A NONLOCAL VECTOR CALCULUS WITH APPLICATION TO NONLOCAL BOUNDARY VALUE PROBLEMS

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Abstract. We develop a calculus for nonlocal operators that mimics Gauss' theorem and the Green's identities of the classical vector calculus. The operators we define do not involve the derivatives. We then apply the nonlocal calculus to define variational formulations of nonlocal "boundary-value" problems that mimic the Dirichlet and Neumann problems for second-order scalar elliptic partial differential equations. For the nonlocal variational problems, we derive fundamental solutions, show how one can derive existence and uniqueness results, and show how, under appropriate limits, they reduce to their classical analogs.

Key words. nonlocal operators, vector calculus, boundary-value problems, diffusion, convection-diffusion-reaction, peridynamics

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1. Introduction.

Gauss's theorem and the Green's identities are crucial for the analysis of the second-order scalar elliptic boundary-value problem

$$-\nabla \cdot \big(\mathbf{D}(\mathbf{x}) \cdot \nabla u(\mathbf{x})\big) = b(\mathbf{x}) \qquad \text{in } \Omega \subset \mathbb{R}^d \tag{1.1}$$

augmented with Dirichlet or Neumann boundary conditions on the boundary $\partial\Omega$, where $\mathbf{D}$ denotes a symmetric, positive definite, second-order tensor, b a scalar-valued data function, and d a positive integer. Gauss's theorem and the ensuing Green's identities provide compatibility relations, a solution operator, and a variational formulation for the boundary-value problem (1.1).

The first contribution of this paper is the development of a calculus for nonlocal analogues of (1.1) that mimics Gauss's theorem and the Green's identities of the classical vector calculus. The nonlocal second-order scalar "elliptic boundary-value" problem is given by

$$\mathcal{L}(u)(\mathbf{x}) := 2\int_{\mathbb{R}^d} \big(u(\mathbf{x}') - u(\mathbf{x})\big)\mu(\mathbf{x},\mathbf{x}')\,d\mathbf{x}' = b(\mathbf{x}) \qquad \text{in } \Omega \subset \mathbb{R}^d \tag{1.2}$$

augmented with nonlocal "Dirichlet" or "Neumann" "boundary" conditions, where $\mu(\mathbf{x},\mathbf{x}')$ denotes a positive, symmetric function of its arguments. The "boundary-value" problem (1.2) characterizes the solution of the formal minimization problem

$$\int_\Omega \int_{\mathbb{R}^d} \big(u(\mathbf{x}') - u(\mathbf{x})\big)^2 \mu(\mathbf{x},\mathbf{x}')\,d\mathbf{x}'\,d\mathbf{x} - \int_\Omega b(\mathbf{x})u(\mathbf{x})\,d\mathbf{x} \to \min! \tag{1.3}$$

augmented with nonlocal "boundary" conditions. The relationship between (1.3) and (1.2) is equivalent to the relationship between (1.1) and the minimization problem

$$\frac{1}{2}\int_\Omega \nabla u(\mathbf{x}) \cdot \mathbf{D}(\mathbf{x}) \cdot \nabla u(\mathbf{x})\,d\mathbf{x} - \int_\Omega b(\mathbf{x})u(\mathbf{x})\,d\mathbf{x} \to \min!$$

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Our second contribution applies the nonlocal calculus to define variational formulations of nonlocal "boundary-value" problems that mimic the Dirichlet and Neumann problems for second-order scalar elliptic partial differential equations. In contrast to their local counterparts, e.g., (1.1), the nonlocal "Dirichlet" and "Neumann" data needed for (1.2) are defined on a nonzero *volume* exterior to Ω. We also establish existence and uniqueness results, and demonstrate how, under appropriate limits, the nonlocal "Dirichlet" and "Neumann" problems reduce to their classical analogs.

Underpinning our analyses are two relatively unknown lemmas established in [14, 15]. The lemmas lead to a closed form expression for a flux such that the divergence of this flux is equal to $\mathcal{L}(u)$.

The second-order elliptic operator associated with (1.1) is local, e.g., $\nabla \cdot \mathbf{D}(\mathbf{x}) \cdot \nabla$ only depends on the point $\mathbf{x}$ whenever $\mathbf{D}(\mathbf{x})$ only depends upon $\mathbf{x}$. In contrast, the operator $\mathcal{L}$ eschews the gradient of the scalar function u, and is nonlocal because points $\mathbf{x}' \neq \mathbf{x}$ can interact with $\mathbf{x}$. The solution operator for (1.2) does not, in general, smooth the data $b(\mathbf{x})$ as does the solution operator associated with (1.1). For example, given homogenous Dirichlet boundary conditions and appropriate conditions on the tensor $\mathbf{D}(\mathbf{x})$, the solution operator for the variational formulation of (1.1) maps $H^{-1}(\Omega)$ to $H_0^1(\Omega)$; see [6]. In contrast, given appropriate conditions on $\mu(\mathbf{x}, \mathbf{x}')$, $\mathcal{L}^{-1}$ maps a subspace of $H^{-1}(\Omega)$ to $H_0^s(\Omega)$, $1/2 < s < 1$; see [4] and also [7].[1] This implies that the solution of (1.2) exhibits multiscale character beyond that achieved by the solution of (1.1). In particular, the solution u of (1.1) is differentiable, albeit in a weak sense, whereas the solution (1.2) is not necessarily weakly differentiable. The minimal regularity of the latter solution suggests that the operator $\mathcal{L}$ is an attractive alternative to $\nabla \cdot \mathbf{D}(\mathbf{x}) \cdot \nabla$ for modeling phenomena exhibiting discontinuities.

The nonlocal operator $\mathcal{L}$ is associated with the jump process of the master equation that generalizes Brownian motion; see [9, chap. 7] for a review. Such a jump process was considered by Einstein in his seminal paper on the origins of diffusion. The operator $\mathcal{L}$ gives rise to nonlocal diffusion that enables improved multiscale modeling; see [3] and [5, chap. 3] for examples and citations to the literature. The nonlocal p-Laplacian diffusion equation with Dirichlet and Neumann boundary conditions (the operator $\mathcal{L}$ corresponds to $p = 2$) are investigated in [1, 2]. Those papers and the citations found therein to related work about Neumann boundary conditions, provide mathematical analyses for the existence and uniqueness of solutions of the nonlocal p-Laplacian diffusion equation, including conditions under which its solution can approximate the solution to the classical p-Laplacian diffusion equation. See also [9, chap. 7] for a review of the Kramers-Moyal and van Kampen asymptotic approximations of a jump process by a Fokker-Plank equation.[2]

In [10], the nonlocal operator $\mathcal{L}$ and its applications to image processing are considered and suggestions are made for its use for modeling physical phenomena. In addition to nonlocal diffusion, the peridynamic continuum theory [16, 17] postulates that the internal force density is given by an integral operator. The nonlocal operator $\mathcal{L}$ results when the deformation is given by a scalar-valued function and the constitutive relation is linear. Our results are directly applicable to the one-dimensional peridynamic equilibrium equation associated with the equation of motion considered in [18, 19]. The nonlocal vector calculus presented in our paper extends the ideas in-

[1]The paper [7] considers the free-space vector-valued formulation of $\mathcal{L}$ and demonstrates that the associated solution operator maps the dual of $[H^s(\mathbb{R}^d)]^d$ to $[H^s(\mathbb{R}^d)]^d$ for $0 \leq s \leq 1$.

[2]The classical diffusion equation is a specialization of the Fokker-Plank equation under the assumption of no drift, i.e., no bias in the associated random walk.

troduced in [10] and applies this calculus to scalar nonlocal boundary-value problems. In a follow-up paper, we consider more general linear peridynamic models for which deformation is given by a vector-valued function.

2. A nonlocal Gauss's theorem. For any mapping[3] $r(\mathbf{x},\mathbf{x}')\colon \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$, it is easily seen that

$$\int_{\widehat{\Omega}}\int_{\widehat{\Omega}} r(\mathbf{x},\mathbf{x}')\,d\mathbf{x}'d\mathbf{x} = \int_{\widehat{\Omega}}\int_{\widehat{\Omega}} r(\mathbf{x}',\mathbf{x})\,d\mathbf{x}'d\mathbf{x} \qquad \forall\,\widehat{\Omega}\subseteq\mathbb{R}^d. \tag{2.1}$$

If $p(\mathbf{x}',\mathbf{x})$ denotes an *anti-symmetric* mapping, i.e., if $p(\mathbf{x}',\mathbf{x}) = -p(\mathbf{x},\mathbf{x}')$ for all $\mathbf{x}$, $\mathbf{x}'\in\mathbb{R}^d$, then (2.1) implies

$$\int_{\widehat{\Omega}}\int_{\widehat{\Omega}} p(\mathbf{x},\mathbf{x}')\,d\mathbf{x}'d\mathbf{x} = 0 \quad \forall\,\widehat{\Omega}\subseteq\mathbb{R}^d. \tag{2.2}$$

Let Ω denote an open bounded subset of $\mathbb{R}^d$. Obviously, if $\Gamma\subseteq\mathbb{R}^d\setminus\Omega$ and $p(\mathbf{x}',\mathbf{x})$ is anti-symmetric, by setting $\widehat{\Omega}=\Omega\cup\Gamma$, (2.2) implies

$$\int_{\Omega}\int_{\Omega\cup\Gamma} p(\mathbf{x},\mathbf{x}')\,d\mathbf{x}'d\mathbf{x} = -\int_{\Gamma}\int_{\Omega\cup\Gamma} p(\mathbf{x},\mathbf{x}')\,d\mathbf{x}'d\mathbf{x}. \tag{2.3}$$

Let $\alpha(\mathbf{x},\mathbf{x}')\colon \Omega\cup\Gamma\times\Omega\cup\Gamma\to\mathbb{R}$ denote a *symmetric mapping*, i.e., $\alpha(\mathbf{x}',\mathbf{x}) = \alpha(\mathbf{x},\mathbf{x}')$ for all $\mathbf{x}$, $\mathbf{x}'\in\Omega\cup\Gamma$, and let $f(\mathbf{x},\mathbf{x}')\colon \Omega\cup\Gamma\times\Omega\cup\Gamma\to\mathbb{R}$ denote a mapping that is not necessarily symmetric or anti-symmetric. Let $\mathcal{D}$ denote the linear operator mapping functions $f(\mathbf{x},\mathbf{x}')$ into functions defined over Ω given by

$$\mathcal{D}(f)(\mathbf{x}) := \int_{\Omega\cup\Gamma}\big(f(\mathbf{x},\mathbf{x}')-f(\mathbf{x}',\mathbf{x})\big)\alpha(\mathbf{x},\mathbf{x}')\,d\mathbf{x}' \qquad \text{for } \mathbf{x}\in\Omega. \tag{2.4}$$

Similarly, let $\mathcal{N}$ denote the linear operator mapping functions $f(\mathbf{x},\mathbf{x}')$ into functions defined over Γ given by

$$\mathcal{N}(f)(\mathbf{x}) := -\int_{\Omega\cup\Gamma}\big(f(\mathbf{x},\mathbf{x}')-f(\mathbf{x}',\mathbf{x})\big)\alpha(\mathbf{x},\mathbf{x}')\,d\mathbf{x}' \qquad \text{for } \mathbf{x}\in\Gamma. \tag{2.5}$$

Then, setting $p(\mathbf{x},\mathbf{x}') = \big(f(\mathbf{x},\mathbf{x}')-f(\mathbf{x}',\mathbf{x})\big)\alpha(\mathbf{x},\mathbf{x}')$ in (2.3) results in the *nonlocal Gauss's theorem*

$$\int_{\Omega}\mathcal{D}(f)\,d\mathbf{x} = \int_{\Gamma}\mathcal{N}(f)\,d\mathbf{x}. \tag{2.6}$$

Note that the operators $\mathcal{D}$ and $\mathcal{N}$ differ only in their domains and signs.[4]

2.1. Relation to the classical Gauss's theorem. Let the vector-valued function $\mathbf{q}\colon\mathbb{R}^d\to\mathbb{R}^d$ be defined by

$$\mathbf{q}(\mathbf{x}) := -\int_{\mathbb{R}^d}(\mathbf{x}'-\mathbf{x})\varphi(\mathbf{x},\mathbf{x}'-\mathbf{x})\,d\mathbf{x}', \tag{2.7}$$

[3]Throughout, vectors in $\mathbb{R}^d$ are denoted in lower-case Roman bold letters, scalar valued functions in lower-case Roman or Greek letters, and second-order tensors in upper-case Roman bold letters.

[4]With $\Omega\cup\Gamma=\Omega$, we have from (2.6) (or directly from (2.2)) that $\int_{\Omega\cup\Gamma}\mathcal{D}(f)\,d\mathbf{x}=0$. In [10, eq. (2.6)], this equation is referred to as the "divergence theorem;" however, we see here that it is a special case of the nonlocal Gauss's theorem (2.6).

where, with $p(\mathbf{x},\mathbf{x}') = \big(f(\mathbf{x},\mathbf{x}') - f(\mathbf{x}',\mathbf{x})\big)\alpha(\mathbf{x},\mathbf{x}')$ and $\mathbf{z} = \mathbf{x}' - \mathbf{x}$, the function $\varphi : \Omega \times \Omega \to \mathbb{R}$ is given by

$$\varphi(\mathbf{x},\mathbf{z}) = \int_0^1 p\big(\mathbf{x} + \lambda\mathbf{z}, \mathbf{x} - (1-\lambda)\mathbf{z}\big)\, d\lambda. \tag{2.8}$$

We also suppose, for this subsection only,[5] that $\Gamma = \mathbb{R}^d \setminus \Omega$ so that $\mathbb{R}^d = \Omega \cup \Gamma$. Then, a formal application of Lemma I in [14, 15] implies

$$\nabla \cdot \mathbf{q}(\mathbf{x}) = \int_{\mathbb{R}^d} \big(f(\mathbf{x},\mathbf{x}') - f(\mathbf{x}',\mathbf{x})\big)\alpha(\mathbf{x},\mathbf{x}')\, d\mathbf{x}' = \mathcal{D}(f) \qquad \text{for } \mathbf{x} \in \Omega, \tag{2.9}$$

where we used the definition (2.4) of the operator $\mathcal{D}$. Lemma II in [14, 15] implies

$$\int_{\partial\Omega} \mathbf{q}(\mathbf{x}) \cdot \mathbf{n}\, dA = \int_\Omega \int_\Gamma \big(f(\mathbf{x},\mathbf{x}') - f(\mathbf{x}',\mathbf{x})\big)\alpha(\mathbf{x},\mathbf{x}')\, d\mathbf{x}' d\mathbf{x}, \tag{2.10}$$

where $\partial\Omega$ denotes the boundary of Ω, dA the surface element on $\partial\Omega$, and $\mathbf{n}$ the outward pointing unit normal vector along $\partial\Omega$. Two successive applications of (2.2), first with $\widehat{\Omega} = \Omega$ and then with $\widehat{\Omega} = \Omega \cup \Gamma$, yields

$$\begin{aligned}\int_\Omega \int_\Gamma f(\mathbf{x},\mathbf{x}')\alpha(\mathbf{x},\mathbf{x}')\, d\mathbf{x}' d\mathbf{x} &= \int_\Omega \int_{\Omega\cup\Gamma} f(\mathbf{x},\mathbf{x}')\alpha(\mathbf{x},\mathbf{x}')\, d\mathbf{x}' d\mathbf{x} \\ &= -\int_\Gamma \int_{\Omega\cup\Gamma} f(\mathbf{x},\mathbf{x}')\alpha(\mathbf{x},\mathbf{x}')\, d\mathbf{x}' d\mathbf{x}.\end{aligned}$$

Combining this result with (2.10) yields

$$\int_{\partial\Omega} \mathbf{q}(\mathbf{x}) \cdot \mathbf{n}\, dA = -\int_\Gamma \int_{\Omega\cup\Gamma} \big(f(\mathbf{x},\mathbf{x}') - f(\mathbf{x}',\mathbf{x})\big)\alpha(\mathbf{x},\mathbf{x}')\, d\mathbf{x}' d\mathbf{x} = \int_\Gamma \mathcal{N}(f)\, d\mathbf{x}, \tag{2.11}$$

where we used the definition (2.5) of the operator $\mathcal{N}(\cdot)$. Substituting (2.9) and (2.11) into the nonlocal Gauss's theorem (2.6) results in

$$\int_\Omega \nabla \cdot \mathbf{q}\, d\mathbf{x} = \int_{\partial\Omega} \mathbf{n} \cdot \mathbf{q}\, dA,$$

i.e., the classical Gauss's theorem for the vector-valued function $\mathbf{q}$. Thus, we have shown that *the nonlocal Gauss's theorem* (2.6) *for the nonlocal scalar-valued mapping f implies the classical Gauss's theorem for the nonlocal vector-valued function $\mathbf{q}$ derived from f through* (2.7).[6] Hence, one can view the right-hand side of (2.3) as a nonlocal "flux." Evidently, Gauss's theorem can be given a meaning without the notions of the divergence operator, unit normal vector, or a surface.

[5] This assumption is made in [14, 15]; it is possible to extend the results of those papers to the case of $f(\cdot,\cdot)$ having compact support, in which case we need not assume that $\Gamma = \mathbb{R}^d \setminus \Omega$.

[6] Under appropriate assumptions, the results of [13] can be invoked to show that the vector field $\mathbf{q}$ solves the minimization problem

$$\inf_{\widehat{\mathbf{q}} \in H_0(\mathrm{div},\mathbb{R}^d)} \frac{1}{2} \int_\Omega |\widehat{\mathbf{q}}|^2 d\mathbf{x} \quad \text{subject to} \quad \nabla \cdot \widehat{\mathbf{q}} = \mathcal{D}(f) \in L^2_0(\mathbb{R}^d),$$

where $H_0(\mathrm{div},\mathbb{R}^d) := \{\, \mathbf{q} \mid \nabla \cdot \mathbf{q} \in L^2_0(\mathbb{R}^d)\}$ and $L^2_0(\mathbb{R}^d) := \{\, \psi \in L^2(\mathbb{R}^d) \mid \int_{\mathbb{R}^d} \psi\, d\mathbf{x} = 0\}$.

2.2. An application of the nonlocal Gauss's theorem. We apply the nonlocal Gauss's theorem (2.6) to the product of two mappings. In particular, for mappings $v(\mathbf{x})\colon\ \Omega\cup\Gamma\to\mathbb{R}$ and $s(\mathbf{x},\mathbf{x}')\colon \Omega\cup\Gamma\times\Omega\cup\Gamma\to\mathbb{R}$, set[7] $f=sv$ in (2.6) to obtain, using (2.4) and (2.5) as well,

$$\int_\Omega\int_{\Omega\cup\Gamma}(sv-s'v')\alpha\,d\mathbf{x}'d\mathbf{x}=-\int_\Gamma\int_{\Omega\cup\Gamma}(sv-s'v')\alpha\,d\mathbf{x}'d\mathbf{x}$$

so that, setting $sv-s'v'=sv-s'v'+s'v-s'v=v(s-s')+s'(v-v')$ in both integrands,

$$\int_\Omega\int_{\Omega\cup\Gamma}v(s-s')\alpha\,d\mathbf{x}'d\mathbf{x}+\int_{\Omega\cup\Gamma}\int_{\Omega\cup\Gamma}s'(v-v')\alpha\,d\mathbf{x}'d\mathbf{x}=-\int_\Gamma\int_{\Omega\cup\Gamma}v(s-s')\alpha\,d\mathbf{x}'d\mathbf{x}.$$

We use (2.4) and (2.5) for the first and third terms and (2.1) for the second term to obtain

$$\int_\Omega v\mathcal{D}(s)\,d\mathbf{x}+\int_{\Omega\cup\Gamma}\int_{\Omega\cup\Gamma}s(v'-v)\alpha\,d\mathbf{x}'d\mathbf{x}=\int_\Gamma v\mathcal{N}(s)d\mathbf{x}. \tag{2.12}$$

Let $\mathcal{G}$ denote the linear operator mapping functions $v:\Omega\cup\Gamma\to\mathbb{R}$ into functions defined over $\Omega\cup\Gamma\times\Omega\cup\Gamma$ given by[8]

$$\mathcal{G}(v):=(v'-v)\alpha \qquad \text{for } \mathbf{x},\mathbf{x}'\in\Omega\cup\Gamma. \tag{2.13}$$

Then, using (2.13) in (2.12) results in

$$\int_\Omega v\mathcal{D}(s)\,d\mathbf{x}+\int_{\Omega\cup\Gamma}\int_{\Omega\cup\Gamma}s\mathcal{G}(v)\,d\mathbf{x}'d\mathbf{x}=\int_\Gamma v\mathcal{N}(s)\,d\mathbf{x}. \tag{2.14}$$

The particular choice $v=$ constant in (2.14) yields

$$\int_\Omega\mathcal{D}(s)\,d\mathbf{x}=\int_\Gamma\mathcal{N}(s)\,d\mathbf{x}, \tag{2.15}$$

which is simply the nonlocal Gauss' theorem (2.6) applied to $f=s$.

3. Nonlinear, nonlocal boundary value problems. Let $U(\Omega\cup\Gamma)$ and $V(\Omega\cup\Gamma)$ denote Banach spaces of scalar-valued functions defined over $\Omega\cup\Gamma$. Let

$$\Gamma:=\Gamma_e+\Gamma_n \qquad\text{with}\qquad \Gamma_e\cap\Gamma_n=\emptyset$$

and define

$$V_0(\Omega\cup\Gamma):=\{v\in V(\Omega\cup\Gamma)\ \ :\ \ v=0\ \text{ for } \mathbf{x}\in\Gamma_e\}.$$

[7]In the sequel, for ease of notation, we define

$$\begin{gathered} v:=v(\mathbf{x}),\quad v':=v(\mathbf{x}'),\qquad \alpha:=\alpha(\mathbf{x},\mathbf{x}'),\quad \alpha':=\alpha(\mathbf{x}',\mathbf{x}),\\ f:=f(\mathbf{x},\mathbf{x}'),\quad f':=f(\mathbf{x}',\mathbf{x}),\qquad s:=s(\mathbf{x},\mathbf{x}'),\quad s':=s(\mathbf{x}',\mathbf{x}),\end{gathered}$$

and analogously for other similar functions yet to be introduced.

[8]In [10, eq.(2.2)], $\mathcal{G}$ is denoted by $\nabla_w u$, a nonlocal gradient, where our α is their $\sqrt{w}$. However, denoting $\nabla_w u$ as a gradient (nonlocal or otherwise) is strictly formal because it is never demonstrated that $\nabla_w u$ corresponds to the best local linear approximation of u.

Define the mappings

$$b : \Omega \to \mathbb{R}, \qquad h_e : \Gamma_e \to \mathbb{R}, \qquad \text{and} \qquad h_n : \Gamma_n \to \mathbb{R}. \tag{3.1}$$

For $u \in U(\Omega \cup \Gamma)$, let $s = \mathcal{A}(u)$ for a possibly nonlinear operator $\mathcal{A}$ that may also depend explicitly on $\mathbf{x}$ and $\mathbf{x}'$. Then, consider the variational problem

$$\begin{cases} \textit{seek } u \in U(\Omega \cup \Gamma) \textit{ such that} \\ \qquad u = h_e \quad \text{for } \mathbf{x} \in \Gamma_e \\ \textit{and} \\ \quad \displaystyle\int_{\Omega\cup\Gamma}\int_{\Omega\cup\Gamma} \mathcal{A}(u)\mathcal{G}(v)\, d\mathbf{x}' d\mathbf{x} = \int_\Omega vb\, d\mathbf{x} + \int_{\Gamma_n} vh_n\, d\mathbf{x} \quad \forall\, v \in V_0(\Omega \cup \Gamma). \end{cases} \tag{3.2}$$

Then, (2.14) and $v = 0$ on Γ_e imply

$$-\int_\Omega v\mathcal{D}\big(\mathcal{A}(u)\big)\, d\mathbf{x} + \int_{\Gamma_n} v\mathcal{N}\big(\mathcal{A}(u)\big)\, d\mathbf{x} = \int_\Omega vb\, d\mathbf{x} + \int_{\Gamma_n} vh_n\, d\mathbf{x} \qquad \forall\, v \in V_0(\Omega \cup \Gamma).$$

Hence, (3.2) can be viewed as a weak formulation of the "boundary-value" problem

$$\begin{cases} -\mathcal{D}\big(\mathcal{A}(u)\big) = b & \text{for } \mathbf{x} \in \Omega \\ \qquad\qquad u = h_e & \text{for } \mathbf{x} \in \Gamma_e \\ \qquad\quad \mathcal{N}(s) = h_n & \text{for } \mathbf{x} \in \Gamma_n. \end{cases} \tag{3.3}$$

The second and third equations of (3.3) are the "Dirichlet boundary" and "Neumann boundary" conditions that are *essential* and *natural*, respectively, for the variational principle (3.2).

If $\Gamma_e = \emptyset$, then the space of test functions $V_0(\Omega)$ in the variational problem (3.2) is replaced by $V(\Omega \cup \Gamma)/\mathbb{R}$ and the compatibility condition[9]

$$\int_\Omega b d\mathbf{x} + \int_\Gamma h_n\, d\mathbf{x} = 0 \tag{3.4}$$

must hold.

4. Linear, nonlocal operators and nonlocal Green's identities. In this section, we specialize the nonlocal Gauss's theorem discussed in Section 2 and, in particular, (2.14) to the case of $U(\Omega \cup \Gamma) = V(\Omega \cup \Gamma)$ and to linear operators. To this end, for $u \in V(\Omega \cup \Gamma)$, let

$$s = \mathcal{A}(u) = \beta\mathcal{G}(u) = (u' - u)\alpha\beta, \tag{4.1}$$

where, at this time, no assumption is made about the symmetry or anti-symmetry of the mapping $\beta(\mathbf{x}, \mathbf{x}') \colon \Omega \cup \Gamma \times \Omega \cup \Gamma \to \mathbb{R}$. Then, substitution into (2.14) results in the *nonlocal Green's first identity*

$$\int_\Omega v\mathcal{D}\big(\beta\mathcal{G}(u)\big)\, d\mathbf{x} + \int_{\Omega\cup\Gamma}\int_{\Omega\cup\Gamma} \beta\mathcal{G}(v)\mathcal{G}(u)\, d\mathbf{x}' d\mathbf{x} = \int_\Gamma v\mathcal{N}\big(\beta\mathcal{G}(u)\big)\, d\mathbf{x}. \tag{4.2}$$

[9]If $\Gamma_e = \emptyset$, (3.4) is a necessary condition for the existence of solutions of the variational problem (3.2) because, in this case, the left-hand side of (3.2) vanishes whenever v = constant. Note that (3.4) states that the data b and h_n have to be orthogonal to the one-dimensional null space of the operator $\mathcal{G}(v) = (v' - v)\alpha$. Correspondingly, we exclude the constant functions from the space of test functions in the variational problem (3.2). This is all entirely analogous to the situation for classical, local Neumann boundary-value problems.

Reversing the roles of u and v in (4.2) and then subtracting the result from (4.2) results in the *nonlocal Green's second identity*

$$\int_\Omega v\mathcal{D}\big(\beta\mathcal{G}(u)\big)\,d\mathbf{x} - \int_\Omega u\mathcal{D}\big(\beta\mathcal{G}(v)\big)\,d\mathbf{x} = \int_\Gamma \Big(v\mathcal{N}\big(\beta\mathcal{G}(u)\big) - u\mathcal{N}\big(\beta\mathcal{G}(v)\big)\Big)\,d\mathbf{x}. \tag{4.3}$$

From (4.2) we have, by setting v =constant,

$$\int_\Omega \mathcal{D}\big(\beta\mathcal{G}(u)\big)\,d\mathbf{x} = \int_\Gamma \mathcal{N}\big(\beta\mathcal{G}(u)\big)\,d\mathbf{x}$$

and, by setting $v = u$, the "energy" identity

$$\int_\Omega u\mathcal{D}\big(\beta\mathcal{G}(u)\big)\,d\mathbf{x} + \int_{\Omega\cup\Gamma}\int_{\Omega\cup\Gamma} \beta\mathcal{G}(u)\mathcal{G}(u)\,d\mathbf{x}'d\mathbf{x} = \int_\Gamma u\mathcal{N}\big(\beta\mathcal{G}(u)\big)\,d\mathbf{x}.$$

See, e.g., [12], for the analogous identities in the classical linear elliptic operator case.

We defer discussion of a nonlocal Green's third identity until after we discuss linear, nonlocal boundary-value problems.

The relation (4.1) is a "constitutive" relation. To define a general form for the constitutive function[10] β, let $\mathbf{K}(\mathbf{x},\mathbf{x}') : \Omega\cup\Gamma\times\Omega\cup\Gamma \to \mathbb{R}^{d\times d}$ denote a tensor. Then, a general constitutive function β is given by

$$\beta = (\mathbf{x}'-\mathbf{x})\cdot\mathbf{K}\cdot(\mathbf{x}'-\mathbf{x}), \tag{4.4}$$

where, for ease of notation, we have suppressed the dependence of β and $\mathbf{K}$ on $\mathbf{x}$ and $\mathbf{x}'$. Note that, at this time, we make no assumptions about the symmetry or positive definiteness of the tensor $\mathbf{K}$ or about the symmetry or positivity of the entries of $\mathbf{K}$.

5. Linear, nonlocal boundary-value problems. With s given by (4.1) and $U(\Omega\cup\Gamma) = V(\Omega\cup\Gamma)$, the variational principle (3.2) reduces to

$$\begin{cases} \textit{seek } u\in V(\Omega\cup\Gamma) \textit{ such that} \\ \qquad u = h_e \quad \text{for } \mathbf{x}\in\Gamma_e \\ \textit{and} \\ \displaystyle\int_{\Omega\cup\Gamma}\int_{\Omega\cup\Gamma}\beta\mathcal{G}(v)\mathcal{G}(u)\,d\mathbf{x}'d\mathbf{x} = \int_\Omega vb\,d\mathbf{x} + \int_{\Gamma_n} vh_n\,d\mathbf{x} \quad \forall\, v\in V_0(\Omega\cup\Gamma) \end{cases} \tag{5.1}$$

and the corresponding nonlocal "boundary-value" problem (3.3) reduces to the linear problem

$$\begin{cases} -\mathcal{D}\big(\beta\mathcal{G}(u)\big) = b & \text{for } \mathbf{x}\in\Omega \\ u = h_e & \text{for } \mathbf{x}\in\Gamma_e \\ \mathcal{N}(\beta\mathcal{G}(u)) = h_n & \text{for } \mathbf{x}\in\Gamma_n, \end{cases} \tag{5.2}$$

where again the second equation is a "Dirichlet boundary" condition that is *essential* for the variational principle (5.1) and the third equation is a "Neumann boundary" condition that is *natural* for that principle.

[10] β plays a role analogous to the diffusion tensor $\mathbf{D}$ for the classical equation (1.1); in fact, in Section 8, we see how they are related.

Substituting the definitions (2.4) and (2.13) for $\mathcal{D}$ and $\mathcal{G}$, respectively, we have the explicit relations

$$\begin{aligned}\int_{\Omega\cup\Gamma}\int_{\Omega\cup\Gamma}\beta\mathcal{G}(v)\mathcal{G}(u)\,d\mathbf{x}'d\mathbf{x} &= \int_{\Omega\cup\Gamma}\int_{\Omega\cup\Gamma}(v'-v)(u'-u)\alpha^2\beta\,d\mathbf{x}'d\mathbf{x}\\ \mathcal{D}\big(\beta\mathcal{G}(u)\big) &= 2\int_{\Omega\cup\Gamma}(u'-u)\alpha^2\beta\,d\mathbf{x}' \qquad \text{for } \mathbf{x}\in\Omega\\ \mathcal{N}\big(\beta\mathcal{G}(u)\big) &= -2\int_{\Omega\cup\Gamma}(u'-u)\alpha^2\beta\,d\mathbf{x}' \qquad \text{for } \mathbf{x}\in\Gamma_n.\end{aligned} \tag{5.3}$$

6. Well posedness of linear nonlocal boundary value problems. We now demonstrate that the variational problem (5.1) is well posed. We assume that $U(\Omega\cup\Gamma) = V(\Omega\cup\Gamma)$. See [4] for a well-posedness result when $U(\Omega\cup\Gamma) = H_0^s(\Omega)$, $1/2 < s < 1$, the two papers [7, 8] for results on the linear peridynamic model for which u in (1.2) is a vector field, and [1, 2] for results about the strong form of the nonlocal boundary value problem (5.2).

6.1. Bilinear forms, norms, and inner products. We now assume that the constitutive function β is positive, i.e., $\beta(\mathbf{x},\mathbf{x}') > 0$ for all $\mathbf{x},\mathbf{x}'\in\Omega\cup\Gamma$. For all $u,v\in V(\Omega\cup\Gamma)$, define the symmetric bilinear form

$$\begin{aligned}B(u,v) &:= \int_{\Omega\cup\Gamma}\int_{\Omega\cup\Gamma}\beta\mathcal{G}(v)\mathcal{G}(u)\,d\mathbf{x}'d\mathbf{x}\\ &= \int_{\Omega\cup\Gamma}\int_{\Omega\cup\Gamma}(v'-v)(u'-u)\beta\alpha^2\,d\mathbf{x}'d\mathbf{x}.\end{aligned} \tag{6.1}$$

Note that $B(u,u)\geq 0$ and let

$$((u,v)) := B(u,v) \qquad \text{and} \qquad |||u||| := \big(B(u,u)\big)^{1/2}.$$

Let the function space $V(\Omega\cup\Gamma)$ be defined by

$$V(\Omega\cup\Gamma) := \{u\ :\ |||u||| < \infty\}.$$

We now show that $|||\cdot|||$ and $((\cdot,\cdot))$ define a norm and an inner product, respectively, on both $V_0(\Omega\cup\Gamma)$ and $V(\Omega\cup\Gamma)\setminus\mathbb{R}$. Note that $|||\cdot|||$ only defines a semi-norm on $V(\Omega\cup\Gamma)$.

Let $u\in V_0(\Omega\cup\Gamma)$ so that $u(\mathbf{x}) = 0$ for all $\mathbf{x}\in\Gamma_e$. Then,

$$\begin{aligned}B(u,u) &= \int_{\Omega\cup\Gamma}\int_{\Omega\cup\Gamma}(u'-u)^2\beta\alpha^2\,d\mathbf{x}'d\mathbf{x}\\ &= \int_{\Omega\cup\Gamma}\int_{\Gamma_e}(u'-u)^2\beta\alpha^2\,d\mathbf{x}'d\mathbf{x} + \int_{\Omega\cup\Gamma}\int_{\Omega\cup\Gamma_n}(u'-u)^2\beta\alpha^2\,d\mathbf{x}'d\mathbf{x}\\ &\geq \int_{\Omega\cup\Gamma}\int_{\Gamma_e}(u'-u)^2\beta\alpha^2\,d\mathbf{x}'d\mathbf{x} = \int_{\Omega\cup\Gamma}u^2\int_{\Gamma_e}\beta\alpha^2\,d\mathbf{x}'d\mathbf{x}\\ &= \int_{\Omega\cup\Gamma_n}u^2\left(\int_{\Gamma_e}\beta\alpha^2\,d\mathbf{x}'\right)d\mathbf{x}\,.\end{aligned}$$

Assuming that $0 < \int_{\Gamma_e}\beta\alpha^2\,d\mathbf{x}' < \infty$ for all $\mathbf{x}\in\Omega\cup\Gamma_n$, we have that

$$B(u,u) = 0 \quad \text{implies that} \quad u = 0 \quad \forall\,\mathbf{x}\in\Omega\cup\Gamma_n.$$

But, $u = 0$ in Γ_e as well so that we have that

$$B(u,u) = 0 \quad \text{implies that} \quad u = 0 \quad \forall\, \mathbf{x} \in \Omega \cup \Gamma.$$

Thus, we have that $|||\cdot|||$ defines a norm and $((\cdot,\cdot))$ defines and inner product on $V(\Omega\cup\Gamma)$.

Also, note that $B(u,u) = 0$ only if $(u'-u)^2\beta\alpha^2 = 0$ for all $\mathbf{x},\mathbf{x}' \in \Omega\cup\Gamma$, i.e., only if $u =$ constant for all $\mathbf{x} \in \Omega\cup\Gamma$. Thus, we again conclude that $|||\cdot|||$ defines a norm and $((\cdot,\cdot))$ defines and inner product on $V(\Omega\cup\Gamma)\setminus\mathbb{R}$.

6.2. Well-posedness of variational problems. Let

$$|||b|||_* := \sup_{v\in V_0(\Omega\cup\Gamma),\, v\neq 0} \frac{\displaystyle\int_\Omega v(\mathbf{x})b(\mathbf{x})\,d\mathbf{x}}{|||v|||}$$

and define the "dual" space

$$V_0^* := \{b \;\;:\;\; |||b|||_* < \infty\}.$$

Next, define the "trace" space

$$V_e := \{\chi_{\Gamma_e} u \;\;:\;\; u \in V(\Omega\cup\Gamma)\},$$

where $\chi_{(\cdot)}$ denotes the characteristic function, along with the norm

$$|||u|||_e := |||\chi_{\Gamma_e} u|||.$$

Finally, define the norm

$$|||h|||_n := \sup_{v\in V_0(\Omega\cup\Gamma),\, v\neq 0} \frac{\displaystyle\int_{\Gamma_n} v(\mathbf{x})h(\mathbf{x})\,d\mathbf{x}}{|||v|||}$$

and the second "trace" space

$$V_n := \{h \;\;:\;\; |||h|||_n < \infty\}.$$

The variational problem (5.1) then takes the form[11]

$$\begin{cases} \textit{given } b \in V_0^*,\ h_e \in V_e, \textit{ and } h_n \in V_n, \textit{ seek } u \in V(\Omega\cup\Gamma) \textit{ such that} \\ \qquad u = h_e \quad \text{for } \mathbf{x} \in \Gamma_e \\ \textit{and} \\ \qquad B(u,v) = F(v) \qquad \forall\, v \in V_0(\Omega\cup\Gamma), \end{cases} \tag{6.2}$$

where the linear functional $F(\cdot)$ is defined by

$$F_n(v) := \int_\Omega vb\,d\mathbf{x} + \int_{\Gamma_n} vh_n\,d\mathbf{x} \qquad \forall\, v \in V_0(\Omega\cup\Gamma). \tag{6.3}$$

[11]Note that the solution of the variational "boundary-value" problem (6.2) corresponds to the solution of the optimization problem

$$\arg\min_{\{v\in V(\Omega\cup\Gamma),\, v\,=\,h_e \text{ for } \mathbf{x}\in\Gamma_e\}} \Big(\frac{1}{2}B(v,v) - F(v)\Big).$$

If Γ_e is empty, we replace the space $V_0(\Omega\cup\Gamma)$ in (6.2) and (6.3) by $V(\Omega\cup\Gamma)\setminus\mathbb{R}$.

Consider the homogeneous essential boundary condition case $h_e=0$. Because $B(\cdot,\cdot)$ defines an inner product on $V_0(\Omega\cup\Gamma)$ or $V(\Omega\cup\Gamma)\setminus\mathbb{R}$, it is continuous and coercive on those spaces. Then, if we assume that the data are such that the functional $F(\cdot)$ is continuous, the Lax-Milgram theorem can be applied to show that (6.2) has a unique solution and, moreover, that solution satisfies

$$|||u|||\le |||b|||_*+|||h_n|||_n.$$

The case $h_e\ne 0$ can be treated in a similar manner after rendering the essential boundary condition in (6.2) homogeneous by subtracting from u a particular solution $\widetilde{u}$ satisfying $\widetilde{u}=h_e$ for $\mathbf{x}\in\Gamma_e$.

6.3. Decomposition of the solution space. Let the space $S(\Omega\cup\Gamma)$ consist of functions $u\in V(\Omega\cup\Gamma)$ that satisfy

$$\begin{cases} \mathcal{D}\big(\beta\mathcal{G}(u)\big)=2\displaystyle\int_{\Omega\cup\Gamma}(u'-u)\beta\alpha^2\,d\mathbf{x}'=0 & \forall\,\mathbf{x}\in\Omega\\ \mathcal{N}\big(\beta\mathcal{G}(u)\big)=-2\displaystyle\int_{\Omega\cup\Gamma}(u'-u)\beta\alpha^2\,d\mathbf{x}'=0 & \forall\,\mathbf{x}\in\Gamma. \end{cases} \tag{6.4}$$

Then, from (4.2), we have that, for all $u\in S(\Omega\cup\Gamma)$ and $v\in V_0(\Omega\cup\Gamma)$,

$$((u,v))=\int_{\Omega\cup\Gamma}\int_{\Omega\cup\Gamma}\mathcal{G}(v)\mathcal{G}(u)\beta\,d\mathbf{x}'d\mathbf{x}=0.$$

Thus, we conclude that

$$V(\Omega\cup\Gamma)=V_0(\Omega\cup\Gamma)\oplus S(\Omega\cup\Gamma), \tag{6.5}$$

i.e., any function in $V(\Omega\cup\Gamma)$ can be written as a sum of two functions that are orthogonal with respect to the inner product $((\cdot,\cdot))$, the first a function that vanishes on Γ_e and the second a function satisfying (6.4).[12]

7. Nonlocal Green's functions and a nonlocal Green's third identity.

7.1. Nonlocal fundamental solutions. For each $\mathbf{y}\in\mathbb{R}^d$, let $g(\mathbf{x};\mathbf{y})$ denote the fundamental solution (or free-space Green's function) for the operator $\mathcal{D}(\beta\mathcal{G}(\cdot))$, formally defined as the solution of[13]

$$\mathcal{D}\big(\beta\mathcal{G}(g(\mathbf{x};\mathbf{y}))\big)=\delta(|\mathbf{x}-\mathbf{y}|)\qquad\forall\,\mathbf{x}\in\mathbb{R}^d,$$

where $\delta(\cdot)$ denotes the Dirac delta function. We assume that α and β are radial functions of $\mathbf{x}$ and $\mathbf{x}'$, e.g., $\alpha(\mathbf{x},\mathbf{x}')=\alpha(\mathbf{x}'-\mathbf{x})$, and that $g(\mathbf{x};\mathbf{y})=g(\mathbf{x}-\mathbf{y})$. In this case, we can assume, without loss of generality, that $\mathbf{y}=\mathbf{0}$. Using (5.3), we then have

$$2\int_{\mathbb{R}^d}\big(g(\mathbf{x}')-g(\mathbf{x})\big)\mu(\mathbf{x}'-\mathbf{x})\,d\mathbf{x}'=\delta(|\mathbf{x}|)\qquad\mathbf{x}\in\mathbb{R}^d, \tag{7.1}$$

[12]If, in (6.4), we set $\beta=1$ and $\Gamma_e=\Gamma$ so that Γ_n is empty, then the space $S(\Omega\cup\Gamma)$ consists of "harmonic" functions; see (7.4). Then, the decomposition (6.5) is entirely analogous to the decomposition of the Sobolev space $H^1(\Omega)$ into functions belonging to $H^1_0(\Omega)$ and harmonic functions.

[13]In this section, we explicitly express the dependences of functions on $\mathbf{x}$ and $\mathbf{x}'$.

where $\mu = \alpha^2\beta$. Assuming, again without loss of generality, that the radial function μ satisfies $\int_{\mathbb{R}^d} \mu\, d\mathbf{x} = 1$, (7.1) can be expressed in the form

$$2\int_{\mathbb{R}^d} g(\mathbf{x}')\mu(\mathbf{x}'-\mathbf{x})\,d\mathbf{x}' - 2g(\mathbf{x}) = \delta(|\mathbf{x}|) \qquad \forall\,\mathbf{x}\in\mathbb{R}^d$$

so that

$$\widehat{g} = \frac{(2\pi)^{-d/2}}{2}\left(\frac{1}{(2\pi)^{d/2}\widehat{\mu}-1}\right),$$

where the Fourier transforms of g and μ are given by

$$\widehat{g}(\mathbf{k}) := (2\pi)^{-d/2}\int_{\mathbb{R}^d} e^{-i\mathbf{k}\cdot\mathbf{x}} g(\mathbf{x})\,d\mathbf{x} \qquad \text{and} \qquad \widehat{\mu}(\mathbf{k}) := (2\pi)^{-d/2}\int_{\mathbb{R}^d} e^{-i\mathbf{k}\cdot\mathbf{x}}\mu(\mathbf{x})\,d\mathbf{x},$$

respectively. Therefore,

$$g(\mathbf{x}) = \frac{(2\pi)^{-d}}{2}\int_{\mathbb{R}^d} e^{i\mathbf{k}\cdot\mathbf{x}}\frac{1}{(2\pi)^{d/2}\widehat{\mu}-1}\,d\mathbf{k}$$

so that, for general $\mathbf{x},\mathbf{y}\in\mathbb{R}^d$, the fundamental solution for (7.1) is given by

$$g(\mathbf{x};\mathbf{y}) = \frac{(2\pi)^{-d}}{2}\int_{\mathbb{R}^d} e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}\frac{1}{(2\pi)^{d/2}\widehat{\mu}-1}\,d\mathbf{k}.$$

Note that the special choice

$$\mu(\mathbf{x}'-\mathbf{x}) = \delta(\mathbf{x}'-\mathbf{x}) + \frac{d^2}{dx^2}\delta(\mathbf{x}'-\mathbf{x})$$

leads to the same fundamental solution as that for the Laplace operator, i.e.,

$$-\frac{(2\pi)^{-d}}{2}\int_{\mathbb{R}^d} e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}|\mathbf{k}|^{-2}\,d\mathbf{k} = \begin{cases} \dfrac{1}{2}|x-y|, & d=1,\\[2mm] \dfrac{1}{2\pi}\ln|\mathbf{x}-\mathbf{y}|, & d=2,\\[2mm] \dfrac{1}{2\omega_d}\dfrac{|\mathbf{x}-\mathbf{y}|^{2-d}}{2-d}, & d\geq 3,\end{cases}$$

where ω_d denotes the volume of the unit ball in $\mathbb{R}^d$.

7.2. Nonlocal Green's third identity. For any $\mathbf{y}\in\Omega\cup\Gamma$, let $G(\mathbf{x};\mathbf{y}):\Omega\cup\Gamma\to\mathbb{R}$ denote any function satifying[14]

$$\mathcal{D}(\beta\mathcal{G}(G(\mathbf{x};\mathbf{y}))) = \delta(|\mathbf{x}-\mathbf{y}|) \qquad \forall\,\mathbf{x}\in\Omega. \tag{7.2}$$

Then, using the nonlocal Green's second identity (4.3) with $v(\cdot) = G(\cdot;\mathbf{y})$, we obtain the *nonlocal Green's third identity*

$$\begin{aligned} u(\mathbf{y}) = &\int_\Omega G(\mathbf{x};\mathbf{y})\mathcal{D}\big(\beta\mathcal{G}(u(\mathbf{x}))\big)\,d\mathbf{x}\\ &-\int_\Gamma \Big(G(\mathbf{x};\mathbf{y})\mathcal{N}\big(\beta\mathcal{G}(u(\mathbf{x}))\big) - u(\mathbf{x})\mathcal{N}\big(\beta\mathcal{G}(G(\mathbf{x};\mathbf{y}))\big)\Big)\,d\mathbf{x} \qquad \forall\,\mathbf{y}\in\Omega.\end{aligned} \tag{7.3}$$

[14]Note that the fundamental solution satisfies this equation.

Suppose that the constitutive function $\beta = 1$ (see (4.4)) and

$$\mathcal{D}\big(\mathcal{G}(u)\big) = 2\int_{\Omega\cup\Gamma}(u'-u)\alpha^2\,d\mathbf{x}' = 0 \qquad \forall\,\mathbf{x}\in\Omega. \tag{7.4}$$

Then, the solution $u(\mathbf{x})$ represents a nonlocal "harmonic" function that, from (7.3), is given by

$$u(\mathbf{y}) = \int_{\Gamma}\Big(u(\mathbf{x})\mathcal{N}\big(\mathcal{G}(G(\mathbf{x};\mathbf{y}))\big) - G(\mathbf{x};\mathbf{y})\mathcal{N}\big(\mathcal{G}(u(\mathbf{x}))\big)\Big)\,d\mathbf{x},$$

i.e., "harmonic" functions are determined by their "boundary" values on Γ. Nonlocal versions of the Poisson integral formula and Gauss's law of the arithmetic mean can also be derived; see [12, Chap. 4] for the classical case.

7.3. Nonlocal Green's functions. Let $g(\mathbf{x};\mathbf{y})$ denote the fundamental solution defined in Section 7.1. For each $\mathbf{y}\in\Omega\cup\Gamma$, define the *nonlocal Green's function* $G(\mathbf{x};\mathbf{y}) : \Omega\cup\Gamma\to\mathbb{R}$ as

$$G(\mathbf{x};\mathbf{y}) = g(\mathbf{x};\mathbf{y}) - \widetilde{g}(\mathbf{x};\mathbf{y}),$$

where $\widetilde{g}(\cdot;\cdot)$ is a solution of

$$\begin{cases} \mathcal{D}\big(\beta\mathcal{G}(\widetilde{g})\big) = 0 & \text{for } \mathbf{x}\in\Omega \\ \mathcal{N}\big(\beta\mathcal{G}(\widetilde{g})\big) = \mathcal{N}\big(\beta\mathcal{G}(g)\big) & \text{for } \mathbf{x}\in\Gamma_n \\ \widetilde{g}(\mathbf{x};\mathbf{y}) = g(\mathbf{x};\mathbf{y}) & \text{for } \mathbf{x}\in\Gamma_e. \end{cases}$$

Then, $G(\cdot;\cdot)$ satisfies (7.2) and the homogeneous "boundary" conditions $G(\mathbf{x};\mathbf{y}) = 0$ for $\mathbf{x}\in\Gamma_e$ and $\mathcal{N}\big(\beta\mathcal{G}(G)\big) = 0$ for $\mathbf{x}\in\Gamma_n$. Applying (7.3), we then have that the solution of the "boundary-value" problem (5.2) is given by

$$\begin{aligned} u(\mathbf{y}) = &-\int_{\Omega} G(\mathbf{x};\mathbf{y})b(\mathbf{x})\,d\mathbf{x} \\ &+\int_{\Gamma_e} h_e(\mathbf{x})\mathcal{N}\big(\beta\mathcal{G}(G(\mathbf{x};\mathbf{y}))\big)\,d\mathbf{x} - \int_{\Gamma_n} G(\mathbf{x};\mathbf{y})h_n(\mathbf{x})\,d\mathbf{x} \qquad \forall\,\mathbf{y}\in\Omega. \end{aligned}$$

Because the operators $\mathcal{D}$ and $\mathcal{N}$ differ only in their signs and domains, it follows that this formula also holds for $\mathbf{y}\in\Gamma_n$.

8. Local smooth limits. We now connect the linear nonlocal "boundary-value" problem of Section 5 to variational formulations of the Dirichlet and Neumann problems for linear, second-order elliptic partial differential equations. To do so, we make two assumptions,[15] one about solutions and the other about the constitutive model, beyond some geometric assumptions and those made in Section 6 for the existence and uniqueness associated with the nonlocal "boundary-value" problem.[16] First, we

[15]We emphasize that these assumptions are made only to make the connection to classical problems for partial differential equations and are not required for the well posedness of the nonlocal "boundary-value" problems; see Section 6. In addition, the nonlocal "boundary-value" problems admit solutions that are not solutions, even in the usual sense of weak solutions, of the partial differential equations. Thus, one can view solutions of the nonlocal "boundary-value" problems as further generalizations of solutions of the partial differential equations, generalized in two ways: they are nonlocal and they lack the smoothness needed for them to be standard weak solutions.

[16]The formal presentation we make here may be made completely rigorous using the procedures and clever scalings introduced in [1, 2].

assume that *solutions of the nonlocal "boundary-value" problems are smooth;* specifically, we assume that[17]

$$u(\mathbf{x}') = u(\mathbf{x}) + \nabla u(\mathbf{x}) \cdot (\mathbf{x}' - \mathbf{x}) + o(\varepsilon) \qquad \text{if} \;\; |\mathbf{x}' - \mathbf{x}| \leq \varepsilon, \tag{8.1}$$

where, for $n \geq 0$, $\varepsilon^{-n} o(\varepsilon^n) \to 0$ as $\varepsilon \to 0$. Second, we assume that *the nonlocal operators are asymptotically local;* specifically, we assume that β is a positive, symmetric function[18] such that, for $\varepsilon > 0$,

$$\beta(\mathbf{x}, \mathbf{x}') = 0 \quad \text{whenever} \quad |\mathbf{x}' - \mathbf{x}| \geq \varepsilon \tag{8.2}$$

and, for some positive constants $\underline{\beta}$ and $\overline{\beta}$,[19]

$$\underline{\beta} < \int_{S_\varepsilon(\mathbf{x})} \beta(\mathbf{x}, \mathbf{x}')\, d\mathbf{x}' < \overline{\beta} \qquad \text{uniformly for } \mathbf{x} \in \Omega, \tag{8.3}$$

where $S_\varepsilon(\mathbf{x}) := \{\mathbf{x}' \in \mathbb{R}^d \mid |\mathbf{x}' - \mathbf{x}| < \varepsilon\}$.

As in [10], we set

$$\alpha(\mathbf{x}, \mathbf{x}') = \frac{1}{|\mathbf{x}' - \mathbf{x}|}. \tag{8.4}$$

Note that, from (8.1), we have that this choice implies that, as $\varepsilon \to 0$, $\mathcal{G}(u) = (u' - u)\alpha$ tends to the directional derivative of u in the direction $\mathbf{x}' - \mathbf{x}$.

The geometric assumptions we make are that,[20] if $\partial\Omega_e := \partial\Omega \cap \partial\Gamma_e$ and $\partial\Omega_n := \partial\Omega \cap \partial\Gamma_n$, then

$$\partial\Omega_e \neq \emptyset, \qquad \partial\Omega_n \neq \emptyset, \qquad \text{and} \qquad \partial\Omega = \partial\Omega_e \cup \partial\Omega_n.$$

Let

$$\Gamma(\varepsilon) := \cup_{\mathbf{x} \in \Omega} \mathrm{supp}(\beta) \setminus \Omega.$$

Then, with $|\Gamma|$ denoting the volume of Γ, we have from (8.2) that

$$|\Gamma(\varepsilon)| = O(\varepsilon) \qquad \text{and} \qquad \Gamma_e \to \partial\Omega_e \quad \text{and} \quad \Gamma_n \to \partial\Omega_n \quad \text{as} \quad \varepsilon \to 0. \tag{8.5}$$

Using (8.1) and (8.2), we now have that

$$\begin{aligned} B(u,v) &= \int_{\Omega\cup\Gamma(\varepsilon)} \int_{\Omega\cup\Gamma(\varepsilon)} \mathcal{G}(v)\mathcal{G}(u)\, \beta\, d\mathbf{x}' d\mathbf{x} \\ &= \int_{\Omega\cup\Gamma(\varepsilon)} \int_{S_\varepsilon(\mathbf{x})\cap(\Omega\cup\Gamma(\varepsilon))} \Big(\nabla v \cdot (\mathbf{x}' - \mathbf{x}) \nabla u \cdot (\mathbf{x}' - \mathbf{x}) + o(\varepsilon^2)\Big) \beta\alpha^2\, d\mathbf{x}' d\mathbf{x} \\ &= \int_{\Omega\cup\Gamma(\varepsilon)} \nabla v \cdot \Big(\mathbf{D}_\varepsilon + o(\varepsilon^0)\Big) \cdot \nabla u\, d\mathbf{x} \end{aligned}$$

[17]Actually, we need only assume that (8.1) holds weakly.

[18]The symmetry and positive definiteness of a tensor $\mathbf{K}$ having elements that are symmetric functions of $\mathbf{x}$ and $\mathbf{x}'$ are sufficient conditions for such a β.

[19]The upper bounds in (8.2) imply the scaling $\beta = \varepsilon^{-d}\widetilde{\beta}$, where, as $\varepsilon \to 0$, $\widetilde{\beta}$ is bounded from above and below uniformly in ε. This is precisely the scaling used in, e.g., [1, 2], to rigorously connect nonlocal diffusion equations to classical diffusion equations.

[20]These assumptions merely state that both Γ_e and Γ_n abut Ω and that the common boundaries of both Γ_e and Γ_n with Ω make up the whole boundary of Ω. Other that this section and Section 2.1, the results presented do not require such assumptions, i.e., neither Γ_e or Γ_n need abut Ω.

where, using (8.4) and (4.4),

$$\begin{aligned}\mathbf{D}_\varepsilon(\mathbf{x}) &= \int_{S_\varepsilon(\mathbf{x})\cap(\Omega\cup\Gamma(\varepsilon))} \frac{(\mathbf{x}'-\mathbf{x})\otimes(\mathbf{x}'-\mathbf{x})}{|\mathbf{x}'-\mathbf{x}|^2}\beta\, d\mathbf{x}' \\ &= \int_{S_\varepsilon(\mathbf{x})\cap(\Omega\cup\Gamma(\varepsilon))} \frac{(\mathbf{x}'-\mathbf{x})\otimes(\mathbf{x}'-\mathbf{x})\mathbf{K}(\mathbf{x}'-\mathbf{x})\otimes(\mathbf{x}'-\mathbf{x})}{|\mathbf{x}'-\mathbf{x}|^2}\, d\mathbf{x}'.\end{aligned}$$

From (8.3) and (8.5), we conclude that if

$$\mathbf{D}(\mathbf{x}) := \lim_{\varepsilon\to 0} \mathbf{D}_\varepsilon(\mathbf{x}), \tag{8.6}$$

then $\mathbf{D}$ exists and is a symmetric, positive definite, non-vanishing tensor and

$$\lim_{\varepsilon\to 0} B(u,v) = \int_\Omega \nabla v\cdot\big(\mathbf{D}\cdot\nabla u\big)\, d\mathbf{x}. \tag{8.7}$$

Using (8.5), we then see that, in the limit $\varepsilon\to 0$, the nonlocal variational problem (5.1) reduces to the local variational problem

$$\begin{cases} \displaystyle\int_\Omega \nabla v\cdot\big(\mathbf{D}\cdot\nabla u\big) = \int_\Omega vb\, d\mathbf{x} + \int_{\partial\Omega_n} v\widetilde{h}_n\, d\mathbf{x} & \text{in } \Omega, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad u = \widetilde{h}_e & \text{on } \partial\Omega_e, \end{cases}$$

where $\widetilde{h}_e$ and $\widetilde{h}_n$ denote traces of the nonlocal data h_e and h_n on $\partial\Omega_e$ and $\partial\Omega_n$, respectively. The corresponding "boundary-value" problem (5.2) reduces to

$$\begin{cases} -\nabla\cdot\big(\mathbf{D}\cdot\nabla u\big) = b & \text{in } \Omega, \\ u = \widetilde{h}_e & \text{on } \partial\Omega_e, \\ \big(\mathbf{D}\cdot\nabla u\big)\cdot\mathbf{n} = \widetilde{h}_n & \text{on } \partial\Omega_n. \end{cases}$$

9. Concluding remarks. We developed a nonlocal vector calculus that consists of a nonlocal Gauss's theorem and nonlocal Green's identities that mimic the corresponding theorem and identities of the classical vector calculus. We defined a nonlocal variational principle and used the nonlocal vector calculus to show that the principle corresponds to nonlocal "boundary-value" problems that mimic the classical Dirichlet and Neumann problems for second-order elliptic partial differential equations. In fact, we showed that, in an appropriate limit, the nonlocal variational principles and the nonlocal boundary-value problems reduce to their classical counterparts. We also derived fundamental solutions and showed how one can derive existence and uniqueness results for the nonlocal "boundary-value" problems.

The nonlocal variational problem (5.1) and the corresponding nonlocal "boundary-value" problem (5.2) mimic the variational setting described by (1.1) along with Dirichlet and Neumann boundary conditions. Nonlocal versions of more general second-order elliptic boundary value problems can also be defined; see [11]. For example, consider the nonlocal variational principle[21]

$$\left\{\begin{array}{l} \textit{seek } u \in V(\Omega\cup\Gamma) \textit{ such that} \\ \qquad u = h_e \quad \text{for } \mathbf{x}\in\Gamma \\ \textit{and} \\ \qquad \displaystyle\int_{\Omega\cup\Gamma}\int_{\Omega\cup\Gamma} \beta\mathcal{G}(v)\mathcal{G}(u)\,d\mathbf{x}'d\mathbf{x} + \int_{\Omega\cup\Gamma} v \int_{\Omega\cup\Gamma} \sigma\mathcal{G}(u)\,d\mathbf{x}'d\mathbf{x} \\ \qquad\quad \displaystyle + \int_{\Omega\cup\Gamma} v \int_{\Omega\cup\Gamma} \omega(u'+u)\,d\mathbf{x}'d\mathbf{x} = \int_{\Omega} vb\,d\mathbf{x} \qquad \forall\, v\in V_0(\Omega\cup\Gamma), \end{array}\right. \tag{9.1}$$

where $\sigma(\mathbf{x},\mathbf{x}')$ and $\omega(\mathbf{x},\mathbf{x}')$ are anti-symmetric and symmetric functions, respectively. The corresponding nonlocal "Dirichlet" boundary-value problems is given by

$$\left\{\begin{array}{ll} -\mathcal{D}(\beta\mathcal{G}(u)) + \sigma\mathcal{G}(u) + \omega(u'+u) = b & \text{for } \mathbf{x}\in\Omega \\ u = h_e & \text{for } \mathbf{x}\in\Gamma. \end{array}\right. \tag{9.2}$$

General problems may be defined by setting β as in (4.4) and setting $\sigma = \mathbf{a}\cdot(\mathbf{x}'-\mathbf{x})$, where $\mathbf{a}(\mathbf{x},\mathbf{x}')$ is a symmetric vector-valued function. We may then proceed as in Section 8 to show that, for smooth solutions u and for asymptotically local operators, (9.2) corresponds to the general linear convection-diffusion-reaction problem

$$-\nabla\cdot\big(\mathbf{D}\cdot\nabla u\big) + \mathbf{w}\cdot\nabla u + cu = b$$

along with a Dirichlet boundary condition, where $\mathbf{w}$ and c are related to σ and ω, respectively, through a limit process analogous to that relating $\mathbf{D}$ to β. Neumann "boundary-value" problems can be defined in a similar manner.

Current work focuses on further refining and extending the results of this paper. In particular, we are

- developing functional analytic characterizations of the solution, trace, and data spaces used in Section 6;
- developing the equivalent multidomain formulations for the linear boundary value problems introduced in Section 5;
- developing and analyzing finite element discretization methods, including discontinuous Galerkin methods, for nonlocal variational problems such as (5.1);
- extending the nonlocal vector calculus to vector-valued functions and developing nonlocal variational problems and their corresponding nonlocal "boundary-value" problems for vector-valued functions; of particular interest is the application of the nonlocal vector calculus to the peridynamic [16, 17] model for materials.

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