha - A_0 \right) \zeta(\alpha) \right\} \, d\Omega.$$

For any $\alpha = (\alpha_1, \alpha_2)$ and corresponding minimizer $(\zeta(\alpha), Q(\alpha))$, let

$$\hat{\zeta}(\alpha)(x) = \zeta(\alpha)(x) e^{i \alpha \cdot x} \quad \forall \, x \in \Omega_P.$$

Then (A.4)–(A.6), along with the related essential and natural conditions, yield the differential equations

$$\left(\frac{i}{\kappa} \text{grad} - A_0 \right) \cdot \left(\frac{i}{\kappa} \text{grad} - A_0 \right) \hat{\zeta}(\alpha) + (|(Q(\alpha)|^2 + |\hat{\zeta}(\alpha)|^2 - 1) \hat{\zeta}(\alpha)$$

$$+ 2Q(\alpha) \cdot \left(\frac{i}{\kappa} \text{grad} - A_0 \right) \hat{\zeta}(\alpha) = 0 \quad \text{in } \Omega_P,$$
\( \text{(A.10)} \) \( \text{curl} \text{ curl} Q(\alpha) + |\hat{\zeta}|^2 Q(\alpha) + \Re \left\{ \hat{\zeta}^* \left( \frac{i}{\kappa} \text{grad} - A_0 \right) \hat{\zeta} \right\} = C(\alpha) \quad \text{in } \Omega_P, \)

and

\( \text{(A.11)} \) \( \text{div} Q(\alpha) = 0 \quad \text{in } \Omega_P, \)

with boundary conditions

\( \text{(A.12)} \) \( Q(\alpha)(x + t_k) = Q(\alpha)(x) \quad \forall x \in \Gamma_{-k}, k = 1, 2, \)

\( \text{(A.13)} \) \( \hat{\zeta}(\alpha)(x + t_k) = \hat{\zeta}(\alpha)(x)e^{i\kappa g_k(x) + i\alpha_k} \quad \forall x \in \Gamma_{-k}, k = 1, 2, \)

\( \text{(A.14)} \) \( \left( \text{grad} |\hat{\zeta}(\alpha)| \right)|_{x + t_k} = \left( \text{grad} |\hat{\zeta}(\alpha)| \right)|_x \quad \forall x \in \Gamma_{-k}, k = 1, 2, \)

\( \text{(A.15)} \) \( \text{curl} (Q(\alpha) - A_0)|_{x + t_k} = \text{curl} (Q(\alpha) - A_0)|_x \quad \forall x \in \Gamma_{-k}, k = 1, 2, \)

and

\( \text{(A.16)} \) \( |\hat{\zeta}(\alpha)| \left( \text{grad} \hat{\omega}(\alpha) - \kappa (Q(\alpha) - A_0) \right) |_{x + t_k} = |\hat{\zeta}(\alpha)| \left( \text{grad} \hat{\omega}(\alpha) - \kappa (Q(\alpha) - A_0) \right) |_x \quad \forall x \in \Gamma_{-k}, k = 1, 2, \)

where \( \hat{\omega}(\alpha) \) denotes the phase of \( \hat{\zeta}(\alpha) \). Also, \( Q(\alpha) \) satisfies (A.6).

We next derive some bounds for \( Q(\alpha), \zeta(\alpha), \) and \( \hat{\zeta}(\alpha) \).

\text{Lemma A.5.} For given \( M > 0 \) and for any \( \alpha \in \mathbb{R}^2 \), let \( J_\alpha(\zeta(\alpha), Q(\alpha)) \leq M \).

Then, there exists a constant \( M_0 > 0 \), independent of \( \alpha \), such that

\[ \|Q(\alpha)\|_{1,2,\Omega_P} \leq M_0, \]

\[ \|\zeta(\alpha)\|_{0,4,\Omega_P} \leq M_0, \]

and

\[ \|\hat{\zeta}(\alpha)\|_{1,2,\Omega_P} \leq M_0. \]

Furthermore, \( |C(\alpha)| \leq M_0. \)

\text{Proof.} From \( J_\alpha(\zeta(\alpha), Q(\alpha)) \leq M \), we may easily deduce that

\[ \|Q(\alpha)\|_{1,2,\Omega_P} \leq M_0 \quad \text{and} \quad \|\zeta(\alpha)\|_{0,4,\Omega_P} \leq M_0. \]

The latter implies that \( \|\hat{\zeta}(\alpha)\|_{0,2,\Omega_P} \leq M_0. \) In addition, we have that

\[ \left\| \left( \frac{i}{\kappa} \text{grad} - \alpha \right) \zeta(\alpha) \right\|_{0,2,\Omega_P} \leq M_0. \]

Therefore, \( \|\hat{\zeta}(\alpha)\|_{1,2,\Omega_P} \leq M_0 \). The bound on \( C(\alpha) \) then follows from (A.7) and the Cauchy–Schwartz inequality. \( \square \)
A.2. Regularity results for minimizers of $J_\alpha$. Similar to the development in §§4.2 and 4.3, we can prove the corresponding regularity results for the solution $(\zeta(\alpha), Q(\alpha))$ of the system (A.6) and (A.9)–(A.16), which is related, through (A.8), to the minimizer $(\zeta(\alpha), Q(\alpha))$ of the functional $J_\alpha$. The existence of minimizers of $J_\alpha$ has been established in §A.1. Here we only quote the following result that will be needed in §A.3.

**Lemma A.6.** Any solution $(\zeta(\alpha), Q(\alpha))$ can be smoothly extended to $\mathbb{R}^2$ using conditions similar to (A.12) and (A.13) that hold for all $x \in \mathbb{R}^2$. Moreover, for given $M > 0$ and for any $\alpha \in \mathbb{R}^2$, if $J(\zeta(\alpha), Q(\alpha), \alpha) \leq M$, then there exists a constant $M_0$, independent of $\alpha$, such that

$$\|\zeta(\alpha)\|_{2,2,\Omega_P} \leq M_0$$

and

$$\|Q(\alpha)\|_{2,2,\Omega_P} \leq M_0.$$  

**Proof.** The first part of the lemma can be verified using a similar approach as that in §§4.1 and 4.2. From Lemma A.5, we obtain

$$\|\zeta(\alpha)\|_{1,2,\Omega_P} \leq M_0,$$

$$\|Q(\alpha)\|_{1,2,\Omega_P} \leq M_0,$$

and

$$|C(\alpha)| \leq M_0.$$

The higher-order estimates in the above lemma now follow from standard regularity estimates that are similar to, e.g., those stated in Lemma 4.3. \qed

A.3. Proof of Theorem A.2. We divide the proof into four steps.

**Step 1.** The functional $J$ is bounded from below so that if $\inf J$ exists in $H^1_P(\Omega_P) \times H^1_{p,0}(\div; \Omega_P) \times \mathbb{R}^2$ and there exists a sequence

$$\{(\zeta_n, Q_n, \alpha_n) \in H^1_P(\Omega_P) \times H^1_{p,0}(\div; \Omega_P) \times \mathbb{R}^2\}$$

such that

$$\lim_{n \to \infty} J(\zeta_n, Q_n, \alpha_n) = \inf J.$$  

By Lemma A.3, for any $\alpha_n$, $\mathcal{J}_{\alpha_n}$ has at least one minimizer $(\zeta(\alpha_n), Q(\alpha_n))$ in $H^1_P(\Omega_P) \times H^1_{p,0}(\div; \Omega_P)$; moreover,

$$\lim_{n \to \infty} J(\zeta(\alpha_n), Q(\alpha_n), \alpha_n) = \inf J,$$

i.e., $\{(\zeta(\alpha_n), Q(\alpha_n), \alpha_n)\}$ is a minimizing sequence.

**Step 2.** Define $\zeta(\alpha)$ by (A.8). Since $J_{\alpha_n}(\zeta(\alpha_n), Q(\alpha_n)) = J(\zeta(\alpha_n), Q(\alpha_n), \alpha_n)$ is uniformly bounded, by Lemma A.6, we have the sequence

$$\{\|\zeta(\alpha_n)\|_{2,2,\Omega_P}\}$$

uniformly bounded. In particular,

$$\{\|\nabla \zeta(\alpha_n)\|_{0,2,\Omega_P}, \|\Delta \zeta(\alpha_n)\|_{0,2,\Omega_P}\}$$
is uniformly bounded. Simple calculations yield that
\[ \nabla \tilde{\zeta}(\alpha) = (\nabla \zeta(\alpha) + i\zeta(\alpha)\alpha) e^{i\alpha \cdot x} \]
and
\[ \Delta \tilde{\zeta}(\alpha) = (\Delta \zeta(\alpha) - |\alpha|^2 \zeta(\alpha) + 2i \nabla \zeta(\alpha) \cdot \alpha) e^{i\alpha \cdot x} \]
so that
\[ \{ \| \nabla \zeta(\alpha_n) + i\alpha_n \zeta(\alpha_n) \|_{0,2,\Omega_P} , \| \nabla \zeta(\alpha_n) \cdot \alpha_n \|_{0,2,\Omega_P} \} \]
is uniformly bounded.

**Step 3.** We now consider the case where \(|\alpha_n\|\) is unbounded. This implies that there exists a subsequence such that \(\lim_{n_k \to \infty} |\alpha_{n_k}| = \infty\). Therefore, to maintain the uniform boundedness obtained in the previous step, we have
\[ \lim_{n_k \to \infty} \| \zeta(\alpha_{n_k}) \|_{1,2,\Omega_P} = 0. \]
This implies that \( J(0,0,0) \leq \lim_{n_k \to \infty} J(\zeta(\alpha_{n_k}), Q(\alpha_{n_k}), \alpha_{n_k}) = \inf J \). So, in this case, the normal state \( \zeta = 0, Q = 0 \) is the minimizer of \( J \) in \( H^1_0(\Omega_P) \times H_{P,0}(\text{div}; \Omega_P) \times R^2 \).

**Step 4.** To conclude the proof, we now consider the remaining case where \(|\alpha_n|\) is bounded. This, in particular, implies that \(|\|\zeta(\alpha_n)\|_{1,2,\Omega_P}\|\) is uniformly bounded. By Lemma A.5, \(|\|Q(\alpha_n)\|_{1,2,\Omega_P}\|\) is also uniformly bounded. Then we can subtract a subsequence \(\lim_{n_k \to \infty} \alpha_{n_k} = \alpha^*\), and \(\{\|\zeta(\alpha_{n_k}), Q(\alpha_{n_k})\|\}\) converges weakly in \( H^1_0(\Omega_P) \times H_{P,0}(\text{div}; \Omega_P) \) to \((\zeta^*, Q^*)\). Using standard compact embedding results and the lower weak semicontinuity of norms, we can verify easily that
\[ J(\zeta^*, Q^*, \alpha^*) \leq \inf J. \]
So, we must have \( J(\zeta^*, Q^*, \alpha^*) = \inf J \), i.e., \((\zeta^*, Q^*, \alpha^*)\) is the minimizer of \( J \) in \( H^1_0(\Omega_P) \times H_{P,0}(\text{div}; \Omega_P) \times R^2 \). This proves Theorem A.2.

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