The Ginzburg-Landau Equations for Superconductivity with Random Fluctuations

Andrei Fursikov, Max Gunzburger, and Janet Peterson

Dedicated to the memory of Sergey L'vovich Sobolev,
one of the greatest mathematicians of the twentieth century

Abstract Thermal fluctuations and material inhomogeneities have a large effect on superconducting phenomena, possibly inducing transitions to the non-superconducting state. To gain a better understanding of these effects, the Ginzburg–Landau model is studied in situations for which the described physical processes are subject to uncertainty. An adequate description of such processes is possible with the help of stochastic partial differential equations. The boundary value problem of Neumann type for the stochastic Ginzburg–Landau equations with additive and multiplicative white noise is investigated. We use white noise with minimal restriction on its independence property. The existence and uniqueness of weak and strong statistical solutions are proved. Our approach is based on using difference schemes for the Ginzburg–Landau equations.

1 Introduction

This paper is dedicated to the memory of Sergey L'vovich Sobolev. His outstanding contributions to the theory for the equations of mathematical physics are extremely deep and influential. Indeed, since the 1960s, practically all investigations in the aforementioned field of mathematics use Sobolev

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Andrei Fursikov
Moscow State University, 119992 Moscow, Russia, e-mail: fursikov@mtu-net.ru

Max Gunzburger
School of Computational Sciences, Florida State University, Tallahassee FL 32306-4120, USA, e-mail: gunzburg@scs.fsu.edu

Janet Peterson
School of Computational Sciences, Florida State University, Tallahassee FL 32306-4120, USA, e-mail: peterson@scs.fsu.edu

spaces and, at the least, are thereby guided by Sobolev's ideas. The present paper, of course, is no exception to this common rule. Moreover, the use of Sobolev spaces in complicated functional constructions for stochastic partial differential equations is especially successful and effective. Note also that being the closest aide to I.V. Kurchatov in the realization of the nuclear project in the Soviet Union after 1943, S.L. Sobolev took part in the numerical solution of huge problems of mathematical physics. From that time on to the end of his life, he had an invariable interest in the discrete approximation of continuum objects, especially in cubature formulas. In the present paper, discrete approximations are not only used, they play a crucial role in obtaining the main results.

This paper is devoted to the mathematical study of a boundary value problem for the stochastic Ginzburg–Landau model of superconductivity; we hope it will promote a better understanding of the transitions that occur between the superconducting and nonsuperconducting states.

In 1908, Kamerlingh–Onnes discovered that when metals such as mercury, lead, and tin are cooled to an absolute temperature below some small but positive critical value, their electrical resistivity completely disappears. This was a great surprise since what was expected is that the resistivity of metals would smoothly tend to zero as the temperature also tended to zero. In addition to this *zero resistance* property, superconductors are characterized by the property of *perfect diamagnetism*. This phenomenon was discovered in 1933 by Meissner and Ochsenfeld and is also known as the Meissner effect. What they observed is that not only is a magnetic field excluded from a superconductor, i.e., if a magnetic field is applied to a superconducting material at a temperature below the critical temperature, it does not penetrate into the material, but also that a magnetic field is expelled from a superconductor, i.e., if a superconductor subject to a magnetic field is cooled through the critical temperature, the magnetic field is expelled from the material. One of the consequences of the Meissner effect is that superconductors cannot be "perfect conductors" which are idealized (and unattainable) materials that have zero resistivity and that can be described by the linear Maxwell equations of electromagnetism. For such materials the magnetic field would not be expelled from the material when it is cooled through the critical temperature.

Superconductivity was not adequately explained until, in 1957, Bardeen, Cooper, and Schrieffer (BCS) [1] published their landmark paper describing a microscopic theory of superconductivity. However, even earlier, several phenomenological *continuum* theories were proposed, most notably by Ginzburg and Landau [20] in 1950. The Ginzburg–Landau theory was itself based on a general theory, introduced by Landau in 1937, for second-order phase transitions in fluids. Ginzburg and Landau thought of conducting electrons as being a "fluid" that could appear in two phases, namely superconducting and normal (non-superconducting). Through a stroke of intuitive genius, Ginzburg and Landau added to the theory of phase transitions certain effects, motivated by quantum-mechanical considerations, to account for how the electron "fluid" motion is affected by the presence of magnetic fields. In 1959, Gor'kov [21] showed that, in an appropriate limit, the macroscopic Ginzburg–Landau theory can be derived from the microscopic BCS theory. Details about the Ginzburg–Landau model can be found in [7, 13, 12, 41], the last of which may also be consulted for details about the BCS model.

The dependent variables of the Ginzburg–Landau model are the complex-valued order parameter ψ and the vector-valued magnetic potential A. Physically interesting variables such as the density of superconducting electrons, the current, and the induced magnetic field can be easily deduced from ψ and A. The Ginzburg–Landau model itself can be expressed as a system of two coupled partial differential equations from which ψ and A can be determined. One of these equations is a vector-valued, nonlinear Maxwell equation that relates the supercurrent, i.e., the current that flows without resistance, to a nonlinear function of ψ, $\nabla\psi$, and A. The second equation is a complex-valued equation that relates spatial and temporal variations of ψ to a nonlinear potential energy term. After appropriate non-dimensionalizations, there are two non-dimensional parameters appearing in the differential equations. One is the ratio of the relaxation times of ψ and A, the other, known as the Ginzburg–Landau parameter, is the ratio of the characteristic lengths over which ψ and A vary. These two length scales are referred to as the coherence and penetration lengths respectively.

In this paper, we consider a simplified Ginzburg–Landau system for ψ in which A is assumed to be a given vector-valued field. There are two situations of paramount practical interest for which the use of this simplified Ginzburg–Landau system can be justified. First, for high values of the Ginzburg–Landau parameter, it can be shown [6, 12] that, to leading order, the magnetic field in a superconductor is simply that given by the linear Maxwell equations so that A may be determined from these equations. Thus, insofar as the other component equation of the Ginzburg–Landau model is concerned, A can be viewed as a given vector field. A similar uncoupling can be shown to occur for thin film samples [5] for all values of the Ginzburg–Landau parameter. Most superconductors of practical interest are characterized by "high" values of the Ginzburg–Landau parameter and superconducting films are of very substantial technological interest; the simplified Ginzburg–Landau system we study can be used to model both of these situations. Furthermore, in the more general case where one has to consider the fully coupled Ginzburg–Landau equations for ψ and A, random fluctuations enter into the system in very much the same way as they do for the simplified system, so much of what is learned about stochastic versions of the simplified system applies to stochastic versions of the full system.

The Ginzburg–Landau theory is applicable only to highly idealized physical contexts that do not take into account factors such as material inhomogeneities and thermal fluctuations due to molecular vibrations. Both these factors play a crucial role in practical superconductivity since the former enables large currents to flow through a superconductor without resistance while the latter can have the opposite effect, especially at temperatures close to critical transition temperature (see, for example, [30, 39]). In [22], it is shown that, within the Ginzburg–Landau framework, thermal fluctuations are properly modeled by an additive white noise term in the Ginzburg–Landau equation for ψ; the amplitude of the noise term grows as the temperature approaches the critical temperature. In [4, 30], it is shown that, again in the Ginzburg–Landau framework, material inhomogeneities can be correctly modeled through the coefficient of the linear (in ψ) term in the Ginzburg–Landau equation for ψ; random variations in the material properties can thus be modeled as random perturbations in this coefficient which results in a multiplicative white noise term in the Ginzburg–Landau equations. In this paper, we treat both the additive and multiplicative noise cases. Studies of the physics of superconductors in the presence of white noise perturbations can be found in [11, 15, 23, 35, 39, 42, 43]; computational studies of the Ginzburg–Landau equations with additive and multiplicative noise are given in [9, 10].

In this paper, we study the stochastic Ginzburg–Landau equation written in the following dimensionless form:

$$d\psi(t,x) + \big((i\nabla + A(x))^2\psi - \psi + |\psi|^2\psi\big)dt = \widehat{r}[\psi]dW, \quad t > 0,\ x \in G \subset \mathbb{R}^d\,, \tag{1.1}$$

where G is a bounded domain, $d = 2, 3$, and an explanation of the notation employed on the right-hand side of (1.1) is given below in (1.3) and (1.4). On the boundary ∂G of G, we set

$$\big(i\nabla + A(x)\big)\psi(t,x) \cdot n = 0, \quad t > 0,\ x \in \partial G\,, \tag{1.2}$$

where n denotes the unit outer normal vector to ∂G.

From the view of the general theory of dynamical systems, the superconducting state is a stable steady-state solution of (1.1) (with zero right-hand side). The disappearance of the superconducting state (when some parameter of the system changes) means that some other steady-state solution of (1.1) arises and becomes stable or either time-periodic or chaotic behavior is realized.

We emphasize that when the dynamical system became unstable, the classical derivation of the equation for the superconducting state, rigorously speaking, looses its correctness. Indeed, in that derivation, as well as in other derivations of such a kind, only the main “forces” controlling the situation are taken into account and all relatively small and unessential “forces” are omitted, implicitly assuming stability in the sense that small fluctuations of “forces” lead to small fluctuations of the state. In unstable situations, this argument is evidently incorrect. The alternative is to replace, in the unstable situation, all small and unessential “forces” by white noise forcing (additive white noise) or perhaps by white noise multiplied by a function proportional to the state (multiplicative white noise). The physical basis of this approach is that, since "values" of white noise at different times are statistically independent, white noise renders a "smoothing" influence on the dynamical system. In more rigorous terms, this means the addition of white noise to the right-hand side of (1.1) leads to the substitution of many steady-state solutions of (1.1) by the unique (ergodic) statistical steady-state solution of (1.1) that is stable, i.e., that satisfies the *mixing property.* We also note that, in stable situations, replacing unessential "forces" by additive (multiplicative) white noise means taking into account thermal (material inhomogenety) fluctuations, as was noted above.

Very important arguments that can be used to justify the physical adequateness of the aforementioned modeling of superconductivity effects with the help of the stochastic problem (1.1) and (1.2) are given by recent results about ergodicity for abstract dynamical systems, including the two-dimensional Navier–Stokes and Ginzburg–Landau equations with random kick forces or additive white noise. The first results in this direction were obtained in [14, 16, 29]. In these papers, ergodicity was proved in stable situations, i.e., when the corresponding dynamical system with random forces omitted is stable. In the case of an unstable dynamical system, ergodicity was established in [36, 37, 38]. A detailed exposition of this topic can be found in [28].

Taking into account all of the above discussion, the following plan for the mathematical investigation of the superconducting state and its possible disappearance in industrial conditions is possible.

- Proof of the existence and uniqueness of weak and strong solutions of the stochastic boundary value problem (1.1) and (1.2).
- Proof of the ergodicity property for the random dynamical system generated by (1.1) and (1.2).
- Investigation of the disappearance of the supercoducting state in terms of the ergodic measure P that corresponds to the stochastic problem (1.1) and (1.2).

This paper is devoted to the proof of the first of these assertions.

The list of investigations of stochastic parabolic partial differential equations is huge because equations of such type arise in many problems of mathematics, physics, biology, and other applications. Here, we cite only the earliest papers in this field and papers closely connected with our paper. Investigations of linear parabolic stochastic partial differential equations were begun in the middle of 1960s [8]. Nonlinear stochastic parabolic equations were studied in [2, 33] and the stochastic Navier–Stokes system was studied in [3, 44, 45]. The paper [27] and the book [34] contain many deep results on these topics as well as a detailed historical review. Lastly, we note the works [25, 32].

In this paper, we study the stochastic boundary value problem (1.1) and (1.2) for the Ginzburg–Landau equation. Note that the right-hand side in

(1.1) should be written in a more detailed way as follows:

$$\widehat{r}[\psi]dW = r\big(\mathrm{Re}\psi(t,x)\big)\, d\mathrm{Re}W(t,x) + ir\big(\mathrm{Im}\psi(t,x)\big)\, d\mathrm{Im}W(t,x), \qquad (1.3)$$

where $dW = dW(t,x)$ is a complex-valued white noise and, as usual, $\mathrm{Re}\, z$ and $\mathrm{Im}\, z$ denote the real and imaginary parts of a complex number z respectively. In addition, $r(\lambda)$, $\lambda \in \mathbb{R}$, is, roughly speaking,[1] the following function:

$$r(\lambda) = \max(\rho_1,\ \rho_2|\lambda|), \qquad \rho_1 > 0,\ \rho_2 \geqslant 0. \qquad (1.4)$$

In particular, when $\rho_2 = 0$, (1.3) reduces to complex-valued additive white noise. Note immediately that the main difficulties we are forced to overcome in this paper are connected with the case $\rho_2 > 0$ which results in some kind of multiplicative white noise. The form (1.3) of the random fluctuations for the Ginzburg–Landau equation is reasonable from our point of view when, describing Ginzburg–Landau flow in instable situation, one replaces all small and unessential "forces" by stochastically independent fluctuations, i.e., by white noise. Indeed, since by the definition of complex-valued white noise $dW(t,x)$, its real $(d\mathrm{Re}W(t,x))$ and imaginary $(d\mathrm{Im}W(t,x))$ parts are mutually independent white noises [19, Chapt. III, Sect. 1]), (1.3) gives the maximal independent form of multiplicative white noise.

In this paper, we provide a detailed exposition of the proof of the existence and uniqueness of weak and strong statistical solutions of the stochastic boundary value problem (1.1) and (1.2). The main feature of our exposition is that, to prove the existence of a weak solution, we use, instead of Galerkin approximations, *approximations by method of lines*, i.e., we introduce a finite difference approximation of the Ginzburg–Landau equation with respect to the spatial variables. Although the method of lines is more complicated in realization than Galerkin's method, it has one important advantage: method of lines approximations inherit the structure of the Ginzburg–Landau equation much better than do Galerkin ones and therefore we can obtain many estimates for method of line approximations that cannot be obtained for Galerkin approximations. All these estimates we essentially use in our proof in order to overcome difficulties arising mostly because of the multiplicative structure of white noise. Nevertheless, one important a priori estimate which can be derived (formally) for the Ginzburg–Landau equation we cannot yet derive for its method of lines approximation. That is why for the three-dimensional Ginzburg–Landau equation with multiplicative white noise, we have proved here only the existence of a weak solution. For the two-dimensional Ginzburg-landau equation with multiplicative white noise as well as for the two- and three-dimensional Ginzburg–Landau equation with additive white noise, we can prove the existence and uniqueness of both weak and strong solutions.

[1] In fact, $r(\lambda)$, is the function (1.4) smoothed at points of discontinuity of its derivative. See the exact definition given below in (3.19).

The structure of the paper can be deduced from its content as described above.

2 The Ginzburg–Landau Equation and Its Finite Difference Approximation

In this section, we formulate the boundary value problem for the (simplified) Ginzburg–Landau equations without fluctuations and define an approximation by the method of lines that will play an important role in our analysis.

2.1 Boundary value problem for the Ginzburg–Landau equation

Let $G \subset \mathbb{R}^d$, $d = 2, 3$, denote a bounded domain with C^∞-boundary ∂G, and let $Q_T = (0, T) \times G$ denote a space-time cylinder. In Q_T, we consider the Ginzburg–Landau equation for the complex-valued function $\psi(t, x)$, referred to as the order parameter,

$$\frac{\partial \psi}{\partial t} + \big(i\nabla + A\big)^2 \psi - \psi + |\psi|^2 \psi = 0 \quad \text{for } (t, x) \in Q_T \tag{2.1}$$

along with the boundary condition

$$(i\nabla + A)\psi \cdot n = 0 \quad \text{on } (0, T) \times \partial G \tag{2.2}$$

and the initial condition

$$\psi(0, x) = \psi_0(x) \quad \text{in } G\,, \tag{2.3}$$

where $\nabla = (\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_d})$ denotes the gradient operator and $A(x) = (A^1, \ldots, A^d)$, the magnetic potential, is a given real-valued vector field such that $\operatorname{div} A = \sum\limits_{j=1}^{d} \frac{\partial A^j}{\partial x_j} = 0$. Also, $n = (n_1, \ldots, n_d)$ denotes the unit outer normal vector to the boundary ∂G and $\psi_0(x)$ is a given initial condition. We have

$$\begin{aligned}(i\nabla + A)^2 \psi &= \big(i\nabla + A, i\nabla + A\big)\psi \\ &= \sum_{j=1}^{d} \Big(i\frac{\partial}{\partial x_j} + A^j(x)\Big)\Big(i\frac{\partial \psi(x)}{\partial x_j} + A^j(x)\psi(x)\Big)\,. \end{aligned} \tag{2.4}$$

We assume that $A(x) \in (C^2(\overline{G}))^d$ and, for any fixed time, $\psi(t,x) \in L^2(G)$.

We want to introduce function spaces within which it is natural to look for the solution of the problem (2.1)–(2.3). The Sobolev space of complex-valued functions defined in G and square integrable there together with their derivatives up to order k is denoted by $H^k(G)$, $k \in \mathbb{N}$. Here, $\mathbb{N}$ denotes the set of positive integers. In addition, we define the space

$$H^2_A(G) = \{\phi(x) \in H^2(G) \ : \ (i\nabla + A)\phi \cdot n = 0 \ \text{ on } \partial G\}\,. \tag{2.5}$$

The space of solutions of (2.1)–(2.3) is defined as follows:

$$\mathcal{Y} = \Big\{\psi(t,x) \in L^2\big(0,T; H^2_A(G)\big) \cap L^6(Q_T) \ : \ \frac{\partial \psi}{\partial t} \in L^2(Q_T)\Big\}\,. \tag{2.6}$$

We also study generalized solutions of the problem (2.1)–(2.3). To obtain a weak formulation, we multiply (2.1) by the complex conjugate of ϕ, denoted by $\overline{\phi}$, and integrate over Q_T. Using the boundary condition (2.2) and integration by parts, we obtain

$$\int\limits_{Q_T} \left[\frac{\partial \psi}{\partial t}\overline{\phi} + \big(i\nabla + A\big)\psi \cdot \overline{(i\nabla + A)\phi} - \psi\overline{\phi} + |\psi|^2\psi\overline{\phi}\right] dxdt = 0\,. \tag{2.7}$$

Here, we will not make more precise the function space used for generalized solutions, defined by (2.3) and (2.7) with arbitrary $\overline{\phi} \in L_2(0,T;H^1(G))$, of the problem (2.1)–(2.3) because just at this moment it is not necessary.

2.2 Approximation by the method of lines

The approximation of the solution of a partial differential equation by the method of lines means that we approximate the continuous space variables $x = (x_1, \ldots, x_d)$ by a discrete grid or mesh so that we approximate the partial differential equation problem by a system of ordinary differential equations. In our case, we use finite difference quotients to approximate spatial derivatives. We assume that the grid is uniform and the scale of the grid, $h > 0$, is a fixed, sufficiently small number. Let an arbitrary point on the grid be denoted by kh, where $k \in \mathbb{Z}^d$, $kh = (k_1h, \ldots, k_dh)$, and $\mathbb{Z}$ denotes the set of integers. Since $\psi(x)$ is a function of the continuous variable x, we let ψ_k, defined on the given grid, denote the approximation to ψ at the point kh.

We now define the corresponding discrete "derivatives" or difference quotients; we distinguish the discrete derivatives from the continuous derivatives $\frac{\partial}{\partial x_j}$ by using the notation $\partial_{j,h}$. Let δ_{jk} denote the Kronecker delta, and let $e_j = (\delta_{j1}, \ldots, \delta_{jd})$, $j = 1, \ldots, d$. We can approximate the derivative $\frac{\partial \psi}{\partial x_j}$ by the forward difference quotient $\partial^+_{j,h}\psi_k = \frac{1}{h}(\psi_{k+e_j} - \psi_k)$ or by the backward difference quotient $\partial^-_{j,h}\psi_k = \frac{1}{h}(\psi_k - \psi_{k-e_j})$. The discrete divergence operator $\mathrm{div}_h{}^{\pm}$, the discrete gradient operator $\nabla^{\pm}_h$, and the discrete Laplace operator $\Delta_h = \mathrm{div}^-_h \nabla^+_h$ are then defined in an obvious manner.

Analogous to (2.4), we define

$$(i\nabla_h + A_k)^2\psi_k = (i\nabla^-_h + A_k, i\nabla^+_h + A_k)\psi_k$$
$$= \sum_{j=1}^{d} \left(i\partial^-_{j,h} + A^j_k\right)\left(i\partial^+_{j,h}\psi_k + A^j_k\psi_k\right), \tag{2.8}$$

where $A_k = A(kh)$ and A^j_k denotes the jth component of the vector $A_k = (A^1(kh), \dots, A^d(kh))$.

We now approximate the domain G and its boundary ∂G.

Definition 2.1. The approximate boundary ∂G_h is the subset of the grid kh, $k \in \mathbb{Z}^d$, that consists of two parts $\partial G_h = \partial G^+_h \cup \partial G^-_h$, where

(i) ∂G^-_h is the set of points $kh \in G$ such that $(k + e_j)h \in \mathbb{R}^d \setminus G$ or $(k - e_j)h \in \mathbb{R}^d \setminus G$ for some $j = 1, \dots, d$

and

(ii) ∂G^+_h the set of points $kh \in \mathbb{R}^d \setminus G$ such that $(k+e_j)h \in G$ or $(k-e_j)h \in G$ for some $j = 1, \dots, d$.

Definition 2.2. The approximate domain G_h is the subset of points $kh \in G$, $k \in \mathbb{Z}^d$; we also set $G^0_h = G_h \setminus \partial G^-_h$.

We introduce the following subsets of the approximate boundary ∂G_h:

$$\begin{aligned} \partial G^+_h(-j) &= \{kh \in \partial G^+_h \,:\, (k + e_j)h \in \partial G^-_h\} \\ \partial G^+_h(+j) &= \{kh \in \partial G^+_h \,:\, (k - e_j)h \in \partial G^-_h\} \end{aligned} \quad \text{for } j = 1, \dots, d \tag{2.9}$$

and

$$\begin{aligned} \partial G^-_h(-j) &= \{kh \in \partial G^-_h \,:\, (k + e_j)h \in \partial G^+_h\} \\ \partial G^-_h(+j) &= \{kh \in \partial G^-_h \,:\, (k - e_j)h \in \partial G^+_h\} \end{aligned} \quad \text{for } j = 1, \dots, d. \tag{2.10}$$

The sets $\partial G^+_h(\pm j)$ and $\partial G^-_h(\pm j)$ are illustrated in Fig. 2.1 for a domain in $\mathbb{R}^2$. In addition, we note that the sets $\partial G^-_h(\pm j)$, $j = 1, \dots, d$, can possess nontrivial pairwise intersections.

We now turn to the approximation of the boundary value problem (2.1)–(2.3) by the method of lines. We have

$$\frac{\partial \psi_k}{\partial t} + (i\nabla_h + A_k)^2\psi_k - \psi_k + |\psi_k|^2\psi_k = 0 \quad \text{for } kh \in G_h \tag{2.11}$$

and

$$\psi_k\big|_{t=0} = \psi_{0,k} \quad \text{for } kh \in G_h, \tag{2.12}$$

where the notation $(i\nabla_h + A_k)^2$ is defined by (2.8).

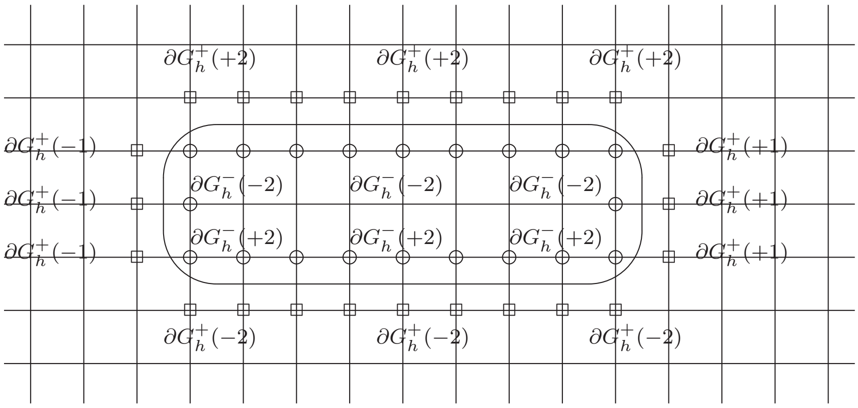

Fig. 2.1 The approximate boundary ∂G_h^+ is denoted by squares, and ∂G_h^- is denoted by circles.

In order to define the analogue of the boundary condition (2.2), we first note that the key property of this condition is that it implies the following formula for integration by parts:

$$\int\limits_G (i\nabla + A)^2 \psi(x)\overline{\phi(x)}\, dx = \int\limits_G (i\nabla + A)\psi(x)\overline{(i\nabla + A)\phi(x)}\, dx \qquad \forall\, \psi \in \mathcal{H}_A^2(G), \phi \in \mathcal{H}^1(G)\,. \tag{2.13}$$

Using (2.13), one can define a weak solution of our problem (2.1)–(2.3) with the aid of (2.7). To define the weak solution for the system (2.11) and (2.12), we need the following discrete analogue of (2.13):

$$h^d \sum_{kh\in G_h} \left(i\nabla_h^- + A_k, i\nabla_h^+ + A_k\right) \psi_k \overline{\phi_k} = h^d \sum_{j=1}^{d} \sum_{kh\in G_h\cup\partial G_h^+(-j)} (i\partial_{j,h}^+\psi_k + A_k^j\psi_k)\overline{(i\partial_{j,h}^+\phi_k + A_k^j\phi_k)}\,. \tag{2.14}$$

We take this formula, which will be proved in the next subsection, as the foundation for the definition of the discrete analogue of the boundary condition (2.2).

2.2.1 Summation by parts formula

In this section, our goal is to prove the discrete analogue of (2.13) given by (2.14).

Lemma 2.3. *Let the discrete functions ϕ_k and ψ_k be defined for $kh \in G_h \cup \partial G_h^+$. Assume that for each function ϕ_k*

$$\sum_{j=1}^{d}\Big(\sum_{kh\in\partial G_h^+(-j)}(iV_k^j + hV_k^j A_k^j)\overline{\phi}_k - \sum_{kh\in\partial G_h^+(+j)} iV_{k-e_j}^j\overline{\phi}_k\Big) = 0\,, \qquad (2.15)$$

where

$$V_k^j = i\frac{\psi_{k+e_j}-\psi_k}{h} + A_k^j\psi_k\,. \qquad (2.16)$$

Then (2.14) *holds.*

Proof. Using (2.16) and setting $r = k - e_j$, we obtain

$$h^d\sum_{kh\in G_h}(i\nabla_h^- + A_k, i\nabla_h^+ + A_k)\psi_k\overline{\phi}_k$$

$$= h^d\sum_{kh\in G_h}\left(\sum_{j=1}^{d} i\frac{V_k^j - V_{k-e_j}^j}{h} + A_k^j V_k^j\right)\overline{\phi}_k$$

$$= h^d\sum_{j=1}^{d}\Bigg[\sum_{kh\in\partial G_h^-(+j)}\frac{(-i)}{h}V_{k-e_j}^j\overline{\phi}_k + \sum_{kh\in\partial G_h^-(-j)}\left(\frac{i}{h}V_k^j + A_k^j V_k^j\right)\overline{\phi}_k$$

$$+ \sum_{rh\in G_h\setminus\partial G_h^-(-j)}\frac{-i}{h}V_r^j\overline{\phi}_{r+e_j} + \sum_{kh\in G_h\setminus\partial G_h^-(-j)}\left(\frac{i}{h}V_k^j + A_k^j V_k^j\right)\overline{\phi}_k\Bigg]$$

$$= h^{d-1}\sum_{j=1}^{d}\Bigg[\sum_{kh\in\partial G_h^-(-j)}\left(iV_k^j + hA_k^j V_k^j\right)\overline{\phi}_k - \sum_{rh\in\partial G_h^+(-j)} iV_r^j\overline{\phi}_{r+e_j}\Bigg]$$

$$+h^d\sum_{j=1}^{d}\sum_{kh\in G_h\setminus\partial G_h^-(-j)} V_k^j\left(\overline{i\frac{\phi_{k+e_j}-\phi_k}{h} + A_k^j\phi_k}\right)\,.$$

We add to the right-hand side of this relation the left-hand side of (2.15), where in the second sum we use the change of variables $r = k - e_j$. After performing this substitution, we arrive at (2.14). □

Thus, the relation (2.15) contains the boundary conditions we need. We only need to write these conditions in a more convenient form.

2.2.2 Boundary conditions for the system (2.11)

Since $G \subset \mathbb{R}^d$, $d = 2, 3$, is a bounded domain with C^∞-boundary ∂G, at each point $x \in \partial G$ the main curvatures of the surface ∂G are well defined. When $d = 2$, the curve ∂G has at $x \in \partial G$ one main curvature (usually called the curvature). We denote the modulus of this curvature as $\kappa(x)$. When $d = 3$, we denote by $\kappa(x) = \max\{|\kappa_1(x)|, |\kappa_2(x)|\}$, where $\kappa_j(x)$, $j = 1, 2$, are the main curvatures of ∂G at the point x. We set

$$\widehat{\kappa} = \max_{x \in \partial G} \kappa(x) .$$

We take a ball of radius $r < 1/\widehat{\kappa}$ and touch this ball at any $x \in \partial G$ from each of the two sides of the surface ∂G. Decreasing the radius r, we can position this ball so that it will not intersect ∂G at any point other than the point x of contact of the ball and ∂G. We denote such a radius $r(x)$ by $r_0(x)$, set $r_0 = \min_{x \in \partial G} r_0(x)$, and assume that

$$h < \frac{r_0}{10} . \tag{2.17}$$

Let $kh \in \partial G_h^+$. The point $\ell h \in \partial G_h^-$ is called the closest to kh if $\mathrm{dist}(\ell h, kh) = h$. The following lemma holds.

Lemma 2.4. *Let $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, be a bounded domain with C^∞-boundary ∂G, and let h satisfy* (2.17)*. Then, if $d = 2$, each point $kh \in \partial G_h^+$ has one or two* (*not more*) *closest points $\ell h \in \partial G_h^-$* (*as illustrated in Fig.* 2.2.) *If $d = 3$, each point $kh \in \partial G_h^+$ has one, two, or three closest points $\ell h \in \partial G_h^-$.*

We have to make more precise what is needed to ensure that the relation (2.14) is valid for every $\{\phi_k, kh \in G_h \cup \partial G_h^+\}$. For this relation, (2.15) should be true for every $\{\phi_k, kh \in \partial G_h^+\}$, i.e., for each $kh \in \partial G_h^+$ the coefficient before $\overline{\phi}_k$ in (2.15) should be equal to zero. First, we calculate these coefficients and write down the boundary conditions for the $d = 2$ case.

(i) *$kh \in \partial G_h^+$ possesses only one closest point from ∂G_h^-.* Then either $kh \in \partial G_h^+(+j)$ or $kh \in \partial G_h^+(-j)$ for some j. In the first case, $V^j_{k-e_j} = 0$ and, by virtue of (2.16),

$$\psi_k = \psi_{k-e_j}(1 + ihA^j_{k-e_j}), \quad kh \in \partial G_h^+(+j) \quad \text{with } j = 1, 2 . \tag{2.18}$$

In the second case, $V^j_k(i + hA^j_k) = 0$ and therefore

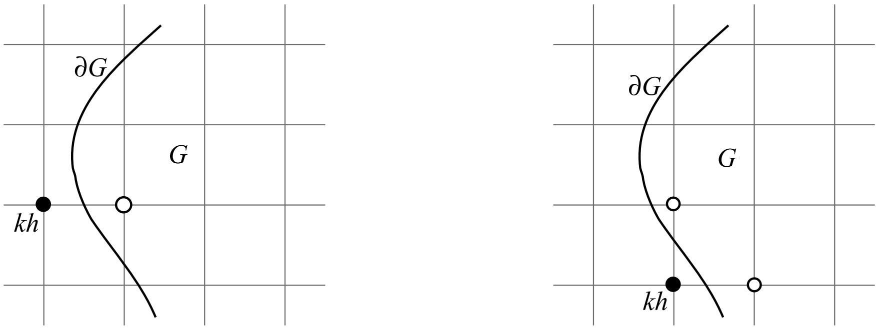

Fig. 2.2 In the figure on the left, the point $kh \in \partial G_h^+$ (filled circle) has one closest point $\ell h \in \partial G_h^-$ (open circle). In the figure on the right, the point $kh \in \partial G_h^+$ has two closest points $\ell h \in \partial G_h^-$.

$$\psi_k = \frac{\psi_{k+e_j}}{1 + ihA_k^j}, \quad kh \in \partial G_h^+(-j) \quad \text{with } j = 1, 2\,. \tag{2.19}$$

(ii) $kh \in \partial G_h^+$ *possesses two closest points from* ∂G_h^-. Then three different cases are possible.

(1) $kh \in \partial G_h^+(+1) \cap \partial G_h^+(+2)$. In this case, $V_{k-e_1}^1 + V_{k-e_2}^2 = 0$ and, by (2.16),

$$2\psi_k = \left(1 + ihA_{k-e_1}^1\right) \psi_{k-e_1} + (1 + ihA_{k-e_2}^2)\psi_{k-e_2}\,. \tag{2.20}$$

(2) $kh \in \partial G_h^+(-1) \cap \partial G_h^+(-2)$. In this case, $V_k^1(i + hA_k^1) + V_k^2(i + hA_k^2) = 0$ and

$$\psi_k = \frac{\psi_{k+e_1}(1 - ihA_k^1) + \psi_{k+e_2}(1 - ihA_k^2)}{2 + h^2\left((A_k^1)^2 + (A_k^2)^2\right)}\,. \tag{2.21}$$

(3) $kh \in \partial G_h^+(-j) \cap \partial G_h^+(+\ell)$ for $1 \leqslant j, \ell \leqslant 2$, $\ell \neq j$. In this case, $V_k^j(1 - ihA_k^j) - V_{k-e_\ell}^\ell = 0$ and

$$\psi_k = \frac{\psi_{k+e_j}(1 - ihA_k^j) + \psi_{k-e_\ell}(1 + ihA_{k-e_j}^\ell)}{2 + h^2(A_k^j)^2}\,. \tag{2.22}$$

For the $d = 3$ case, the derivation of the boundary conditions is absolutely the same, but the number of distinct cases is larger. Note that, for our purposes, we need only two things from the boundary conditions. First, that the formula (2.14) holds and second, that for each $kh \in \partial G_h^+$, ψ_k is expressed in terms of ψ_ℓ, $\ell h \in \partial G_h^-$. That is why it is quite enough for us to write down boundary conditions for both the $d = 2$ and $d = 3$ cases as follows. We have

$$\psi_k = \sum_{j=1}^{3} \left(a_{k,j}^{+} \psi_{k+e_j} + a_{k,j}^{-} \psi_{k-e_j} \right) \quad \forall\, kh \in \partial G_h^{+}\,, \tag{2.23}$$

where $a_{k,j}^{\pm}$ are certain coefficients (that can be written down explicitly) such that if $a_{k,j}^{+} \neq 0$, ($a_{k,j}^{-} \neq 0$) then $h(k+e_j) \in \partial G_h^{-}$ (correspondingly $h(k-e_j) \in \partial G_h^{-}$). Moreover, $0 < \sum_{j=1}^{3} |a_{k,j}^{+}|^2 + |a_{k,j}^{-}|^2 < c$, where c does not depend on h.

3 The stochastic Ginzburg–Landau Equation

In this section, we provide the formal definition of the Wiener process, the Wiener measure, and some related concepts. Then these results are used in the formulation of the stochastic problem for the Ginzburg–Landau equation.

3.1 Wiener process

We have an abstract probability space $(\Omega, \Sigma, m(d\omega))$, where Ω is the set of elementary events; Σ is a σ-algebra of subsets of Ω (if Ω is a metric space, Σ is a Borel σ-algebra, i.e., $\Sigma = \mathcal{B}(\Omega)$ is the σ-algebra generated by all open subsets of Ω); and $m(d\omega)$ is a probability measure defined on Σ. Recall that a set A is of m-measure zero if there exists $B \in \Sigma$ such that $m(B) = 0$ and $A \subset B$. The σ-algebra Σ^m is called the completion of Σ with respect to m if Σ^m is the family of all subsets of the form $A \cup B$, where A is of m-measure zero and $B \in \Sigma$. In the sequel, we change Σ on Σ^m, i.e., we will consider the σ-algebra Σ that is complete with respect to m.

Let

$$W : \Omega \to C(0,\infty; L^2(G)) \equiv \mathcal{C}$$

be a measurable mapping, i.e., for all $B \in \mathcal{B}(\mathcal{C})$, $\{\omega : W(\cdot,\cdot,\omega) \in B\} \in \Sigma$. The probability distribution of W is the measure Λ defined on $\mathcal{B}(\mathcal{C})$ by the formula

$$\Lambda(B) \equiv m\big(\{\omega \in \Omega : W(\cdot,\cdot,\omega) \in B\}\big) \quad \forall\, B \in \mathcal{B}(\mathcal{C})\,. \tag{3.1}$$

$W(t,x,\omega)$ is called a Wiener process if $\Lambda(B)$ is a Wiener measure. In the following definition, we assume that $\mathcal{C}$ consists of real-valued functions.

Definition 3.1. $\Lambda(B)$ for $B \in \mathcal{B}(\mathcal{C})$ is called a Wiener measure if its Fourier transform $\widetilde{\Lambda}$ is of the form

$$\widetilde{\Lambda}(v) = \int e^{i[w,v]} \Lambda(dW) = e^{-\frac{1}{2}B(v,v)} \quad \forall\, v \in \mathcal{C}_0^\infty \equiv C_0^\infty((0,\infty)\times G)\,, \tag{3.2}$$

where

$$[w, v] = \int_0^\infty \int_G w(t, x) v(t, x)\, dxdt\,. \tag{3.3}$$

Here, $B(v, v)$ is the quadratic form

$$B(v, v) = \int_0^\infty \int_0^\infty t \wedge s \Big\langle \mathcal{K}\big(v(t, \cdot), v(s, \cdot)\big) \Big\rangle\, dtds\,, \tag{3.4}$$

where $t \wedge s = \min(t, s)$ and $\langle f, g \rangle = \int_G f(x) g(x)\, dx$. Here, $\mathcal{K}$ is a self-adjoint, nonnegative trace class operator in $L^2(G)$ called the correlation operator of Λ; we have

$$\mathcal{K}^* = \mathcal{K} \geqslant 0, \quad S = S_P \mathcal{K} = \sum_{j=1}^{\infty} \lambda_j < \infty \quad (S_P \text{ is the spur-trace})\,, \tag{3.5}$$

where $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_k \geqslant \cdots \geqslant 0$ is the set of all eigenvalues of the operator $\mathcal{K}$.

Evidently, (3.1)–(3.4) imply that

$$\int W(t, x, \omega) W(s, y, \omega) m(d\omega) = t \wedge s\, \mathcal{K}(x, y)\,, \tag{3.6}$$

where $W(t, x, \omega)$ is a Wiener process and $\mathcal{K}(x, y)$ is the kernel of the operator $\mathcal{K}$ from (3.4) and (3.5).

Lemma 3.2. *The following conditions hold.*

1. *For any operator* $\mathcal{K} : L^2(G) \to L^2(G)$ *satisfying* (3.5) *there exists a unique Wiener measure* Λ *on* $\mathcal{C}$ *with the correlation operator* $\mathcal{K}$.

2. *For any* $\phi, \psi \in L^2(G)$

$$\int_{\mathbb{C}} \langle W(t, \cdot)\phi(\cdot)\rangle \langle W(s, \cdot)\psi(\cdot)\rangle \Lambda(dW) = t \wedge s \langle \mathcal{K}\phi, \psi\rangle. \tag{3.7}$$

3. *Let* $S = S_P \mathcal{K}$ *be defined by* (3.5)*. Then*

$$\int \|W(t, \cdot)\|^2_{L^2(G)} \Lambda(dW) = tS \quad \forall\, t > 0\,. \tag{3.8}$$

4. $W(t, x, \omega)$ *is a process with independent increments, i.e., for any* $0 \leqslant \tau \leqslant s \leqslant t$,

$$\Lambda(\{W : W(t,\cdot) - W(s,\cdot) \in B_1, W(\tau,\cdot) \in B_2\})$$
$$= \Lambda(\{W : W(t,\cdot) - W(s,\cdot) \in B_1\})\Lambda(\{W : W(\tau,\cdot) \in B_2\}) \quad \forall\, B_1, B_2 \in \mathcal{B}(L^2(G))\,. \tag{3.9}$$

For the proof, see [18].

Recall that, given a Wiener measure $\Lambda(B)$, $B \in \mathcal{B}(\mathcal{C})$, one can easily construct a Wiener process for which $\Lambda(B)$ is a probability distribution. Indeed, we take the probability space $(\Omega, \Sigma, m(dW)) = (\mathcal{C}, \mathcal{B}(\mathbb{C}), \Lambda(dW))$ and define a Wiener process $W(t,x,\omega)$ as follows: for each $W \in \mathcal{C}$, $W(t,x,\omega) = W(t,x)$. Clearly, this map $W(t,x,\omega)$ satisfies the definition of a Wiener process.

Below we use Wiener processes $W(t,x,\omega)$ defined on the space $\mathbb{C} = \mathcal{C} + i\mathcal{C}$ of complex valued functions, where recall that $\mathcal{C} = C(0,\infty; L^2(G))$. Taking into account [19, Chapt. III, Sect. 1], we give the following definition.

Definition 3.3. The random process $W(t,x,\omega)$, $t \geqslant 0$, $x \in G$, $\omega \in \Omega$, is called a complex Wiener process if

$$W(t,x,\omega) = \mathrm{Re}\, W(t,x,\omega) + i\mathrm{Im}\, W(t,x,\omega)\,, \tag{3.10}$$

where $\mathrm{Re}\, W(t,x,\omega)$ and $\mathrm{Im}\, W(t,x,\omega)$ are real-valued Wiener processes on $(\Omega, \Sigma, m(d\omega))$ and $W(t,x)$ satisfies the equality

$$\int W(t,x,\omega) W(s,y,\omega) m(d\omega) \equiv 0 \quad \forall\, t \geqslant 0,\ s \geqslant 0, \quad \text{a.e. } x, y \in G\,. \tag{3.11}$$

It is clear that (3.11) is equivalent to the conditions

$$\begin{aligned} t \wedge s\, \mathcal{K}_{11}(x,y) &\equiv \int \mathrm{Re}\, W(t,x,\omega) \mathrm{Re}\, W(s,y,\omega) m(d\omega) \\ &= \int \mathrm{Im}\, W(t,x,\omega) \mathrm{Im}\, W(s,y,\omega) m(dw) \end{aligned} \tag{3.12}$$

and

$$\begin{aligned} t \wedge s\, \mathcal{K}_{12}(x,y) &\equiv \int \mathrm{Re}\, W(t,x,\omega) \mathrm{Im}\, W(s,y,\omega) m(d\omega) \\ &= -\int \mathrm{Im}\, W(t,x,\omega) \mathrm{Re}\, W(s,y,\omega) m(dw)\,, \end{aligned} \tag{3.13}$$

where the first identities in (3.12) and (3.13) are the definitions of $\mathcal{K}_{11}(x,y)$ and $\mathcal{K}_{12}(x,y)$ respectively. By virtue of (3.13), $\mathcal{K}_{12}(x,x) \equiv 0$ and therefore the Wiener processes $\mathrm{Re}\, W(t,x)$ and $\mathrm{Im}\, W(t,x)$ are independent. Moreover, (3.11) implies that

$$t \wedge s\, \mathcal{K}(x,y) \equiv \int W(t,x,\omega) \overline{W(s,y,\omega)} m(d\omega)$$

$$= 2t \wedge s\Big(\mathcal{K}_{11}(x,y) - i\mathcal{K}_{12}(x,y)\Big)\,, \qquad (3.14)$$

where the first identity is the definition of $\mathcal{K}(x,y)$. The function $\mathcal{K}(x,y)$ is a non-negative definite kernel; this means that

$$\int\limits_G \int\limits_G \mathcal{K}(x,y)z(y)\overline{z(x)}\; dxdy \geqslant 0 \quad \forall\; z(x) \in L^2(G)\,. \qquad (3.15)$$

Here, $z(x)$ is a complex-valued function. As in the real-valued case, we suppose that the operator $\mathcal{K}z = \int\limits_G \mathcal{K}(x,y)z(y)\; dy$ is not only non-negative self-adjoint, but is a trace class operator in $L^2(G)$, i.e.

$$\int\limits_G \mathcal{K}(x,x)\; dx < \infty. \qquad (3.16)$$

Moreover, we assume that the kernel $\mathcal{K}$ satisfies the inequality:

$$\int\limits_G \Big(\sum_{j=1}^{d} \frac{\partial^2 \mathcal{K}(x,y)}{\partial_{x_j}\partial_{y_j}}\Big)\Big|_{y=x}\; dx < \infty\,. \qquad (3.17)$$

Finally, we denote by $\varLambda(\mathbb{B})$, $\mathbb{B} \in \mathcal{B}(\mathbb{C})$, the Wiener measure, i.e., the distribution of a complex Wiener process $W(t,x)$ from (3.10) and by $\varLambda_R(B_R)$, $B_R \in \mathcal{B}(\mathcal{C})$, and $\varLambda_I(B_I)$, $B_I \in \mathcal{B}(\mathcal{C})$, we respectively denote the Wiener measures of the Wiener processes $\operatorname{Re} W(t,x)$ and $\operatorname{Im} W(t,x)$. It was mentioned above that the Wiener processes $\operatorname{Re} W$ and $\operatorname{Im} W$ are independent. Therefore

$$\varLambda(\mathbb{B}) = \varLambda_R(B_R)\varLambda_I(B_I) \quad \forall\, \mathbb{B} = B_R + iB_I, \quad B_R, B_I \in \mathcal{B}(\mathbb{C})\,. \qquad (3.18)$$

3.2 The stochastic problem for the Ginzburg–Landau equation

Let $r(\lambda)$ be the function $\max\{\rho_1, \rho_2|\lambda|\}$, $\lambda \in \mathbb{R}^1$, smoothed in a neighborhood of the points $\lambda = \pm\rho_1/\rho_2$, where $\rho_1 > 0$ and $\rho_2 \geqslant 0$ are given scalars. More precisely, we define $r(\lambda)$ as

$$\begin{cases} r(\lambda) \in C^2(\mathbb{R}^1), \quad r(\lambda) = r(|\lambda|), \\ r'(\lambda) > 0 \text{ for } \lambda > \dfrac{\rho_1}{2\rho_2}, \quad r''(\lambda) > 0 \text{ for } \dfrac{\rho_1}{2\rho_2} < \lambda < \dfrac{3\rho_1}{2\rho_2}, \\ r(\lambda) = \max\{\rho_1, \rho_2|\lambda|\} \text{ for } \lambda \in \mathbb{R}^1 \setminus \left\{ \dfrac{\rho_1}{2\rho_2} < |\lambda| < \dfrac{3\rho_1}{2\rho_2} \right\}\,. \end{cases} \qquad (3.19)$$

For each real-valued function $f(\lambda)$, $\lambda \in \mathbb{R}^1$, and complex number $\psi = \mathrm{Re}\,\psi + i\mathrm{Im}\,\psi$, we denote

$$f[\psi] = f(\mathrm{Re}\,\psi) + if(\mathrm{Im}\,\psi)\,. \tag{3.20}$$

Moreover, we set, for each complex $z = \mathrm{Re}\,z + i\mathrm{Im}\,z$,

$$\widehat{f}[\psi]z = f(\mathrm{Re}\,\psi)\mathrm{Re}\,z + if(\mathrm{Im}\,\psi)\mathrm{Im}\,z\,. \tag{3.21}$$

This notation will be used throughout the paper. Using this notation, the stochastic Ginzburg–Landau equation we consider has the form

$$d\psi(t,x) + (i\nabla + A)^2\psi - \psi + |\psi|^2\psi = \widehat{r}[\psi(t,x)]dW(t,x)\,, \tag{3.22}$$

where, as in (2.1), $(t,x) \in Q_T \equiv (0,T)\times G$ and the operator $(i\nabla + A)^2$ is defined in (2.4). $W(t,x)$ on the right-hand side of (3.22) is a complex Wiener process introduced in the previous subsection, i.e., $W(t,x) = \mathrm{Re}\,W(t,x) + \mathrm{Im}\,W(t,x)$ and $dW(t,x)$ is the corresponding white noise. $r(\cdot)$ is the function defined in (3.19). The solution $\psi(t,x)$ of (3.22) is a complex-valued random function defined on the same probability space (Ω,Σ,m) in which the Wiener process $W(t,x) \equiv W(t,x,\omega)$, $\omega \in \Omega$, is defined, i.e.,

$$\psi(t,x) = \mathrm{Re}\,\psi(t,x,\omega) + i\mathrm{Im}\,\psi(t,x,\omega), \quad \omega \in \Omega,$$

is a Σ-measurable function with respect to ω.

Note that we interpret the right-hand side $\widehat{r}(\psi)dW$ of (3.22) in the sense of (3.21), i.e.,

$$\begin{aligned}&\widehat{r}\big[\psi(t,x)\big]dW(t,x)\\&\quad = r\big(\mathrm{Re}\,\psi(t,x)\big)d\mathrm{Re}\,W(t,x) + ir\big(\mathrm{Im}\,W(t,x)\big)d\mathrm{Im}\,W(t,x)\,.\end{aligned} \tag{3.23}$$

Each component of the random force should be proportional to the corresponding component of the solution. We introduce ρ_1 in the definition of $r(\lambda)$ given in (3.19) because, should the solution be sufficiently small, the consideration of additive white noise as a random force is more natural. Formally, the function defined in (3.19) multiplying the white noise dW allows us to consider the case of additive white noise (when $\rho_1 > 0$, $\rho_2 = 0$) and multiplicative white noise (when $\rho_1 > 0$, $\rho_2 > 0$). However, note that the majority of the difficulties we are forced to overcome are connected with multiplicative white noise.

Equation (3.22) is supplied with the boundary condition (2.2) and the initial condition (2.3). In this case, the initial function $\psi_0(x) = \psi_0(x,\omega)$, $\omega \in \Omega$, is a random function, defined on the same probability space as the Wiener process $W(t,x)$, that has values in $L^1(G)$; $\psi_0 : \Omega \to L^1(G)$. Moreover, we assume that $\psi_0(x,\omega)$ and $W(t,x,\omega)$ are independent.

Finally, note that Equation (3.22) is understood as an Ito differential equation. This means that, by definition, (3.22) is equivalent to the equation

$$\psi(t,x) + \int_0^t \Big[(i\nabla + A)^2\psi(s,x) - \psi(s,x) + |\psi|^2\psi(s,x)\Big]\, ds \\ = \int_0^t \widehat{r}\big[\psi(s,x)\big] dW(s,x) + \psi_0(x)\,. \tag{3.24}$$

A more precise definition of the stochastic integral on the right-hand side of (3.24) will be given later.

4 Discrete Approximation of the Stochastic Problem

To prove the main result about the existence of a solution for the stochastic Ginzburg–Landau problem, we approximate this problem by the method of lines. In this section, we study these approximations. We begin with the approximation of the Wiener process defined in Sect. 3. For this we need some preliminaries.

4.1 Definition of a projector P_h in $L^2(G)$

For each point $kh \in G_h^0$, $k = (k_1, \dots, k_d)$, we define

$$Q_k = \{x = (x_1, \dots, x_d) \in G \,:\, h(k_j - \frac{1}{2}) \leqslant x_j < h(k_j + \frac{1}{2}),\ \ j = 1, \dots, d\}\,. \tag{4.1}$$

If $kh \in \partial G_h^-(-m)$ and $kh \neq \partial G_h^-(\pm n)$ for each $n \neq m$, we set

$$Q_k = \{x = (x_1, \dots, x_d) \in G \,: \\ x_m \in [h(k_m - \tfrac{1}{2}), h(k_m + 1)),\ x_j \in [h(k_m - \tfrac{1}{2}), h(k_m + \tfrac{1}{2})), \forall j \neq m\}\,. \tag{4.2}$$

Analogously, for $kh \in \partial G_h^-(+m)$ such that $kh \neq \partial G_h^-(\pm n)$ for all $n \neq m$, we set

$$Q_k = \{x = (x_1, \dots, x_d) \in G \,: \\ x_m \in [h(k_m - 1), h(k_m + \tfrac{1}{2})),\ x_j \in [h(k_m - \tfrac{1}{2}), h(k_m + \tfrac{1}{2})), \forall j \neq m\}\,. \tag{4.3}$$

Remark 4.1. We note that the change from (4.1) to (4.2) consists of increasing the interval $x_m \in [h(k_m - \frac{1}{2}), h(k_m + \frac{1}{2}))$ from the right and, in (4.3), this interval is increased from the left.

For each $kh \in \partial G_h^-(-m) \cap \partial G_h^-(-n)$, $kh \neq \partial G_h^-(\pm p)$, if $p \neq n$, $p \neq m$, we define

$$Q_k = \{x = (x_1, \dots, x_d) \in G : \\ x_j \in [h(k_j - \tfrac{1}{2}), h(k_j + 1)), j = n, m;\ x_p \in [h(k_p - \tfrac{1}{2}), h(k_p + \tfrac{1}{2}))\}\,. \tag{4.4}$$

The sets Q_k for $kh \in \partial G_h^-(+m) \cap \partial G_h^-(\pm n)$, $kh \neq \partial G_h^-(\pm p)$ for $p \neq n$, $p \neq m$, and for $kh \in \partial G_h^-(-m) \cap \partial G_h^-(+n)$, $kh \neq \partial G_h^-(\pm p)$, $p \neq n$, $p \neq m$, are defined analogously to (4.4), but with the changes noted in Remark 4.1.

Finally, if $d = 3$, then for each $kh \in \partial G_h^-(-m) \cap \partial G_h^-(-n) \cap \partial G_h^-(-p)$, we set

$$Q_k = \{x = (x_1, \dots, x_3) \in G : x_j \in (h(k_j - \frac{1}{2}), h(k_j + 1)), j = 1, 2, 3\}\,. \tag{4.5}$$

In the other cases when $kh \in \partial G_h^-(\pm m) \cap \partial G_h^-(\pm n) \cap \partial G_h^-(\pm p)$, the set Q_k is defined analogously by taking into account Remark 4.1. Important properties of the sets Q_k defined in (4.1)–(4.5) are as follows:

a. for each $k, \ell \in \mathbb{Z}^p$ such that $kh \in G_h$, $\ell h \in G_h$, and $k \neq \ell$, the relation $Q_k \cap Q_\ell = \varnothing$ is true;

b. $\displaystyle\bigcup_{kh \in G_h} Q_k = G$.

For each set Q_k defined in (4.1)–(4.5) we put

$$V(Q_k) = \int_{Q_k} dx\,.$$

Clearly, $V(Q_k) = h^d$ for Q_k defined in (4.1) and, if h is small enough, which is the situation we consider, then

$$\frac{h^d}{4} \leqslant V(Q_k) \leqslant \left(\frac{3}{2}\right)^2 h^d \quad \text{for } Q_k \text{ defined by (4.2)–(4.4)} \tag{4.6}$$

and

$$\frac{h^d}{8} \leqslant V(Q_k) \leqslant \left(\frac{3}{2}\right)^3 h^3 \quad \text{for } Q_k \text{ defined by (4.5).} \tag{4.7}$$

The space $L^{2,h} \equiv L^{2,h}(G_h)$ is defined as the set of lattice functions $\mathbf{f} = \{f_k, kh \in G_h\}$ supplied with the scalar product and norm given by

$$(\mathbf{f}, \mathbf{g})_{L^{2,h}} = h^d \sum_{kh \in G_h} f_k \overline{g_k} \qquad \text{and} \qquad \|\mathbf{f}\|^2_{L^{2,h}} = h^d \sum_{kh \in G_h} |f_k|^2\,, \tag{4.8}$$

respectively. We introduce the operator P_h as follows:

$$P_h : L^2(G) \to L^{2,h}(G_h) \text{ such that } (P_h f)_k = V^{-1}(Q_k) \int_{Q_k} f(x)\,dx\,. \tag{4.9}$$

Then, taking into account (4.6) and (4.7), we obtain

$$\|P_h f\|^2_{L^{2,h}} = h^d \sum_{kh\in G_h} V^{-2}(Q_k) \left| \int\limits_{Q_k} f(x)\,dx \right|^2$$

$$\leqslant h^d \sum_{kh\in G_h} V^{-1}(Q_k) \int\limits_{Q_k} |f(x)|^2\,dx$$

$$\leqslant 8 \int\limits_{G} |f(x)|^2\,dx = 8\|f\|^2_{L^2(G)}\,. \tag{4.10}$$

4.2 Approximation of Wiener processes

Now let (Ω, Σ, m) be the probability space $\Omega \ni \omega \to W(t,x,\omega) \in \mathbb{C}$, where $W(t,x,\omega)$ is the complex-valued Wiener process introduced in Sect. 3.1; recall that we defined $\mathbb{C} \equiv C(0,\infty; L^2(G))$. In a similar manner, we let $\mathbb{C}_h$ denote $\mathbb{C}_h = C(0,\infty; L^{2,h}(G_h))$. Then the operator (4.9) defines the operator $P_h : \mathbb{C} \to \mathbb{C}_h$. Using this operator, we introduce the projection of the Wiener process on the space $\mathbb{C}_h$ as follows:

$$\mathbf{W}(t,\omega) \equiv \{W_k(t,\omega), kh \in G_h\} = P_h W(t,\cdot,\omega)\,, \tag{4.11}$$

where $W(t,\cdot,\omega) = W(t,x,\omega)$ is the initial Wiener process. We will show that $W_k(t,\omega)$ is a scalar Wiener process and $\mathbf{W}(t,\omega)$ is a vector-valued Wiener process by calculating their probability distributions. Let Λ be the distribution defined by (3.1). Recall that, by definition (see [44]),[2]

$$P_h^T \Lambda(\mathcal{B}_h) \equiv P_h^* \Lambda(\mathcal{B}_h) = \Lambda(P_h^{-1}\mathcal{B}_h) \quad \forall\, \mathcal{B}_h \in \mathcal{B}(\mathbb{C}_h)\,, \tag{4.12}$$

where $P_h^{-1}\mathcal{B}_h = \{\omega \in \mathbb{C} : P_h\omega \in \mathcal{B}_h\}$. This definition is equivalent to the expression

$$\int\limits_{\mathbb{C}_h} F(\mathbf{W}) P_h^* \Lambda(d\mathbf{W}) = \int\limits_{\mathbb{C}} F(P_h \mathrm{W}) \Lambda(d\mathrm{W}) = \int\limits_{\Omega} F(P_h \mathrm{W}(\cdot,\omega)) m(d\omega) \tag{4.13}$$

for every F for which at least one integral from (4.13) is well-defined. Note that the operator $P_h^* : L^{2,h}(G_h) \to L^2(G)$ is the adjoint of the operator (4.9) and is defined as

[2] In addition to the standard notation $P_h^*\Lambda$, we also introduce $P_h^T\Lambda$ in order to avoid confusion in (4.15).

$$(P_h^*\mathbf{f})(x) = f_h(x) = \sum_{kh\in G_h} f_k h^d V^{-1}(Q_k)\mathcal{X}_{Q_k}(x), \quad x \in G\,, \tag{4.14}$$

where $\mathbf{f} = \{f_k\} \in L^{2,h}(G_h)$ and $\mathcal{X}_{Q_k}(x)$ is the characteristic function of the set Q_k, i.e., $\mathcal{X}_{Q_k}(x) = 1$ for $x \in Q_k$ and $\mathcal{X}_{Q_k}(x) = 0$ for $x \neq Q_k$.

Taking into account (3.2), (4.13), and (4.14), we have

$$\begin{aligned}\widetilde{P_h^T\Lambda}(\mathbf{v}) \equiv \widetilde{P_h^*\Lambda}(\mathbf{v}) &= \int\limits_{\mathbb{C}_h} e^{i\int_0^\infty (\mathbf{W}(t),\mathbf{v}(t))_{L^{2,h}}\,dt} P_h^*\Lambda(dW)\\ &= \int\limits_{\mathbb{C}_h} e^{i\int_0^\infty (P_h\mathbf{W})(t),\mathbf{v}(t))_{L^{2,h}}\,dt}\Lambda(dW) = \int\limits_{\mathbb{C}} e^{i[\mathrm{W},P_h^*\mathbf{v}]}\Lambda(dW)\\ &= e^{-\frac{1}{2}B(P_h^*\mathbf{v},P_h^*\mathbf{v})}\,.\end{aligned} \tag{4.15}$$

By virtue of (3.4) and (4.14),

$$B_h(\mathbf{v},\mathbf{v}) \equiv B(P_h^*\mathbf{v},P_h^*\mathbf{v}) = \int\limits_0^\infty\int\limits_0^\infty t\wedge s\; h^{2d} \sum_{\substack{\mathbf{j}h\in G_h\\ kh\in G_h}} \mathcal{K}_{jk} v_k(t)\overline{v_j(s)}\;dtds\,, \tag{4.16}$$

where

$$\mathcal{K}_{jk} = V^{-1}(Q_j)V^{-1}(Q_k)\int\limits_{Q_j}\int\limits_{Q_k} \mathcal{K}(x,y)\mathcal{X}_{Q_j}(x)\mathcal{X}_{Q_k}(y)\;dxdy \tag{4.17}$$

and $\mathcal{K}(x,y)$ is the kernel defined in (3.14). The corresponding correlation operator $\mathcal{K}$ is defined by the equality

$$\int\limits_G\int\limits_G \mathcal{K}(x,y)u(y)\overline{v}(x)\;dxdy = (\mathcal{K}u,v)_{L^2(G)} \quad \forall\, u,v \in L^2(G)\,. \tag{4.18}$$

The equality $\mathcal{K} = \mathcal{K}^*$ implies that $\mathcal{K}(x,y) = \overline{\mathcal{K}(y,x)}$ and therefore $\mathcal{K}_{jk} = \overline{\mathcal{K}}_{kj}$. Formulas (4.15)–(4.18) show that $\mathbf{W}(t,\omega)$ is defined. In other words, the matrix $\widehat{\mathcal{K}} = \|h^d\mathcal{K}_{ij}\|$ is reduced to diagonal form by the unitary transformation $\widehat{\Theta} = \|\Theta_{ij}\|$, i.e.,

$$\widehat{\Theta}^*\widehat{\mathcal{K}}\widehat{\Theta} = \widehat{L}\,, \qquad \text{where } \widehat{L} = \|\widehat{L}_{ik}\| = \|\delta_{jk}\mu_k\|\,. \tag{4.19}$$

Here, μ_k are the eigenvalues of the operator $\widehat{\mathcal{K}} = \|h^d\mathcal{K}_{ij}\|$. Since (3.15) implies the positive semidefiniteness of $\widehat{\mathcal{K}}$, the inequalities $\mu_k \geqslant 0$ hold.

Lemma 4.1. *The bound*

$$\sum_{jh\in G_h} \mu_j \leqslant C \int\limits_G \mathcal{K}(x,x)\,dx < \infty \tag{4.20}$$

holds, where $\mathcal{K}(x,y)$ *is the kernel* (3.14) *and* $C>0$ *does not depend on* h*.*

Proof. By virtue of (4.17),

$$\sum_j \mu_j = h^d \sum_j \mathcal{K}_{jj} = h^d \sum_j V^{-2}(Q_j) \int\limits_{Q_j}\int\limits_{Q_j} \mathcal{K}(x,y)\mathcal{X}_{Q_j}(x)\mathcal{X}_{Q_j}(y)\,dxdy\,. \tag{4.21}$$

It is well known that

$$\mathcal{K}(x,y) = \sum_{j=1}^{\infty} \lambda_j e_j(x)\overline{e_j(y)}\,, \tag{4.22}$$

where $e_j(x), \lambda_j$ are the eigenfunctions and eigenvalues corresponding to $\mathcal{K}(x,y)$. From this equality we have

$$\begin{aligned} |\mathcal{K}(x,y)| &\leqslant \sum_{j=1}^{\infty} \lambda_j |e_j(x)|\,|e_j(y)| \\ &\leqslant \frac{1}{2}\sum_{j=1}^{\infty} \lambda_j \Big(|e_j(x)|^2 + |e_j(y)|^2\Big) = \frac{1}{2}\Big(\mathcal{K}(x,x) + \mathcal{K}(y,y)\Big)\,. \end{aligned} \tag{4.23}$$

Substituting this inequality into (4.21), we find

$$\begin{aligned} \sum_j \mu_j &\leqslant h^d \sum_j V^{-1}(Q_j)\frac{1}{2}\Big(\int\limits_{Q_j} \mathcal{K}(x,x)\mathcal{X}_j(x)\,dx + \int\limits_{Q_j} \mathcal{K}(y,y)\mathcal{X}_j(y)\,dy\Big) \\ &\leqslant C\int\limits_G \mathcal{K}(x,x)\,dx < \infty\,. \end{aligned} \tag{4.24}$$

The lemma is proved □

We set

$$\widetilde{\mathbf{v}}(t) = \Theta^*\mathbf{v}(t) \qquad \text{and} \qquad \widetilde{\mathbf{W}}(t,\omega) = \Theta^*\mathbf{W}(t,\omega)\,. \tag{4.25}$$

Since $\Theta^* = \Theta^{-1}$, we have, by (4.15), (4.16), and (4.19), that

$$\begin{aligned} \widetilde{P_h^*\Lambda}(\mathbf{v}) &= e^{-\frac{1}{2}\int_0^\infty\int_0^\infty t\wedge s(\widehat{\mathcal{K}}\mathbf{v}(t),\mathbf{v}(s))_{L^{2,h}}\,dtds} \\ &= e^{-\frac{1}{2}\int_0^\infty\int_0^\infty t\wedge s(\widehat{\mathcal{K}}\Theta\widetilde{\mathbf{v}}(t),\Theta\widetilde{\mathbf{v}}(s))_{L^{2,h}}\,dtds} \end{aligned}$$

$$= e^{-\frac{1}{2}\int\limits_0^\infty\int\limits_0^\infty t\wedge s\, h^d \sum\limits_k \mu_k \widetilde{v}_k(t)\overline{\widetilde{v}_k(s)}\, dtds}$$

$$= \prod_k e^{-\frac{h^d}{2}\int\limits_0^\infty\int\limits_0^\infty t\wedge s \mu_k v_k(t)\overline{\widetilde{v}_k(s)}\, dtds}$$

$$= \prod_k \int_\Omega e^{ih^d\int\limits_0^\infty \widetilde{W_k}(t,\omega)\overline{\widetilde{v}_k(t)}dt} m(d\omega)\,.$$

Hence,

$$\int_\Omega e^{i\int\limits_0^\infty(\widetilde{\mathbf{W}}(t,\omega),\widetilde{\mathbf{v}}(t))dt} m(d\omega) = \prod_k \int_\Omega e^{ih^d\int\limits_0^\infty \widetilde{W_k}(t,\omega)\overline{\widetilde{v_k}(t)}dt} m(d\omega)\,. \tag{4.26}$$

This equality implies that the scalar Wiener processes $\widetilde{W_k}(t,\omega)$ for $kh \in G_h$ are independent. For the definition of independence of scalar Wiener processes, see [26, p. 55].

4.3 The Ito integral

Together with the probability space (Ω, Σ, m) and the Wiener process $W(t,x)$ introduced in Sect. 3, we consider the increasing filtration Σ_t (see [26, p. 52]), i.e., a collection of σ-fields $\Sigma_t \subset \Sigma$, defined for each t, such that $\Sigma_s \subset \Sigma_t$ for $t \geqslant s$. Also, we assume that $W(t,\cdot)$ is Σ_t-measurable for every t and $W(t+h,\cdot) - W(t,\cdot)$ is independent on Σ_t. The last statement means that for every $\mathcal{A} \in \Sigma_t$ and $B \in \mathcal{B}(L^2(G))$

$$m\Big(\mathcal{A} \cap \{W(t+h,\cdot) - W(t,\cdot) \in B\}\Big) = m(\mathcal{A}) m\Big(\{W(t+h,\cdot) - W(t,\cdot) \in B\}\Big)\,.$$

Then $W(t,x)$ is called the Wiener process relative to the filtration Σ_t and the pair (W$(t,\cdot), \Sigma_t$) is called a Wiener process.

The operator P_h defined in (4.9) generates the operator $P_h : \mathcal{B}(L^2(G)) \to \mathcal{B}(L^{2,h}(G_h))$ and therefore generates the operator of filtrations

$$P_h : \Sigma_t \to \Sigma_{h,t}\,, \tag{4.27}$$

where, by definition, $B_h \in \Sigma_{h,t}$ if there exists a set $B \in \Sigma_t$ such that $B_h = P_h B$. It is clear that the pair $(\mathbf{W}(t), \Sigma_{h,t})$ is a Wiener process.

Recall (see [26, p. 66]) that a vector-valued function $\mathbf{f}(t,\omega)$ given on $(0,\infty) \times \Omega$ is called $\Sigma_{h,t}$ adapted if it is $\Sigma_{h,t}$-measurable for each $t > 0$. By Υ we denote the set of all $\Sigma_{h,t}$ adapted vector-valued functions which are $\mathcal{B}(0,\infty) \otimes \Sigma_h$ measurable (recall that $\Sigma_h = P_h\Sigma$) and satisfy

$$E\int_0^\infty \mathbf{f}^2(t)\,dt \equiv \int_\Omega \int_0^\infty \mathbf{f}(t,\omega)^2\,dt\,m(d\omega) < \infty\,,$$

where we have used the definition of the mathematical expectation. Here $\mathbf{f} = (f_1, \ldots, f_K)$ where K is the number of points in the grid kh belonging to G_h: $K = \#\{k \in \mathbb{Z}^d : kh \in G_h\}$.

It is well known (see [26, p. 68]) that the Ito integral of a $\Sigma_{h,t}$-adapted function is defined as follows:

$$\int_0^\infty \widehat{\mathbf{f}}(t)d\mathbf{W}(t) = \lim \sum_{j=0}^\infty \mathbf{f}(t_j)(\mathbf{W}(t_{j+1}) - \mathbf{W}(t_j))\,, \tag{4.28}$$

where $\sup_j |t_{j+1} - t_j| \to 0$ and this limit is understood in the sense of the space $L^2(\Omega, m)$. Here, in accordance with (3.20) and (3.21),

$$\widehat{\mathbf{f}}(t)d\mathbf{W} = \sum_{k=1}^K (\mathrm{Re}\, f_k(t)d\mathrm{Re}\, W_k(t) + i\mathrm{Im} f_k(t)d\mathrm{Im} W_k(t)).$$

By the definition of $\Sigma_{h,t}$-adaptiveness of $\mathbf{f}(t)$, we have, since $E\mathbf{W}(t) = 0$,

$$\begin{aligned} E\int_0^\infty \widehat{\mathbf{f}}(t)d\mathbf{W}(t) &= \lim \sum_{i=0}^\infty E\left(\widehat{\mathbf{f}}(t_j)(\mathbf{W}(t_{j+1}) - \mathbf{W}(t_j))\right) \\ &= \sum_{i=0}^\infty E\left(\widehat{\mathbf{f}}(t_j)\right) E\left((\mathbf{W}(t_{j+1}) - \mathbf{W}(t_j))\right) = 0\,. \end{aligned} \tag{4.29}$$

4.4 The discrete stochastic system

We consider the following discrete analogue of the stochastic Ginzburg–Landau equation given in (3.22):

$$\begin{aligned} d\psi_k(t) + \big\{(i\nabla_h + A_k)^2\psi_k(t) - \psi_k(t) + |\psi_k(t)|^2\psi_k(t)\big\}\,dt \\ = \widehat{r}\big[\psi_k(t)\big]dW_k(t)\,, \end{aligned} \tag{4.30}$$

where $W_k(t) = W_k(t,\omega)$ are the scalar Wiener processes introduced in Sect. 4.2, $dW_k(t)$ is white noise, $\boldsymbol{\psi}(t) = \{\psi_k(t), kh \in G_h\}$ is the unknown stochastic vector-valued process that we seek, and $r(\lambda)$ is the function given in (3.19). As was the case for (3.22), the right-hand side of (4.30) is interpreted in accordance with (3.20) and (3.21). If $\psi_k(t) = \mathrm{Re}\,\psi_k(t) + i\mathrm{Im}\,\psi_k(t)$ and $dW_k(t) = d\mathrm{Re}\, W_k(t) + id\mathrm{Im}\, W_k(t)$, then, by definition, we have

$$\widehat{r}\big[\psi_k(t)\big]dW_k(t) = r(\mathrm{Re}\,\psi_k(t))d\mathrm{Re}\, W_k(t) + ir(\mathrm{Im}\,\psi_k(t))d\mathrm{Im}\, W_k(t)\,. \tag{4.31}$$

We assume that the solution $\boldsymbol{\psi}(t) = \{\psi_k(t)\}$ of the system (4.30) satisfies the initial condition (2.12) and the boundary condition (2.23).

The problem (4.30), along with (2.12) and (2.23), is the *differential form* of the Ito system that by definition is equivalent to the integral form

$$\begin{aligned}\psi_k(t) = \psi_{0,k} - \int_0^t \big\{(i\nabla_h + A_k)^2\psi_k(\tau) - \psi_k(\tau) + |\psi_k(\tau)|^2\psi_k(\tau)\big\}\ d\tau \\ + \int_0^t \widehat{r}\big[\psi_k(\tau)\big]dW_k(\tau) \quad kh \in G_h\end{aligned} \tag{4.32}$$

combined with the boundary condition (2.23). The Ito integral from (4.32) is defined by (4.28).

4.5 The Ito formula

To derive a priori estimates, we use the Ito formula written in convenient form; it is formulated as follows. Let $(\mathbf{W}(t), \Sigma_{h,t})$ be a Wiener process, where $\mathbf{W}(t)$ is defined by (4.11) and $\Sigma_{h,t}$ is defined by (4.27). Suppose that $\boldsymbol{\sigma}(t,\omega)$ is a $K \times K$-matrix-valued function[3] with elements $\boldsymbol{\sigma}_{k,\ell}(t,\omega)$ that are 2×2 real-valued matrices, i.e., $\boldsymbol{\sigma}_{k,\ell}(t,\omega) = \sigma_{k,\ell,i,j}(t,\omega),\ i,j = 1,2$. The functions $\sigma_{k,\ell,i,j}(t,\omega)$ are assumed to be $\Sigma_{h,t}$-adapted random functions, $B(0,\infty) \times \Sigma_h$ measurable, and satisfy

$$E\int_0^\infty |\sigma_{k,\ell}(t,\omega)|^2\ dt \equiv \int_\Omega \int_0^\infty |\sigma_{k,\ell}(t,\omega)|^2 |dt\ m(d\omega) < \infty\,.$$

We set, by definition,

$$\boldsymbol{\sigma}(t)d\{W\}(t) = \Big\{\sum_{\ell h \in G_h} \widehat{\boldsymbol{\sigma}}_{k,\ell}(t)dW_\ell(t), \quad kh \in G_h\Big\}\,,$$

where

$$\begin{aligned}\widehat{\boldsymbol{\sigma}}_{k,\ell}dW_\ell = \big(\sigma_{k,\ell,1,1}d\mathrm{Re}\,W_\ell + \sigma_{k,\ell,1,2}d\mathrm{Im}\,W_\ell\big) \\ + i\big(\sigma_{k,\ell,2,1}d\mathrm{Re}\,W_\ell + \sigma_{k,\ell,2,2}d\mathrm{Im}\,W_\ell\big)\,.\end{aligned}$$

[3] Recall that $K = \#\{k \in \mathbb{Z}^d\ :\ kh \in G_h\}$.

Let $\mathbf{b}(t,\omega)$ be a K-dimensional vector-valued random process (with complex components $b_i(t,\omega)$) that is jointly measurable in (t,ω), $\Sigma_{h,t}$-adapted, and $\int\limits_0^T |b_s|\,dx < \infty$ a.s.

Definition 4.2. A continuous, $\Sigma_{h,t}$-adapted $\mathbb{C}^K$-valued random process $\boldsymbol{\psi}(t,\omega) = (\psi_1,\dots,\psi_K)$ has the stochastic differential

$$d\boldsymbol{\psi}(t) = \widehat{\boldsymbol{\sigma}}(t)d\mathbf{W}(t) + \mathbf{b}(t)dt \tag{4.33}$$

if and only if, a.s. for all t,

$$\boldsymbol{\psi}(t) = \boldsymbol{\psi}_0 + \int\limits_0^t \widehat{\boldsymbol{\sigma}}(s)d\mathbf{W}(s) + \int\limits_0^t \mathbf{b}(s)\,ds\,. \tag{4.34}$$

Theorem 4.3 (Ito's formula). *Let $u(x)$ be a real-valued, twice continuously differentiable function of $x \in \mathbb{C}^K$, and let $\boldsymbol{\psi}(t)$ be the random process from Definition* 4.2. *Then $u(\boldsymbol{\psi}(t))$ has a stochastic differential and*

$$\begin{aligned} du(\boldsymbol{\psi}(t)) = \sum_j \frac{\partial u(\boldsymbol{\psi}(t)}{\partial \psi_j} d\psi_j + \frac{\partial u(\boldsymbol{\psi}(t)}{\partial \overline{\psi_j}} d\overline{\psi_j} \\ + \frac{1}{2}\sum_{j,n}\Big(\frac{\partial^2 u(\boldsymbol{\psi}(t))}{\partial \psi_j \partial \psi_n} d\psi_j d\psi_n + \frac{\partial^2 u(\boldsymbol{\psi}(t))}{\partial \psi_j \partial \overline{\psi_n}} d\psi_j d\overline{\psi}_n \\ + \frac{\partial^2 u(\boldsymbol{\psi}(t))}{\partial \overline{\psi_j} \partial \psi_u} d\overline{\psi_j} d\psi_n + \frac{\partial^2 u(\boldsymbol{\psi}(t))}{\partial \overline{\psi_j} \partial \overline{\psi_n}} d\overline{\psi_j} d\overline{\psi_n}\Big)\,. \end{aligned} \tag{4.35}$$

In addition to calculating the products $d\psi_j d\psi_n$, $d\overline{\psi_j} d\psi_n$, $d\psi_j d\overline{\psi_n}$, and $d\overline{\psi_j} d\overline{\psi_n}$, one has to take into account the following rules for calculating products of independent Wiener processes $\widetilde{W}_j(t) \equiv \mathrm{Re}\,\widetilde{W}_j + i\mathrm{Im}\,\widetilde{W}_j$:

$$\begin{aligned} &d\mathrm{Re}\,\widetilde{W}_j(t)d\mathrm{Re}\,\widetilde{W}_k(t) = d\mathrm{Im}\,\widetilde{W}_j(t)d\mathrm{Im}\,\widetilde{W}_k(t) = \mu_k\delta_{jk}dt \\ &d\mathrm{Re}\,\widetilde{W}_j(t)dt = d\mathrm{Im}\,\widetilde{W}_j(t)dt = 0, \quad d\mathrm{Re}\,\widetilde{W}_j(t)d\mathrm{Im}\,\widetilde{W}_k(t) = 0\,. \end{aligned} \tag{4.36}$$

The proof of the Ito formula is given in [26] for the case of real-valued functions. One can easily reduce the case of complex-valued functions to that of real-valued functions by treating $\mathbb{C}^K$ as $\mathbb{R}^{2K}$.

To derive a priori estimates, we will need the following corollary of (4.36). (Below we will use the definition (4.31).)

Lemma 4.4. *The following relationships hold:*

$$\left(\widehat{r}[\psi_k]dW_k\right)^2 = \left(\overline{\widehat{r}[\psi_k]dW_k}\right)^2 = \left(r^2(\operatorname{Re}\psi_k) - r^2(\operatorname{Im}\psi_k)\right)\sum_j |\Theta_{kj}|^2 \mu_j \quad (4.37)$$

and

$$(\widehat{r}[\psi_k]dW_k)\left(\overline{\widehat{r}(\psi_k)dW_k}\right) = \left(r^2(\operatorname{Re}\psi_k) + r^2(\operatorname{Im}\psi_k)\right)\sum_j |\Theta_{kj}|^2 \mu_j\,, \quad (4.38)$$

where μ_k *are the eigenvalues of the correlation operator* (4.17) *for the Wiener process* $\mathbf{W}(t)$ *and* Θ_{kj} *are elements of the unitary matrix given in* (4.19) *that reduces* (4.17) *to diagonal form.*

Proof. We begin from the proof of the following corollaries of (4.36):

$$(d\operatorname{Re} W_k)^2 = (d\operatorname{Im} W_k)^2 = \sum_j |\Theta_{kj}|^2\mu_j dt \quad \text{and} \quad (d\operatorname{Re} W_k)(d\operatorname{Im} W_k) = 0\,. \quad (4.39)$$

By virtue of (4.25), $W_k = \sum_j \Theta_{kj} d\widetilde{W}_j$ and therefore, using (4.36), we obtain

$$\begin{aligned}(d\operatorname{Re} W_k)^2 &= \Big(\sum_j (\operatorname{Re}\Theta_{kj} d\operatorname{Re}\widetilde{W}_j - \operatorname{Im}\Theta_{kj} d\operatorname{Im}\widetilde{W}_j)\Big)^2 \\ &= \sum_j \Big((\operatorname{Re}\Theta_{kj})^2 + (\operatorname{Im}\Theta_{kj})^2\Big)\mu_j dt = \sum_j |\Theta_{kj}|^2\mu_j dt\,.\end{aligned}$$

The second and third equalities in (4.39) are proved in the same manner.

By (4.31), (4.36), and (4.39), we have

$$\begin{aligned}\left(\widehat{r}[\psi_k]dW_k\right)^2 =& \Big(r(\operatorname{Re}\psi_k)d\operatorname{Re} W_k + ir(\operatorname{Im}\psi_k)d\operatorname{Im} W_k\Big)^2 \\ =& r^2(\operatorname{Re}\psi_k)(d\operatorname{Re} W_k)^2 - r^2(\operatorname{Im}\psi_k)(d\operatorname{Im} W_k)^2 \\ &+ 2ir(\operatorname{Re}\psi_k)r(\operatorname{Im}\psi_k)d\operatorname{Re} W_k d\operatorname{Im} W_k \\ =& \left(r^2(\operatorname{Re}\psi_k) - r^2(\operatorname{Im}\psi_k)\right)\sum_j |\Theta_{kj}|^2\mu_j dt\,.\end{aligned}$$

This equality and the fact that its right-hand side is a real function prove (4.37). The relation (4.38) is proved analogously. □

5 A Priori Estimates

In order to prove the solvability not only of the discrete stochastic system (4.30), (2.12), and (2.23), but also of the main stochastic problem (3.22), (2.2), and (2.3), we have to establish a number of a priori estimates for the system (4.30).

5.1 Application of the Ito formula

We take the function $u(\boldsymbol{\psi})$ from Theorem 4.3 as

$$u(\boldsymbol{\psi}) = h^d \sum_{hk \in G_h} |\psi_k(t)|^{2p} \equiv \|\boldsymbol{\psi}(t)\|^{2p}_{L^{2p,h}} , \tag{5.1}$$

where $p = 1$ or $p = 2$. Applying (5.1) in the Ito formula with the stochastic differential du defined in (4.35) and using (4.37) and (4.38), we obtain

$$d\|\boldsymbol{\psi}(t)\|^{2p}_{L^{2p,h}} = h^d \sum_k \Big\{ p|\psi_k|^{2p-2} \left(\overline{\psi}_k d\psi_k + \psi_k d\overline{\psi}_k \right)$$

$$+ \frac{1}{2} p \cdot (p-1) |\psi_k|^{2(p-2)} \left(\overline{\psi}_k^2 d\psi_k d\psi_k + \psi_k^2 d\overline{\psi}_k d\overline{\psi}_k \right)$$

$$+ p^2 |\psi_k|^{2(p-1)} d\psi_k d\overline{\psi}_k \Big\}$$

$$= h^d \sum_k \Big\{ p|\psi_k|^{2p-2} \Big\{ \left(-\overline{\psi}_k (i\nabla + A_k)^2 \psi_k + 2|\psi_k|^2 - 2|\psi_k|^4 \right) dt$$

$$+ \overline{\psi}_k \widehat{r}[\psi_k] dW_k - \psi_k \overline{(i\nabla_h + A_k)^2 \psi_k} dt$$

$$+ \psi_k \overline{\widehat{r}[\psi_k] dW_k} \Big\} + \frac{1}{2} p(p-1) |\psi_k|^{2(p-2)}$$

$$\Big(\overline{\psi}_k^2 \left\{ \left(-(i\nabla_h + A_k)^2 \psi_k + \psi_k - |\psi_k|^2 \psi_k \right) dt + \widehat{r}[\psi_k] dW_k \right\}^2$$

$$+ \psi_k^2 \Big\{ \Big(-\overline{(i\nabla_h + A_k)^2 \psi_k} + \overline{\psi_k} - |\psi_k|^2 \overline{\psi_k} \Big) dt + \overline{\widehat{r}[\psi_k] dW_k} \Big\}^2 \Big)$$

$$+ p^2 |\psi_k|^{2(p-1)} \Big\{ \left(-(i\nabla_h + A_k)^2 \psi_k + \psi_k - |\psi_k|^2 \psi_k \right) dt$$

$$+ \widehat{r}[\psi_k] dW_k \left\{ \Big(-\overline{(i\nabla_h + A_k)^2 \psi_k} + \overline{\psi_k} - |\psi_k|^2 \overline{\psi_k} \Big) dt + \overline{\widehat{r}[\psi_k] dW_k} \right\} \Big\}$$

so that

$$
\begin{aligned}
d\|\boldsymbol{\psi}(t)\|^{2p}_{L^{2p,h}} \\
&= h^d \sum_k p|\psi_k|^{2p-2}\Big\{\big(-2\mathrm{Re}\,(\overline{\psi}_k(i\nabla_h + A_k)^2\psi_k) + 2|\psi_k|^2 - 2|\psi_k|^4\big)\,dt \\
&+ 2\mathrm{Re}\,(\overline{\psi}_k\widehat{r}[\psi_k]dW_k)\Big\} \\
&+ h^d \sum_k \Big\{p(p-1)|\psi_k|^{2(p-2)}\mathrm{Re}\,(\psi_k^2)(r^2(\mathrm{Re}\,\psi_k) \\
&- r^2(\mathrm{Im}\,\psi_k)) \sum_j |\Theta_{kj}|^2\mu_j \\
&+ p^2|\psi_k|^{2(p-1)}\Big(r^2(\mathrm{Re}\,\psi_k) + r^2(\mathrm{Im}\,\psi_k)]\Big) \sum_j |\Theta_{kj}|^2\mu_j\Big\}dt\,,
\end{aligned}
\tag{5.2}
$$

where $\sum_k = \sum_{kh\in G_h}$. Applying (2.14) with $\overline{\phi}_k = |\psi_k|^{2p-2}\overline{\psi}_k$ to the first term on the right-hand side of (5.2) results in

$$
\begin{aligned}
-h^d \sum_{kh\in G_h} 2p|\psi_k|^{2p-2}\mathrm{Re}\,\big(\overline{\psi}_k(i\nabla_h + A_k)^2\psi_k\big) \\
= -h^d \widetilde{\sum_{jk}} 2p\mathrm{Re}\,\left\{\Big((i\partial^+_{j,h} + A^j_k)\psi_k, \overline{(i\partial^+_{j,h} + A^j_k)(|\psi_k|^{2p-2}\psi_k)}\Big)\right\}\,,
\end{aligned}
\tag{5.3}
$$

where, for brevity, we use the following notation:

$$
\begin{aligned}
\widetilde{\sum_{jk}}\Big((i\partial^+_{j,h} + A^j_k)\psi_k, \overline{(i\partial^+_{j,h} + A^j_k)\phi_k}\Big) \\
= \sum_{j=1}^{d} \sum_{kh\in G_h\cup\partial G^+_h(-j)} \big(i\partial^+_{j,h}\psi_k + A^j_k\psi_k\big)\overline{\big(i\partial^+_{j,h}\phi_k + A^j_k\phi_k\big)}\,.
\end{aligned}
\tag{5.4}
$$

Below, we will also use the notation $\widetilde{\sum_{jk}}$ when in (5.4), $A_k = \{A^j_k\}$ is absent. Moreover, in the next subsection we use the following notation which is closely related to (5.4):

$$
\|\nabla^+_h\psi\|^2_{L^{2,h}} = \widetilde{\sum_{j,k}}|\partial^+_{j,h}\psi_k|^2 \equiv \sum_{j=1}^{d} \sum_{kh\in G_h\cup\partial G^+_h(-j)} |\partial^+_{j,h}\psi_k|^2\,. \tag{5.5}
$$

5.2 A priori estimate for $p = 1$

The following assertion holds.

Theorem 5.1. *Let a random process $\{\boldsymbol{\psi}(t)\} = \{\psi_k\}$ have the stochastic differential* (4.30)*. Then $\boldsymbol{\psi}$ satisfies the estimate*

$$E\|\boldsymbol{\psi}(t)\|^2_{L^{2,h}} + E\int\limits_0^t \left(\|\nabla_h^+\boldsymbol{\psi}(\tau)\|^2_{L^{2,h}} + \|\boldsymbol{\psi}(\tau)\|^4_{L^{4,h}}\right) d\tau \leqslant C_2\left(E\|\boldsymbol{\psi}_0\|^2_{L^{2,h}} + 1\right) e^{C_1 t}, \tag{5.6}$$

where the constants C_1 and C_2 do not depend on h.

Proof. The equality (5.3) for $p = 1$ can be rewritten as follows:

$$-h^d \sum_k 2\mathrm{Re}\,\left\{\overline{\psi}_k (i\nabla_h + A_k)^2 \psi_k\right\} = -h^d \widetilde{\sum_{jk}} 2\left|(i\partial^+_{j,h} + A^j_k)\psi_k\right|^2 .$$

Here and in the sequel, we use the notation $\sum\limits_k = \sum\limits_{kh\in G_h}$ as well as the notation (5.4). We substitute this equality into the right-hand side of (5.2) to obtain

$$\begin{aligned} &d\|\boldsymbol{\psi}\|^2_{L^{2,h}} \\ &\quad = -2h^d\left[\widetilde{\sum_{jk}}\left|(i\partial^+_{j,h} + A^j_k)\psi_k)\right|^2 - \sum_k\left(|\psi_k|^2 - |\psi_k|^4\right)\right] dt \\ &\quad +2h^d \mathrm{Re}\sum_k(\overline{\psi}_k \widehat{r}[\psi_k] dW_k) \\ &\quad +h^d \sum_k\Big(r^2(\mathrm{Re}\,\psi_k) + r^2(\mathrm{Im}\,\psi_k)\Big)\sum_j |\Theta_{kj}|^2\mu_j \; dt\,. \end{aligned} \tag{5.7}$$

By virtue of the definition (3.19) for the function $r(\lambda)$ and (3.20), we have

$$|r[\psi_k]|^2 \equiv |r(\mathrm{Re}\,\psi_k)|^2 + |r(\mathrm{Im}\,\psi_k)|^2 \leqslant C^2(1 + |\psi_k(t)|)^2\,. \tag{5.8}$$

An equivalent integral form of the Ito differential is written as

$$\|\boldsymbol{\psi}(t)\|^2_{L^{2,h}} + 2\int\limits_0^t h^d\left(\widetilde{\sum_{jk}}|(i\partial^+_{j,h} + A^j_k)\psi_k|^2 + \sum_k|\psi_k|^4\right) d\tau$$
$$- 2\int\limits_0^t h^d\sum_k \mathrm{Re}\,(\overline{\psi}_k\widehat{r}[\psi_k] dW_k)$$

$$= \int_0^t \left(h^d \sum_k (2|\psi_k|^2 + \sum_j |\Theta_{kj}|^2 \mu_j |r[\psi_k]|^2 \right) d\tau + \|\boldsymbol{\psi}_0\|^2_{L^{2,h}} . \tag{5.9}$$

Thus, assuming that $\boldsymbol{\psi}(t)$ is a $\Sigma_{h,t}$ adaptive vector function, we apply the mathematical expectation to (5.9). Then, taking into account (5.8), (4.29), the bound $\sum\limits_j |\Theta_{kj}|^2 \mu_j \leqslant \sum\limits_j \mu_j$, and (4.23), we obtain

$$E\|\boldsymbol{\psi}(t)\|^2_{L^{2,h}} + 2E \int_0^t h^d \left(\widetilde{\sum_{jk}} |(i\partial^+_{j,h} + A^j_k)\psi_k(t)|^2 + \sum_k |\psi_k(t)|^4 \right) d\tau$$
$$\leqslant E \int_0^t C \left(\|\boldsymbol{\psi}(t)\|^2_{L^{2,h}} + 1 \right) d\tau + E\|\boldsymbol{\psi}_0\|^2_{L^{2,h}} . \tag{5.10}$$

Using the fact that

$$|(i\nabla^+_h + A_k)\psi_k|^2 \geqslant |\nabla^+_h \psi_k|^2 - C|\psi_k|^2 , \tag{5.11}$$

we obtain from (5.10) that

$$E\|\boldsymbol{\psi}(t)\|^2_{L^{2,h}} + 2E \int_0^t \left(\|\nabla^+_h \boldsymbol{\psi}\|^2_{L^{2,h}} + \|\boldsymbol{\psi}(t)\|^4_{L^{4,h}} \right) d\tau$$
$$\leqslant C_1 \left(E \int_0^t \|\boldsymbol{\psi}(t)\|^2_{L^{2,h}} \, d\tau + t \right) + E\|\boldsymbol{\psi}_0\|^2_{L^{2,h}} . \tag{5.12}$$

Note that the term $|\psi_k|^2$ from (5.11) with $kh \in \partial G^+_h$ can be estimated by $\|\boldsymbol{\psi}\|_{L^{2,h}}$ by virtue of (2.23) and the bounds following that inequality. Now, by applying the Gronwall inequality to (5.12), we finally obtain the desired estimate (5.6). □

5.3 A priori estimate for $\mathbf{p} = 2$

We now establish the following bound.

Theorem 5.2. *Let a random process $\boldsymbol{\psi}(t) = \{\psi_k\}$ have the stochastic differential* (4.30). *Then $\boldsymbol{\psi}$ satisfies the estimate*

$$E(\|\boldsymbol{\psi}(t)\|^4_{L^{4,h}} + E \int_0^t \left(\|\boldsymbol{\psi}(t)\|^6_{L^{6,h}} + h^d \widetilde{\sum_{j,k}} |\partial^+_{j,h}\psi_k|^2 |\psi_k|^2 \right) d\tau$$

$$\leqslant C_1 \left(1 + E\|\boldsymbol{\psi}_0\|_{L^4}^4\right) e^{Ct} \,, \tag{5.13}$$

where C and C_1 do not depend on h.

Proof. Taking into account that

$$\psi_{k+e_j} - \psi_k = h\partial_j^+ \psi_k \,, \tag{5.14}$$

we obtain

$$|\psi_{k+e_j}|^2\overline{\psi}_{k+e_j} - |\psi_k|^2\overline{\psi}_k = |\psi_{k+e_j}|^2 \left(\overline{\psi}_{k+e_j} - \overline{\psi}_k\right) + \overline{\psi}_k \left(|\psi_{k+e_j}|^2 - |\psi_k|^2\right)$$

$$= |\psi_{k+e_j}|^2 h\overline{\partial_j^+ \psi_k} + \overline{\psi}_k \left(\overline{\psi}_{k+e_j}(\psi_{k+e_j} - \psi_k) + \psi_k(\overline{\psi}_{k+e_j} - \overline{\psi}_k)\right)$$

and therefore

$$\begin{aligned} &\operatorname{Re}\left\{\left(\partial_j^+\psi_k\right)\partial_j^+(|\psi_k|^2\overline{\psi}_k)\right\} \\ &\quad = |\psi_{k+e_j}|^2|\partial_j^+\psi_k|^2 + \operatorname{Re}\left((\partial_j^+\psi_k)^2\overline{\psi}_k\overline{\psi}_{k+e_j}\right) + |\psi_k|^2|\partial_j^+\psi_k|^2 \\ &\quad \geqslant |\partial_j^+\psi_k|^2\left(|\psi_{k+e_j}|^2 + |\psi_k|^2 - |\psi_k||\psi_{k+e_j}|\right) \\ &\quad \geqslant \frac{3}{4}|\partial_j\psi_k|^2|\psi_k|^2 \,. \end{aligned} \tag{5.15}$$

In addition,

$$\begin{aligned} \operatorname{Im}\sum_j \left(A_k^j(\partial_j^+\psi_k)|\psi_k|^2\overline{\psi}_k\right) &\geqslant -|A_k||\nabla^+\psi_k||\psi_k|^3 \\ &\geqslant -C_\varepsilon|\psi_k|^4 - \varepsilon|\nabla^+\psi_k|^2|\psi_k|^2 \end{aligned} \tag{5.16}$$

so that

$$\begin{aligned} &\operatorname{Im}\left(\psi_k(A_k, \nabla_h^+)(|\psi_k|^2\overline{\psi}_k)\right) \\ &\quad = \operatorname{Im}\left(\psi_k \sum_j \frac{A_k^j}{h}(|\psi_{k+e_j}|^2\overline{\psi}_{k+e_j} - |\psi_k|^2\overline{\psi}_k)\right) \\ &\quad = \operatorname{Im}\left(\psi_k \sum_j \frac{A_k^j}{h}\left((\psi_{k+e_j} - \psi_k)\overline{\psi}_{k+e_j}^2\right.\right. \\ &\qquad\qquad \left.\left. + \psi_k(\overline{\psi}_{k+e_j} - \overline{\psi}_k)(\overline{\psi}_{k+e_j} + \overline{\psi}_k)\right)\right) \end{aligned} \tag{5.17}$$

$$= \ \mathrm{Im}\Big(\sum_j A_k^j\big((\partial_j^+\psi_k)\psi_k\overline{\psi}^2_{k+e_j} + \psi_k^2\overline{\partial_j^+\psi_k}(\overline{\psi}_{k+e_j}+\overline{\psi}_k)\big)\Big)$$

$$\geqslant \ -C\sum_j |\partial_j^+\psi_k||\psi_k|\Big(|\psi_{k+e_j}|^2 + |\psi_k||\psi_{k+e_j}| + |\psi_k|^2\Big)$$

$$\geqslant \ -\varepsilon\sum_j |\partial_j^+\psi_k|^2|\psi_k|^2 - C_\varepsilon\sum_j\Big(|\psi_{k+e_j}|^4 + |\psi_k|^4\Big)\,.$$

Using (5.15)–(5.17), we obtain

$$\begin{aligned}
&\mathrm{Re}\,\Big((i\nabla_h^+ + A_k)\psi_k, \overline{(i\nabla_h^+ + A_k)(|\psi_k|^2\psi_k)}\Big)\\
&= \mathrm{Re}\,(\nabla_h^+\psi_k, \nabla_h^+(|\psi_k|^2\overline{\psi}_k)) - \mathrm{Im}\,(\nabla_h^+\psi_k, A_k|\psi_k|^2\overline{\psi}_k)\\
&\quad + \mathrm{Im}\,(\psi_k(A_k\nabla_h^+)(|\psi_k|^2\overline{\psi}_k)) + |A_k|^2|\psi_k|^4\\
&\geqslant \frac{3}{4}\sum_{j=1}^d |\partial_j^+\psi_k|^2|\psi_k|^2 - C_\varepsilon(|\psi_k|^4 + \sum_{j=1}^d |\psi_{k+e_j}|^4) - \varepsilon|\nabla_h^+\psi_k|^2|\psi_k|^2\,.
\end{aligned} \tag{5.18}$$

Now we substitute (5.18) into (5.3) and subsequently use this inequality in (5.2). As a result, taking into account (5.8), we obtain the inequality

$$\begin{aligned}
d\|\boldsymbol{\psi}\|^4_{L^{4,h}} \leqslant h^d\widetilde{\sum_{jk}}&\Big(-(3-4\varepsilon)|\partial_{j,h}^+\psi_k|^2|\psi_k|^2\Big)dt\\
&+\sum_k\big\{C|\psi_k|^4 + \varepsilon|\psi_k|^2 - 4|\psi_k|^6\big\}\,dt\\
&+\sum_k\Big(2\mathrm{Re}\,(\overline{\psi}_k\widehat{r}[\psi_k]dW_k) + C\big(|\psi_k|^2 + |\psi_k|^4\big)dt\Big)\,.
\end{aligned} \tag{5.19}$$

Rewriting (5.19) in integral form and taking the mathematical expectation of the obtained inequality, we obtain the estimate

$$\begin{aligned}
E\|\boldsymbol{\psi}(t)\|^4_{L^{4,h}} + E\int_0^t h^d&\left(\widetilde{\sum_{jk}}|\partial_{j,h}^+\psi_k|^2|\psi_k|^2 + \sum_k|\psi_k|^6\right)\,d\tau\\
&\leqslant CE\int_0^t h^d\sum_k\big(|\psi_k|^4 + |\psi_k|^2 + 1\big)\,d\tau + E\|\boldsymbol{\psi}_0\|^4_{L^4}\,.
\end{aligned} \tag{5.20}$$

Applying the bound (5.6) to the right-hand side of (5.20) and applying after that the Gronwall inequality, we obtain the final estimate (5.13). □

Note that, in addition to the estimates (5.6) and (5.13) corresponding to the cases $p = 1$ and $p = 2$, one can prove by induction analogous estimates for arbitrary natural p; specifically, we have

$$E\|\boldsymbol{\psi}(t)\|^{2p}_{L^{2p,h}} + \int_0^t \left(\|\boldsymbol{\psi}(\tau)\|^{2(p+1)}_{L^{2(p+1)},h} + h^d \widetilde{\sum_{jk}} |\partial^+_{j,h}\psi_k|^2 |\psi_k|^{2(p-1)} \right) d\tau$$
$$\leqslant C_p \left(1 + E\|\boldsymbol{\psi}_0(t)\|^p_{L^{p,h}}\right) e^{Ct} . \tag{5.21}$$

We will not prove the estimate (5.21) for $p \geqslant 3$ because, for our purposes, the estimates (5.6) and (5.20) will suffice.

5.4 Auxiliary Wiener process

We will need a more general projection of the initial Wiener process than (4.11). Roughly speaking, the new projection contains not only coordinates from (4.11), but also their difference gradients at points kh. To be precise, in a manner similar to (4.9), we define for, $f(x) \in L^2(G)$,

$$p^0_k(f) = V^{-1}(Q_k) \int_{Q_k} f(x)\, dx, \quad \text{for } kh \in G_h\,, \tag{5.22}$$

$$p^0_k(f) \text{ for } kh \in \partial G^+_h \text{ is calculated by } p^0_k(f) \text{ with } kh \in \partial G^-_h \text{ using (2.23)}\,, \tag{5.23}$$

and

$$p^j_k(f) = i\frac{p^0_{k+e_j}(f) - p^0_k(f)}{h} + A^j_k p^0_k(f) \quad \text{for } kh \in G_h \cup \partial G^+_h(-j) \tag{5.24}$$

for $j = 1, \ldots, d$. We denote

$$\widehat{G_h} = \{(j,k) \,:\, j = 0, kh \in G_h;\, j = 1, \ldots, d, kh \in G_h \cup \partial G^+_h(-j)\}$$

and introduce the projector

$$P^A_h : L^2(G) \to L^2(\widehat{G_h}); \quad P^A_h(f) = \{p^j_k(f), (j,k) \in \widehat{G_h}\}\,, \tag{5.25}$$

where the scalar product in $L^2(\widehat{G_h})$ is defined in the standard way:

$$\text{for} \quad u = \{u^j_k, (j,k) \in \widehat{G_h}\}, \quad v = \{v^j_k, (j,k) \in \widehat{G_h}\},$$

$$(u,v)_{L^2(\widehat{G}_h)} = h^d \sum_{(j,k)\in\widehat{G}_h} u_k^j \overline{v_k^j}\,.$$

Note that the components u_k^j of $u \in L^2(\widehat{G}_h)$ with $j \neq 0$ are expressed via the components u_m^0 by the formula analogous to (5.24):

$$u_k^j = \frac{u_{k+e_j}^0 - u_k^0}{h} + A_k^j u_k^0 \quad \text{for } j = 1,\ldots,d, \quad kh \in G_h \cup \partial G_h^+(-j)\,.$$

We can calculate the operator $(P_h^A)^* : L^2(\widehat{G_h}) \to L^2(G)$ which is adjoint to (5.25) using (2.14) which is summation by parts:

$$\begin{aligned}
(P_h^A f, g)_{L^2(\widehat{G}_h)} &= h^d \sum_{(j,k)\in\widehat{G}_h} p_k^j(f) g_k^j \\
&= h^d \widetilde{\sum_{jk}} (i\partial_{j,h}^+ + A_k^j) p^0(f) \overline{(i\partial_{j,h}^+ + A_h^j) g_k^0} + h^d \sum_{kh\in G_h} p_k^0(f)\overline{g_k^0} \\
&= h^d \sum_{kh\in G_h} p_k^0(f) \Big(\overline{g_k^0 + (i\nabla_h^- + A_k, i\nabla_h^+ + A_k) g_k^0} \Big)
\end{aligned} \tag{5.26}$$

so that

$$\big((P_h^A)^* g\big)(x) = \sum_{kh\in G_h} h^d V^{-1}(Q_k) \big(g_k^0 + (i\nabla_h^- + A_k, i\nabla_h^+ + A_k) g_k^0\big) \mathcal{X}_k(x)\,. \tag{5.27}$$

Analogous to (4.11), we introduce the vector-valued process

$$\mathbf{AW}(t,\omega) = P_h^A W(t,\cdot,\omega) = \{p_k^j(W(t,\cdot,\omega) \equiv AW_k^j(t), (j,k) \in \widehat{G_h}\}\,. \tag{5.28}$$

Here, $p_k^0(W(t,\cdot,\omega)) = W^k(t,\omega)$ for $kh \in G_h$, where $W^k(t,\omega)$ is the Wiener process from (4.11). In order to define $p_k^j(W(t,\cdot,\omega))$ by (5.24), one has to know $W^k(t,\omega)$ with $kh \in \partial G^+$. These Wiener processes are defined by formula (2.23) via $W^m(t,\omega)$ with $mh \in G_h$. Repeating the calculation in (4.13) and (4.15), where the projector (4.9) is changed to the projector (5.25) and L^{2h} is changed to $L^2(\widehat{G}_h)$, we find that the process (5.28) is a vector-valued Wiener process. Moreover,

$$\begin{aligned}
&B\big((P_h^A)^* v, (P_h^A)^* v\big) \\
&= \int_0^\infty \int_0^\infty t \wedge s \int_{G\times G} \mathcal{K}(x,y) (P_h^A)^*(v(s))(y) \overline{(P_h^A)^*(v(t))(x)}\, dxdydsdt
\end{aligned}$$

$$= \int\limits_0^\infty \int\limits_0^\infty t \wedge s \sum_{(j_1,k_1)\in\widehat{G}_h} \sum_{(j_2,k_2)\in\widehat{G}_h} \mathcal{K}^{j_1,j_2}_{k_1,k_2} \overline{v^{j_1}_{k_1}(t)} v^{j_2}_{k_2}(s)\, dtds\,, \tag{5.29}$$

where $\mathcal{K}^{0,0}_{j,k}$ are defined by (4.17) with the upper indices (0,0) omitted,

$$\mathcal{K}^{j_1,j_2}_{k_1,k_2} = \big(i\partial^+_{j_1,h} + A^{j_1}_{k_1}\big)\overline{\big(i\partial^+_{j_2,h} + A^{j_2}_{k_2}\big)}\mathcal{K}^{0,0}_{k_1,k_2}, \quad (j_\ell,k_\ell)\in\widehat{G_h}, j_\ell\neq 0, \ell=1,2\,, \tag{5.30}$$

and $\mathcal{K}^{j_1,0}_{k_1,k_2}$ and $\mathcal{K}^{0,j_2}_{k_1,k_2}$ are defined similarly (correspondingly, the second or first operator $\big(i\partial^+_{j,h} + A^j_k\big)$ in (5.30) should be omitted). It is clear that

$$s\wedge t\, \mathcal{K}^{j_1,j_2}_{k_1,k_2} = \int AW^{j_1}_{k_1}(t,\omega)\overline{AW^{j_2}_{k_2}(s,\omega)} m(d\omega)\,, \tag{5.31}$$

where the scalar Wiener processes $AW^j_k(t,\omega)$ are defined in (5.28). Definitions (4.17), (5.30), and (5.31) of the operator $A\mathcal{K} = \{h^d \mathcal{K}^{j_1,j_2}_{k_1,k_2}\}$ imply that $\mathcal{K}^{j_1,j_2}_{k_1,k_2} = \overline{\mathcal{K}^{j_2,j_1}_{k_2,k_1}}$ and therefore there exists a unitary transformation $A\theta = \{\theta^{j_1,j_2}_{k_1,k_2}\}$ (i.e., $A\theta^* \equiv \{\overline{\theta^{j_2,j_1}_{k_2,k_1}}\} = (A\theta)^{-1}$) that reduces the operator $A\mathcal{K}$ to diagonal form:

$$A\theta^* A\mathcal{K} A\theta = AL, \quad \text{where} \quad AL = \{L^{j_1,j_2}_{k_1,k_2}\} = \{\delta_{j_1,j_2}\delta_{k_1,k_2}\mu^{j_1}_{k_1}\}\,. \tag{5.32}$$

We set

$$\widetilde{\mathbf{AW}}(t,\omega) = A\theta^* \mathbf{AW}(t,\omega) = \{\widetilde{W}^j_k(t,\omega)\}\,. \tag{5.33}$$

Then calculations analogous to (4.25) and (4.26) show that the scalar Wiener processes $\widetilde{W}^j_k(t,\omega)$ are independent and therefore, for their differentials, the following Ito table analogous to (4.36) is true:

$$\begin{cases} d\mathrm{Re}\, \widetilde{W}^{j_1}_{k_1}(t) d\mathrm{Re}\, \widetilde{W}^{j_2}_{k_2}(t) = d\mathrm{Im}\, \widetilde{W}^{j_1}_{k_1}(t) d\mathrm{Im}\, \widetilde{W}^{j_2}_{k_2}(t) = \mu^{j_1}_{k_1}\delta_{j_1,j_2}\delta_{k_1,k_2} dt \\ \\ d\mathrm{Re}\, \widetilde{W}^j_k(t)dt = d\mathrm{Im}\, W^j_k(t)dt = d\mathrm{Re}\, W^{j_1}_{k_1}(t) d\mathrm{Im}\, W^{j_2}_{k_2}(t) = 0\,. \end{cases} \tag{5.34}$$

Now we are in a position to prove the following analogue of (4.39).

Lemma 5.3. *For scalar Wiener processes $AW^j_k(t)$ defined in* (5.28) *the following relationships hold*:

$$(d\mathrm{Re}\, AW^j_k)^2 = (d\mathrm{Im}\, AW^j_k)^2 = \sum_{(m,\ell)\in\widehat{G}_h} \mu^\ell_m |\theta^{j\ell}_{km}|^2 dt, \quad d\mathrm{Re}\, AW^j_k\; d\mathrm{Im}\, AW^j_k = 0\,. \tag{5.35}$$

Proof. By virtue of (5.33),

$$AW_k^j = \sum_{(\ell,m)\in\widehat{G_h}} \theta_{k,m}^{j,\ell} \widetilde{W}_m^\ell \,. \tag{5.36}$$

Therefore, taking into account (5.34) and the fact that the transformation $A\theta = \|\theta_{k,m}^{j,l}\|$ is unitary, we obtain

$$\begin{aligned}(d\mathrm{Re}\, AW_k^j)^2 &= \Big(\sum_{\ell,m}\Big[\mathrm{Re}\,\theta_{k,m}^{j,\ell} d\mathrm{Re}\,\widetilde{W}_m^\ell - \mathrm{Im}\,\theta_{k,m}^{j,\ell} d\mathrm{Im}\,\widetilde{W}_m^\ell\Big]\Big)^2 \\ &= \sum_{\ell,m}\sum_{\ell_1,m_1}\Big(\mathrm{Re}\,\theta_{k,m}^{j,\ell}\mathrm{Re}\,\theta_{k,m_1}^{j,\ell_1} d\mathrm{Re}\,\widetilde{W}_m^\ell d\mathrm{Re}\,\widetilde{W}_{m_1}^{\ell_1} \\ &\qquad + \mathrm{Im}\,\theta_{k,m}^{j,\ell}\mathrm{Im}\,\theta_{k,m_1}^{j,\ell_1} d\mathrm{Im}\,\widetilde{W}_m^\ell d\mathrm{Im}\,\widetilde{W}_{m_1}^{\ell_1}\Big) \\ &= \sum_{\ell,m}\mu_m^\ell|\theta_{k,m}^{j,\ell}|^2\, dt\,.\end{aligned} \tag{5.37}$$

The other relations in (5.35) are proved in a similar manner. □

Lemma 5.4. *The following equalities hold*:

$$\begin{cases} d\mathrm{Re}\, AW_k^j d\mathrm{Re}\, AW_k^0 = d\mathrm{Im}\, AW_k^j d\mathrm{Im}\, AW_k^0 = \sum_{l,m}\mu_m^\ell \mathrm{Re}\left(\theta_{k,m}^{j,\ell}\overline{\theta_{k,m}^{0,\ell}}\right)\, dt \\ d\mathrm{Re}\, AW_k^j d\mathrm{Im}\, AW_k^0 = 0\,, \end{cases} \tag{5.38}$$

where $j = 1, \ldots, d$.

Proof. Similar to (5.37), we find

$$\begin{aligned} d\mathrm{Re}\, AW_k^j d\mathrm{Re}\, AW_k^0 &= \sum_{\ell,m}\mu_m^\ell\Big(\mathrm{Re}\,\theta_{k,m}^{j,\ell}\mathrm{Re}\,\theta_{k,m}^{0,\ell} + \mathrm{Im}\,\theta_{k,m}^{j,\ell}\mathrm{Im}\,\theta_{k,m}^{0,\ell}\Big)\, dt \\ &= \sum_{\ell,m}\mu_m^\ell \mathrm{Re}\left(\theta_{k,m}^{j,\ell}\overline{\theta_{k,m}^{0,\ell}}\right)\, dt\,.\end{aligned} \tag{5.39}$$

All the other equalities from (5.38) are proved in a similar manner. □

We will need the following lemma.

Lemma 5.5. *Let* μ_m^ℓ *denote the eigenvalues from* (5.32). *Then* $\mu_m^\ell \geqslant 0$ *and the following estimates hold*:

$$\sum_{(\ell,m)\in\widehat{G_h}} \mu_m^\ell |\theta_{k,m}^{j,\ell}|^2 \leqslant \sum_{(\ell,m)\in\widehat{G_h}} \mu_m^\ell\,, \tag{5.40}$$

$$\sum_{(\ell,m)\in\widehat{G}_h} \mu_m^\ell \operatorname{Re}\big(\theta_{k,m}^{j,\ell}\overline{\theta_{j,m}^{0,\ell}}\big) \leqslant \sum_{(\ell,m)\in\widehat{G}_h} \mu_m^\ell\,. \tag{5.41}$$

Proof. To show that $\mu_m^\ell \geqslant 0$, we have to prove that the operator $A\mathcal{K} = \{h^d\mathcal{K}_{k_1,k_2}^{j_1,j_2}\}$ is positive semi-definite. Let $v=\{v_k^j\}\in L^2(\widehat{G}_h)$. Then, by virtue of (5.30) and (2.14), we obtain

$$\begin{aligned}
&h^{2d}\sum_{(j,k)\in\widehat{G}_h}\sum_{(\ell,m)\in\widehat{G}_h}\mathcal{K}_{k,m}^{j,\ell}v_m^\ell\overline{v_k^j} = h^{2d}\sum_{\substack{kh\in G_h\\ mh\in G_h}}\mathcal{K}_{k,m}^{0,0}v_m^0\overline{v_k^0}\\
&\quad+h^{2d}\widetilde{\sum_{j,k}}\widetilde{\sum_{\ell,m}}\Big((i\partial_{j,h}^+ + A_k^j)\overline{(i\partial_{\ell,h}^+ + A_m^\ell)}\mathcal{K}_{k,m}^{0,0}\Big)(i\partial_{\ell,h}^+ + A_m^\ell)v_m^0\overline{(i\partial_{j,h}^+ + A_k^j)v_k^0}\\
&\quad+h^{2d}\widetilde{\sum_{j,k}}\sum_{mh\in G_h}\Big((i\partial_{jh}^+ + A_k^j)\mathcal{K}_{k,m}^{0,0}v_m^0\overline{(i\partial_{j,h}^+ + A_k^j)v_k^0}\\
&\quad+h^{2d}\sum_{kh\in G_h}\widetilde{\sum_{\ell,m}}\overline{(i\partial_{\ell,h}^+ + A_m^\ell)}\mathcal{K}_{k,m}^{0,0}(i\partial_{\ell,h}^+ + A_m^\ell)v_m^0\overline{v_k^0}\\
&= h^{2d}\sum_{kh\in G_h}\sum_{mh\in G_h}\mathcal{K}_{k,m}^{0,0}\big(1+(i\nabla_m + A_m)^2\big)v_m^0\overline{\big(1+(i\nabla_k + A_k)^2\big)v_k^0}\\
&= \int\limits_{G\times G}\mathcal{K}(x,y)\big((P_h^A)^*v\big)(y)\overline{\big(P_h^A)^*v\big)(x)}\,dxdy \geqslant 0
\end{aligned} \tag{5.42}$$

because the positive semi-definiteness of the operator $\mathcal{K}$ was assumed in (3.5). To prove (5.40), it is enough to note that since the matrix $\{\theta_{k,m}^{j,\ell}\}$ is unitary, we have

$$\sum_{(\ell,m)\in\widehat{G}_h}\big|\theta_{k,m}^{j,\ell}\big|^2 = \sum_{(\ell,m)\in\widehat{G}_h}\theta_{k,m}^{j,\ell}\big(\theta_{m,k}^{\ell,j}\big)^* = 1$$

and therefore $|\theta_{k,m}^{j,\ell}|^2 \leqslant 1$ for each (j,k), (ℓ,m). Thus

$$\big|\operatorname{Re}\big(\theta_{k,m}^{j,\ell}\overline{\theta_{k,m}^{0,\ell}}\big)\big| \leqslant \big|\theta_{k,m}^{j,\ell}\big|\,\big|\theta_{k,m}^{0,\ell}\big| \leqslant 1$$

which implies (5.40) and (5.41). □

Lemma 5.6. *The following bound is valid*:

$$\sum_{(\ell,m)\in\widehat{G}_h}\mu_m^\ell \leqslant C\Big(\int\limits_G\mathcal{K}(x,x)\,dx + \sum_{j=1}^d\int\limits_G\partial_{x_j}\partial_{y_j}\mathcal{K}(x,y)\Big|_{y=x}dx + 1\Big)\,, \tag{5.43}$$

where the constant C does not depend on h and $\mathcal{K}(x,x)$ is the kernel (3.14).

Proof. By virtue of (4.17) and (5.30), we have

$$\sum_{(j,k)\in\widehat{G}_h} \mu_k^j = h^d \sum_{(j,k)\in\widehat{G}_h} \mathcal{K}_{k,k}^{j,j} = I_1 + I_2\,, \tag{5.44}$$

where

$$I_1 = h^d \sum_k \mathcal{K}_{k,k}^{0,0}, \qquad I_2 = h^d \widetilde{\sum_{jk}} \mathcal{K}_{k,k}^{j,j}\,. \tag{5.45}$$

By virtue of Lemma 4.1, we have

$$I_1 = \sum_{jh\in G_h} \mu_j \leqslant C \int_G \mathcal{K}(x,x)\,dx\,. \tag{5.46}$$

From (5.30), we have

$$I_2 = h^d \widetilde{\sum_{(j,k)}} \Big(\big(i\partial_{j,h}^+ + A_k^j\big) \overline{\big(i\partial_{j,h}^+ + A_m^j\big)} \mathcal{K}_{k,m}^{0,0} \Big)\Big|_{m=k}\,. \tag{5.47}$$

Note that, in fact, the summation in (5.47) is performed over (j,k) such that $kh \in G_h$ and $(k+e_j)h \in G_h$ because, by virtue of (5.23) and (2.15), all other summands in (5.47) vanish. Therefore, taking into account that $\mathcal{K}_{k,m}^{0,0} = \mathcal{K}_{km}$ is defined by (4.17) and after changing variables in the integrals (4.17) in the appropriate terms connected with $i\partial_{j,h}^+ \mathcal{K}_{k,m}^{0,0}$, we obtain

$$I_2 = \int_{G^0(h)} \big(i\partial_{j,h}^-(x) + A^j(x)\big) \overline{\big(i\partial_{j,h}^-(y) + A^j(y)\big)} \mathcal{K}(x,y)\big|_{y=x} dx + J\,. \tag{5.48}$$

Here, $\partial_{j,h}^-(x)\mathcal{K}(x,y) = \big(\mathcal{K}(x,y) - \mathcal{K}(x-e_jh,y)\big)/h$, $\partial_{j,h}^-(y)\mathcal{K}(x,y) = \big(\mathcal{K}(x,y) - \mathcal{K}(x,y-e_jh)\big)/h$, and $G^0(h) = \sum_{kh\in G_h^0} Q_k$, where $G_h^0 = G_h \backslash \partial^- G_h$ (see Definition 2.1 in Sect. 2.2), and Q_k are the sets are defined in (4.1). The term J arises because of the summation of some terms connected with $\mathcal{K}_{k,k}^0$ with $kh \in \partial^- G_h$. It is easy to see that

$$|J| \leqslant C\,, \tag{5.49}$$

where C does not depend on h. Using the representation

$$\mathcal{K}(x,y) = \sum_r \lambda_r e_r(x) e_r(y),$$

we obtain from (5.48) and (5.49)

$$
\begin{aligned}
|I_2| &\leqslant \sum_r \lambda_r \int\limits_{G^0(h)} |\big(i\nabla_h^- + A(x)\big)e_r(x)|^2 \, dx + |J| \\
&\leqslant C\Big(1 + \sum_r \lambda_r \int\limits_{G^0(h)} \big(|\nabla_h^- e(x)|^2 + |e(x)|^2\big) \, dx\Big) \\
&\leqslant C\Big(1 + \sum_r \lambda_r \int\limits_G \big(|\nabla e(x)|^2 + |e(x)|^2\big) \, dx\Big) ,
\end{aligned}
\tag{5.50}
$$

where the last inequality estimating the finite difference by the derivative can be obtained by using the elementary equality $u(x+h) - u(x) = \int\limits_x^{x+h} u'(y) \, dy$. The bounds (5.46) and (5.50) imply (5.43). □

5.5 A priori estimates for $\Delta_h \psi_k$

In addition to (5.1) and (5.5), we introduce the notation

$$
\|\Delta_h \psi\|_{L^{2,h}}^2 = \sum_{kh \in G_h} |\Delta_h \psi_k|^2 , \tag{5.51}
$$

where the values ψ_k with $kh \in \partial G_h^+$ (we need these values to define $\Delta_h \psi_k$) are defined with the help of (2.23). We will also need the following estimate.

Theorem 5.7. *Let a random process $\boldsymbol{\psi}(t) = \{\psi_k\}$ have the stochastic differential* (4.30). *Then $\boldsymbol{\psi}$ satisfies the bound*

$$
\begin{aligned}
&E\Big(\|\nabla_h^+ \boldsymbol{\psi}(t)\|_{L^{2,h}}^2 + \int\limits_0^t \|\Delta_h \boldsymbol{\psi}(\tau)\|_{L^{2,h}}^2 \, d\tau\Big) \\
&\qquad \leqslant E(\|\nabla_h^+ \boldsymbol{\psi}_0\|_{L^{2,h}}^2) + C_3 e^{Ct}\Big(E(\|\boldsymbol{\psi}_0\|_{L^{4,h}}^4) + 1\Big) ,
\end{aligned}
\tag{5.52}
$$

with constants C_3 and C independent of h.

Proof. We apply the Ito formula to the function $u(\boldsymbol{\psi}) = h^d \widetilde{\sum_{jk}} |(i\partial_{j,h}^+ + A_k^j)\psi|^2$ to obtain

$$
du(\boldsymbol{\psi}) = h^d \widetilde{\sum_{j,k}} \big((i\partial_{j,h}^+ + A_k^j) d\psi_k, \overline{(i\partial_{j,h}^+ + A_k^j)\psi_k}\big)
$$

$$
\begin{aligned}
&+h^d\widetilde{\sum_{j,k}}\big((i\partial_{j,h}^{+}+A_k^j)\psi_k,\overline{(i\partial_{j,h}^{+}+A_k^j)d\psi_k}\big)\\
&+\frac{h^d}{2}\widetilde{\sum_{j,k}}\big((i\partial_{j,h}^{+}+A_k^j)d\psi_k,\overline{(i\partial_{j,h}^{+}+A_k^j)d\psi_k}\big) \qquad (5.53)\\
&=\mathrm{I}+\bar{\mathrm{I}}+\mathrm{II}\,,
\end{aligned}
$$

where I, $\bar{\mathrm{I}}$, and II are the first, second and third terms of the right-hand side of (5.53) respectively. Applying (2.14) and (4.30) and using the notation $(i\nabla_h^- + A_k, i\nabla_h^+ + A_k) = (i\nabla_h + A_k)^2$, we obtain

$$
\begin{aligned}
\mathrm{I}=h^d\sum_k d\psi_k\overline{(i\nabla+A_k)^2\psi_k}&=\Big\{-h^d\sum_k|(i\nabla+A_k)^2\psi_k|^2\\
&+h^d\widetilde{\sum_{j,k}}|(i\partial_{j,h}^{+}+A_k^j)\psi_k|^2-\big((i\partial_{j,h}^{+}+A_k^j)(|\psi_k|^2\psi_k),\overline{(i\partial_{j,h}^{+}+A_k^j)\psi_k}\big)\Big\}dt\\
&+h^d\sum_k\Big\{\widehat{r}[\psi_k]dW_k\overline{(i\nabla_h+A_k)^2\psi_k}\Big\}\,.
\end{aligned}
\qquad (5.54)
$$

Since $\bar{\mathrm{I}}$ is the complex conjugate to I, we obtain from (5.54) that

$$
\begin{aligned}
\mathrm{I}+\bar{\mathrm{I}}=-2h^d\sum_{kh\in G_h}|(i\nabla+A_k)^2\psi_k|^2dt+\widetilde{\sum_{j,k}}C_{jk}(\psi_k,\partial_{j,h}^{+}\psi_k)dt\\
+h^d\sum_{kh\in G_h}\Big\{\widehat{r}[\psi_k]dW_k\overline{(i\nabla_h+A_k)^2\psi_k}\Big\}\,,
\end{aligned}
\qquad (5.55)
$$

where $C_{jk}(\psi_k,\partial_{j,h}^{+}\psi_k)$ admits the bound

$$
|C_{jk}(\psi_k,\partial_{j,h}^{+}\psi_k)|\leqslant C\Big(|\psi_k|^6+|\partial_{j,h}^{+}\psi_k|^2|\psi_k|^2+|\partial_{j,h}^{+}\psi_k|^2+1\Big) \qquad (5.56)
$$

with constant C independent of j,k,h.

Let us consider the term II. Applying (2.14), (4.30), and using the notation $\mathcal{D}_k dt$ for the term with the differential dt in (4.30) and taking into account (4.36) and (4.25), we have

$$
2\mathrm{II}=h^d\widetilde{\sum_{jk}}\Big((i\partial_{jh}^{+}+A_k^j)(\mathcal{D}_k dt+\widehat{r}[\psi_k]dW_k)
$$

$$\overline{(i\partial_{jh}^{+} + A_k^j)(\mathcal{D}_k dt + \widehat{r}[\psi_k]dW_k)}\Big)$$

$$= h^d \widetilde{\sum_{jk}} \Big(\big(i\partial_{jh}^{+} + A_k^j\big)\big(\widehat{r}[\psi_k]dW_k\big) \overline{\big(i\partial_{jh}^{+} + A_k^j\big)\big(\widehat{r}[\psi_k]dW_k\big)} \Big) \,. \tag{5.57}$$

Using the equality

$$a_{k+1}b_{k+1} - a_k b_k = a_{k+1}(b_{k+1} - b_k) + (a_{k+1} - a_k)b_k$$

and the definitions (3.20) and (3.21), we obtain

$$(i\partial_{j,h}^{+} + A_k^j)(\widehat{r}[\psi_k]dW_k)$$

$$= \widehat{r}[i\overline{\psi}_{k+e_j}](i\partial_{j,h}^{+} + A_k^i)dW_k + (i\partial_{j,h}^{+}\widehat{r}[\psi_k] - A_k^j\widehat{r}[i\overline{\psi}_{k+e_j}] + A_k^j\widehat{r}[\psi_k])dW_k \,. \tag{5.58}$$

Using (5.58) and the definition (5.28) of the scalar Wiener process $AW_k^j(t)$, we obtain, from (5.57),

$$2II = h^d \widetilde{\sum_{j,k}} \Big| \widehat{r}[i\overline{\psi}_{k+e_j}]dAW_k^j + \big(i\partial_{j,h}^{+}\widehat{r}[\psi_k] - A_k^j\widehat{r}[i\overline{\psi}_{k+e_j}] + A_k^j\widehat{r}[\psi_k]\big)dAW_k^0 \Big|^2$$

$$= J_1 + J_2 + J_3 \,, \tag{5.59}$$

where

$$J_1 = h^d \widetilde{\sum_{j,k}} \widehat{r}[i\overline{\psi}_{k+e_j}]dAW_k^j \; \overline{\widehat{r}[i\overline{\psi}_{k+e_j}]dAW_k^j} \,, \tag{5.60}$$

$$J_2 = h^d \widetilde{\sum_{j,k}} 2\mathrm{Re} \Big\{ \widehat{r}[i\overline{\psi}_{k+e_j}]dAW_k^j \overline{\big(i\partial_{j,h}^{+}\widehat{r}[\psi_k] - A_k^j\widehat{r}[i\overline{\psi}_{k+e_j}] + A_k^j\widehat{r}[\psi_k]\big)dAW_k^0} \Big\} \,, \tag{5.61}$$

and

$$J_3 = h^d \widetilde{\sum_{j,k}} \big| \big(i\partial_{j,h}^{+}\widehat{r}[\psi_k] - A_k^j\widehat{r}[i\overline{\psi}_{k+e_j}] + A_k^j\widehat{r}[\psi_k]\big)dAW_k^0 \big|^2 \,. \tag{5.62}$$

By virtue of (3.20), (3.21), and (5.35), we obtain from (5.60) that

$$J_1 = h^d \widetilde{\sum_{j,k}} \big(r^2(\mathrm{Im}\,\psi_{k+e_j}) + r^2(\mathrm{Re}\,\psi_{k+e_j})\big) \sum_{(\ell,m)\in\widehat{G}_h} \mu_m^\ell |\theta_{k,m}^{j,\ell}|^2 dt \,. \tag{5.63}$$

Similarly, (5.38) and (5.61) imply

$$\begin{aligned}
J_2 = 2h^d\widetilde{\sum_{j,k}}\Big\{ & r(\mathrm{Im}\,\psi_{k+e_j})d\mathrm{Re}\,AW_k^j\big(-(\partial_{j,h}^+ r(\mathrm{Im}\,\psi_k))d\mathrm{Im}\,AW_k^0 \\
& + A_k^j(r(\mathrm{Re}\,\psi_k) - r(\mathrm{Im}\,\psi_{k+e_j}))d\mathrm{Re}\,AW_k^0\big) + r(\mathrm{Re}\,\psi_{k+e_j})d\mathrm{Im}\,AW_k^j \\
& \cdot\Big(-\partial_{j,h}^+ r(\mathrm{Re}\,\psi_k)d\mathrm{Re}\,AW_k^0 + A_k^j(r(\mathrm{Im}\,\psi_k) - r(\mathrm{Re}\,\psi_{k+e_j}))d\mathrm{Im}\,AW_k^0\Big)\Big\} \\
= 2h^d\widetilde{\sum_{j,k}}\Big\{ & A_k^j\Big(r(\mathrm{Im}\,\psi_{k+e_j})r(\mathrm{Re}\,\psi_k) - r^2(\mathrm{Im}\,\psi_{k+e_j}) \\
& + r(\mathrm{Re}\,\psi_{k+e_j})r(\mathrm{Im}\,\psi_k) - r^2(\mathrm{Re}\,\psi_{k+e_j})\Big)\sum_{(\ell,m)\in\widehat{G_h}}\mu_m^\ell \mathrm{Re}\,(\theta_{k,m}^{j,\ell}\overline{\theta_{k,m}^{0,\ell}})\Big\}dt\,.
\end{aligned} \tag{5.64}$$

In addition, by (5.35) and (5.62), we have

$$\begin{aligned}
J_3 = h^d\widetilde{\sum_{j,k}}\Big\{ & \big(-\partial_{j,h}^+ r(\mathrm{Im}\,\psi_k)d\mathrm{Im}\,AW_k^0 \\
& + A_k^j\big(r(\mathrm{Re}\,\psi_k) - r(\mathrm{Im}\,\psi_{k+e_j})\big)d\mathrm{Re}\,AW_k^0\big)^2 \\
& + \big(\partial_{j,h}^+ r(\mathrm{Re}\,\psi_k)d\mathrm{Re}\,AW_k^0 + A_k^i(r(\mathrm{Im}\,\psi_k) - r(\mathrm{Re}\,\psi_{k+e_j}))d\mathrm{Im}\,AW_k^0\big)^2\Big\} \\
= h^d\widetilde{\sum_{j,k}}\Big\{ & \big(\partial_{j,h}^+ r(\mathrm{Im}\,\psi_k)\big)^2 + \big(\partial_{j,h}^+ r(\mathrm{Re}\,\psi_k)\big)^2 \\
& + (A_k^j)^2\big((r(\mathrm{Re}\,\psi_k) - r(\mathrm{Im}\,\psi_{k+e_j}))^2 \\
& + (r(\mathrm{Im}\,\psi_k) - r(\mathrm{Re}\,\psi_{k+e_j}))^2\Big\}\sum_{\ell,m}\mu_m^\ell|\theta_{k,m}^{0,\ell}|^2 dt\,.
\end{aligned} \tag{5.65}$$

Now relations (5.59), (5.63), (5.64), and (5.65) and Lemmas 5.5 and 5.6 imply that

$$II = \widetilde{\sum_{j,k}} d_{jk}(\psi_k, \psi_{k+e_j}.\partial_{j,h}^+\psi_k)dt\,, \tag{5.66}$$

where

$$|d_{jk}(\psi_k, \psi_{k+e_j}, \partial_{j,h}^+\psi_k)| \leqslant C\big(1 + |\psi_k|^2 + |\psi_{k+e_j}|^2 + |\partial_{j,h}^+\psi_k|^2\big) \tag{5.67}$$

with constant C independent of j, k, h.

Relations (5.53), (5.55), and (5.66) give

$$
\begin{aligned}
d\big(h^d\widetilde{\sum_{j,k}}|(i\partial^+_{j,h}+A^j_k)\psi_k|^2\big) &+ 2h^d\sum_{kh\in G_h}|(i\nabla^+_h+A_k)^2\psi_k|^2 \\
&= \widetilde{\sum_{j,k}}\big(C_{jk}(\psi_k,\partial^+_{j,h}\psi_k)+d_{jk}(\psi_k,\psi_{k+e_j},\partial^+_{j,h}\psi_k)\big)dt \\
&\quad + h^d\sum_{khG_h}\left\{\widehat{r}[\psi_k]dW_k\overline{(i\nabla_h+A_k)^2\psi_k}\right\}.
\end{aligned}
\tag{5.68}
$$

Writing the differential Ito formula (5.68) in integral form and applying the mathematical expectation, we obtain

$$
\begin{aligned}
&E\Big(\|(i\nabla^+_h+\mathbf{A})\boldsymbol{\psi}(t)\|^2_{L^{2,h}}+2\int_0^t\|(i\nabla_h+A)^2\boldsymbol{\psi}(\tau)\|^2_{L^{2,h}}\,d\tau\Big) \\
&= E\Big(\int_0^t h^d\widetilde{\sum_{j,k}}\Big\{(C_{jk}(\psi_k,\partial^+_{j,h}\psi_k)+d_{jk}(\psi_k,\psi_{k+e_j},\partial^+_{j,h}\psi_k)\Big\}\,d\tau\Big) \\
&\quad +E\big(\|(i\nabla^+_h+A)\boldsymbol{\psi}_0\|^2_{L^{2,h}}\big),
\end{aligned}
\tag{5.69}
$$

where $\mathbf{A}=\{A^j_k, kh\in G_h\cup\partial G^+_h(-j)\}$. Doing a simple transformation with the first term on the left-hand side of (5.69), applying the bounds (5.56) and (5.67) to the right-hand side of (5.69), and then applying to the result the inequalities (5.6) and (5.13) results in (5.52). □

6 Existence Theorem for Approximations

The aim of this section is to prove an existence theorem for the stochastic system (4.30), (2.12), and (2.23). First, we recall a well-known existence theorem for stochastic equations which we will use in our analysis.

6.1 Preliminaries

Recall the existence theorem for stochastic equations proved in [24, pp. 165-173]. Let $W(t)=W(t,\omega)$ be a d_1-dimensional real-valued Wiener process on (Ω,Σ,m), $\Sigma_t\subset\Sigma$ be the increasing filtration (see Sect. 4.3) complete with respect to σ-algebra m-measurable sets Σ_m and coordinated with $W(t)$, i.e., $(W(t),\Sigma_t)$ is a Wiener process.

We consider the stochastic equation

$$d\xi(s) = \sigma(s,\xi(s))dW(s) + b(s,\xi(s))ds, \quad s \geqslant t, \quad \text{and} \quad \xi(t) = \xi^0(t)\,, \tag{6.1}$$

where $t \geqslant 0$ is fixed and $\xi^0(t)$ is a Σ_t-measurable d-dimensional vector. The integral form of (6.1) is

$$\xi(s) = \xi^0(t) + \int_t^s \sigma(r,\xi(r))\, dW(r) + \int_t^s b(r,\xi(r))\, dr\,. \tag{6.2}$$

By the solution of (6.2) we mean a d-dimensional process $\xi(s) = \xi(s,\omega)$ that is Σ_s-measurable in ω for all $s \geqslant t$, is continuous in s and defined for $\omega \in \Omega$ and $s \in (t,\infty)$, and satisfies (6.2) for all $s \in [t,\infty)$ almost everywhere. Additionally, $\sigma(s,x) \in L^{2,\mathrm{loc}}$ (in s), $b(s,x) \in L^{1,\mathrm{loc}}$ (in s), and they are defined on $\Omega \times (t,\infty)$ for all $x \in \mathbb{R}^d$ and have values in $(d \times d_1)$-matrices and in $\mathbb{R}^d$ correspondingly. We assume that σ and b are continuous on x for all (s,ω) and, for each $T, R \in [0,\infty)$ and $\omega \in \Omega$, the bound

$$\int_0^T \sup_{|x|\leqslant R} \Big[\|\sigma(s,x)\|^2 + |b(s,x)|\Big] ds < \infty \tag{6.3}$$

holds.

Theorem 6.1 (see [24, p. 166]). *Let the following conditions hold.*

(i) *Lipschitz condition: For any $R > 0$ there exists a function $K_r(R) > 0$ belonging to $L^{1,\mathrm{loc}}$ as a function of (ω, r) such that for all $|x|, |y| \leqslant R$, $r > 0$, and $\omega \in \Omega$,*

$$2\big(x - y, b(r,x) - b(r,y)\big) + \|\sigma(r,x) - \sigma(r,y)\|^2 \leqslant K_r(R)|x-y|^2\,. \tag{6.4}$$

(ii) *Growth condition: For all $x \in E^d$, $r > 0$, and $\omega \in \Omega$*

$$2(x, b(r,x)) + \|\sigma(r,x)\|^2 \leqslant K_r(1)(1 + |x|^2)\,. \tag{6.5}$$

Then the stochastic equation (6.1) *has a solution and any two solutions are identical.*

6.2 Bounded approximations

Theorem 6.1 is not applicable to the problem (4.30), (2.12), and (2.23) because (4.30) has the term $|\psi(t)|^2\psi(t)$ that does not satisfy the growth condition (6.5). Moreover, (4.30) holds for $k \in G_h$; due to the boundary conditions (2.23), the function $k \to \psi_k(\cdot)$ should be defined for $k \in \partial G_h^+$ as well.

The requirement that ψ_k is defined for $kh \in \partial G_h^+$ does not bring any difficulties because it is enough for us to put into (4.30) an expression for ψ_k

with $kh \in \partial G_h^+$ given in (2.23) and after that to solve the Cauchy problem (4.30) and (2.12).

Temporarily, we modify (4.30) to an equation that satisfies the conditions of Theorem 6.1. To this end, we introduce the function $\gamma_N \in C^\infty(0,\infty)$ such that

$$\gamma_N(t) = \begin{cases} t, & t \in [0, N], \\ \text{increases monotonically}, & t \in (N, N+1), \\ N+1, & t \geqslant N+1, \end{cases} \tag{6.6}$$

and consider the system

$$\begin{aligned} d\psi_k(t) + \Big\{(i\nabla_h + A_k)^2\psi_k(t) - \psi_k(t) + \gamma_N(|\psi_k(t)|^2)\psi_k(t)\Big\}dt \\ = \widehat{r}[\psi_k(t)]dW_k(t) \end{aligned} \tag{6.7}$$

instead of (4.30). We consider the problem (6.7) and (2.12). In this problem, the functions $\psi_k(t)$ and $W_k(t)$ are complex-valued. (Recall that $\widehat{r}[\psi_k(t)]W_k(t)$ in (6.7) is understood in the sense of (4.31).) If we introduce the real and imaginary parts of these functions, substitute them into (6.2), and separate the real and imaginary parts of the resulting equations, we obtain a system that satisfies all the conditions of Theorem 6.1. Therefore, the following theorem holds.

Theorem 6.2. *The problem* (6.7) *and* (2.12) *has a solution, and any two solutions with identical initial data* (2.12) *are identical.*

We apply to (6.7) the same arguments that were applied to (4.30) that led us to the bound (5.6). Then we obtain the following bounds for the solution $\psi_k(t) \equiv \psi_k^N(t)$ of (6.7) and (2.12):

$$\begin{aligned} \|\boldsymbol{\psi}^N(t)\|^2_{L^{2,h}} + 2\int_0^t \Big[\|(i\nabla_h + \mathbf{A})\psi^N\|^2_{L^{2,h}} \\ + h^d \sum_k \gamma_N\big(|\psi_k^N(s)|^2\big)|\psi_k^N(s)|^2\Big]ds \\ -2\int_0^t h^d \sum_k \mathrm{Re}\left(\overline{\psi_k^N}\widehat{r}[\psi_k^N]dW_k\right) \\ = \int_0^t h^d \sum_k \left(2|\psi_k^N|^2 + \sum_j |\Theta_{kj}|^2\mu_j|\widehat{r}[\psi_k^N]|^2\right) ds + \|\boldsymbol{\psi}_0\|^2_{L^{2,h}} . \end{aligned} \tag{6.8}$$

This is the analogue of (5.9); after some transformations, we obtain the final inequality

$$E\|\boldsymbol{\psi}^N(t)\|^2_{L^{2,h}} + E\int_0^t \Big(\|\nabla_h^+\boldsymbol{\psi}^N\|^2_{L^{2,h}} + h^d\sum_k \gamma_N(|\psi_k^N(\tau)|^2)|\psi_k^N(\tau)|^2\Big)d\tau \leqslant C_2 e^{C_1 t}\left(E\|\boldsymbol{\psi}_0\|^2_{L^{2,h}} + 1\right) . \tag{6.9}$$

6.3 Solvability of the discrete stochastic system

Recall (see [26, p. 54]) that a random variable $\tau = \tau(\omega)$, $\omega \in \Omega$, that takes values in $[0,\infty]$ is called the *stopping* time (relative to Σ_t) if $\{\omega : \tau(\omega) > t\} \in \Sigma_t$ for every $t \in (0,\infty)$.

Let $M < N$. We introduce the (random) Markov moment

$$\tau_M(\omega) = \begin{cases} \inf\{\tau > 0 \,:\, \|\boldsymbol{\psi}^N(\tau,\omega)\|^2_{L^{2,h}} \geqslant M\} \text{ for } \omega \in \Omega \\ \infty \quad \text{if } \|\boldsymbol{\psi}^N(\tau,\omega)\|_{L^{2,h}} \leqslant M \quad \forall\, \tau > 0\,. \end{cases} \tag{6.10}$$

Clearly, $\tau_M(\omega)$ is the stopping time. For fixed $t > 0$ we set $t_M = t_M(\omega) = t \wedge \tau_M(\omega)$, which is the stopping time as well.

We substitute $t = t_M(\omega)$ with $M/h^d < N$ into (6.8) and obtain

$$\|\psi_k^N(t_M))\|^2_{L^{2,h}} + 2\int_0^{t_M}\Big(\|(i\nabla_h^+ + A)\boldsymbol{\psi}^N\|^2_{L^{2,h}} + \|\boldsymbol{\psi}^N\|^4_{L^{4,h}}\Big)\,dt - 2\int_0^{t_M}\sum_k \mathrm{Re}\,\Big(\overline{\psi_k^N}\widehat{r}[\psi_k^N]dW_k\Big) = \int_0^{t_M}\sum_k\left(2|\psi_k^N|^2 + \mu_k|\widehat{r}[\psi_k^N]|^2\right)dt + \|\boldsymbol{\psi}_0\|^2_{L^{2,h}}\,. \tag{6.11}$$

We note that we have changed the term $h^d\sum_k \gamma_N(|\psi_k^N(s)|^2)|\psi_k^N(s)|^2$ to $\|\boldsymbol{\psi}^N(s)\|^4_{L^{2,h}}$ because, for $s < t_M$, $\|\boldsymbol{\psi}^N(s)\|^2_{L^{2,h}} \leqslant M/h^d < N$ and therefore for every k, $|\psi_k^N(s)|^2 < N$. This justifies the aforementioned change. Therefore, repeating the derivation of (6.9) from (6.8), we find that (6.11) implies the bound

$$E\|\boldsymbol{\psi}^N(t_M)\|^2_{L^{2,h}} + E\int_0^{t_M}\Big(\|\nabla_h\boldsymbol{\psi}^N(s)\|^2_{L^{2,h}} + \|\boldsymbol{\psi}^N(s)\|^4_{L^{4,h}}\Big)\,ds \leqslant C_2 e^{C_1 t}\left(E\|\boldsymbol{\psi}_0\|^2_{L^{2,h}} + 1\right)\,, \tag{6.12}$$

where C_1 and C do not depend on N.

Taking into account the definition of $t_M(\omega)$ and the arguments written before (6.12), we see that $\boldsymbol{\psi}^N(s)$ satisfies not only (6.7), but also the equation

$$\begin{aligned}\psi_k^N(t_M) + \int_0^{t_M} \left[(i\nabla_h + A_k)^2\psi_k^N(s) - \psi_k^N(s) + |\psi_k^N(s)|^2\psi_k^N(s)\right] ds \\ = \int_0^{t_M} r_k[\boldsymbol{\psi}^n(s)]dW_k^N(s) + \boldsymbol{\psi}_0 \,.\end{aligned} \tag{6.13}$$

It is clear that for each N_1 satisfying $M < N < N_1$ the vector-valued function $\boldsymbol{\psi}^{N_1}(s) = \boldsymbol{\psi}^{N_1}(s,\omega)$ (that evidently exists) satisfies (6.13) as well for almost all $\omega \in \Omega$ and $s \in (0, t_M(\omega))$. This implies that for almost all $\omega \in \Omega$

$$\psi_k^N(s,\omega) = \psi_k^{N_1}(s,\omega) \quad \forall\, kh \in G_h \text{ for } s \in (0, t_M(\omega)) \;. \tag{6.14}$$

Indeed, $\boldsymbol{\psi}^N(s)$, as well as $\boldsymbol{\psi}^{N_1}(s)$, satisfies (6.13) in which the term $|\psi_k^N(s)|^2\psi_k^N(s)$ is changed to $\gamma_{N_1}\left(|\psi_k^N(s)|^2\right)\psi_k^N(s)$. But for this equation, all solutions are indistinguishable.

The equality (6.14) permits us to define the vector-valued function $\boldsymbol{\psi}(s,\omega)$ as follows:

$$\psi_k(s,\omega) = \psi_k^N(s,\omega), \quad kh \in G_h \quad \forall\, N > M/h^d,\; s \in (0, t_M(\omega))\,. \tag{6.15}$$

By virtue of (6.12), the function $\boldsymbol{\psi}(s,\omega)$ defined in (6.15) satisfies

$$\begin{aligned}E\|\boldsymbol{\psi}(t_M)\|^2_{L^{2,h}} + E\int_0^{t_M}\left(\|\nabla_h\boldsymbol{\psi}(s)\|^2_{L^{2,h}} + \|\boldsymbol{\psi}(s)\|^4_{L^{4,h}}\right) ds \\ \leqslant C_2 e^{C_1 t} E\left(\|\boldsymbol{\psi}_0\|^2_{L^{2,h}} + 1\right)\end{aligned} \tag{6.16}$$

and the inequality in (6.16) is true since, by definition, $t_M \leqslant t$.

Lemma 6.3. *For almost all $\omega \in \Omega$, $t_M \nearrow t$ as $M \to \infty$.*

Proof. The definitions of τ_M and t_M imply that for each $M_1 > M$ the inequalities

$$\tau_M \leqslant \tau_{M_1}, \quad t_M \leqslant t_{M_1} \leqslant t \tag{6.17}$$

hold. Then, by the monotone convergence theorem, there exists $t_\infty(\omega) \leqslant t$ and $\tau_\infty(\omega) \leqslant \infty$ such that $\tau_M(\omega) \to \tau_\infty(\omega)$ and $t_M(\omega) \to t_\infty(\omega) \leqslant t$ as $M \to \infty$ for almost any $\omega \in \Omega$. Suppose that there exists a set $b \subset \Sigma$ satisfying $m(b) > 0$ such that $t_\infty(\omega) < t$ for all $\omega \in b$. This means that for each $M > 0$, $\tau_M(\omega) = t_M(\omega) < t$ for $\omega \in b$ and therefore $\tau_\infty(\omega) = t_\infty(\omega)$, $\omega \in b$. The definition (6.15) of $\psi_k(s,\omega)$ and (6.8) imply that for almost all $\omega \in \Omega$, $\|\boldsymbol{\psi}(s)\|^2_{L^{2,h}}$ is continuous for $s \in (0, \tau_\infty(\omega))$. Due to the continuity for almost all $\omega \in \Omega$, $\tau_M(\omega) < \tau_{M+1}(\omega) < \cdots < \tau_{M+K}(\omega) < \cdots$. Recall that for $\omega \in b$, $\tau_M(\omega) \to \tau_\infty(\omega) < t$. Hence, by (6.10), we obtain

$$\int_b \|\boldsymbol{\psi}(\tau_M(\omega),\omega)\|^2_{L^{2,h}} m(d\omega) \geqslant (M-1)\int_b m(d\omega) \to \infty \quad \text{as } M \to \infty \ . \tag{6.18}$$

Since for $\omega \in b$, $\tau_M(\omega) = t_M(\omega)$, we obtain, by (6.16),

$$\int_b \|\boldsymbol{\psi}(\tau_M(\omega),\omega)\|^2_{L^{2,h}} m(d\omega) \leqslant E\|\boldsymbol{\psi}(t_M)\|^2_{L^{2,h}} \leqslant C_1 e^{Ct}\|\boldsymbol{\psi}_0\|^2_{L^{2,h}} \tag{6.19}$$

for $M \to \infty$. But (6.19) contradicts (6.18) and therefore the proof is complete. □

By Lemma 6.3, (6.10), and the fact that $t_M = t \wedge \tau_M$ for almost all $\omega \in \Omega$ the function

$$G(t_M,\omega) = \|\boldsymbol{\psi}(t_M(\omega),\omega\|^2_{L^{2,h}} + \int_0^{t_M(\omega)} \left(\|\nabla_h^+ \boldsymbol{\psi}(s)\|^2_{L^{2,h}} + \|\boldsymbol{\psi}(s)\|^4_{L^{4,h}}\right) ds$$

increases monotonically as $M \to \infty$. By (6.16) and the Beppo Levi theorem, the function $G(t,\omega)$ is well-defined for a nonrandom value t. Hence,

$$E\|\boldsymbol{\psi}(t)\|^2_{L^{2,h}} + E\int_0^t \left(\|\nabla_h^+ \boldsymbol{\psi}(s)\|^2_{L^{2,h}} + \|\boldsymbol{\psi}(s)\|^4_{L^{4,h}}\right) ds \leqslant C_2 e^{C_1 t}\|\boldsymbol{\psi}\|^2_{L^{2,h}} \ . \tag{6.20}$$

Therefore, the function $\psi_k(s,\omega)$ defined in (6.15) can be extended up to a function defined for every nonrandom $t > 0$, and this function satisfies (4.32) and is equivalent to (4.30). Uniqueness of the obtained solution of (4.32) follows from (6.15) and the uniqueness of $\psi_k^N(s,w)$. Applying the arguments of Sects. 5.3 and 5.5 to $\psi_k(s,\omega)$, we find that $\psi_k(s,\omega)$ satisfies the estimates (5.13) and (5.52).

Thus, we have proved the following theorem.

Theorem 6.4. *There exists a continuous $\Sigma_{h,t}$-adapted random process $\{\boldsymbol{\psi}(t,\omega)\} = \{\psi_k(t,\omega), kh \in G_h\}$ given for $t \geqslant 0$ and such that* (4.32) *holds for all $t \geqslant 0$ with probability one. This process $\boldsymbol{\psi}(t,\omega)$ satisfies the inequalities* (5.6), (5.13), *and* (5.52). *The process $\boldsymbol{\psi}$ that satisfies the aforementioned properties is unique.*

Definition 6.5. The random process $\{\boldsymbol{\psi}(t,\omega)\} = \{\psi_k(t,\omega), kh \in G_h\}$ that satisfies all the properties mentioned in Theorem 6.4 is called the *strong solution* of (4.30), (2.12), and (2.23) or (what is equivalent) the strong solution of (4.32).

To prove the solvability of the stochastic problem for the Ginzburg–Landau equation, we need certain additional bounds for the strong solution of (4.32). These bounds will be proved in the next section.

7 Smoothness of the Strong Solution with respect to t

We establish two estimates for the solution of the problem (4.30), (2.23), and (2.12). Specifically, we estimate the mean maximum and the mean modulus of continuity. In both estimates we follow [44, pp. 352-360].

7.1 Estimate of the mean maximum

In this subsection, we present a result for the mean maximum of the solution of the problem (4.30) and (2.12).

Proposition 7.1. *Let $\boldsymbol{\psi}(t)$ be the strong solution of* (4.30) *and* (2.12). *Then*

$$E(\|\boldsymbol{\psi}(t)\|_{L^\infty(0,T;L^{2,h})}) \leqslant C(T) < \infty \quad \text{for any } T > 0, \tag{7.1}$$

where $C(T)$ does not depend on h.

Proof. We obtain from (5.8) and (5.9) that

$$\begin{aligned}\|\boldsymbol{\psi}(t)\|^2_{L^{2,h}} \leqslant \|\boldsymbol{\psi}_0\|^2_{L^{2,h}} + \int_0^t 2\left(\|\boldsymbol{\psi}\|^2_{L^{2,h}} + 1\right)\sum_j \mu_j \, d\tau \\ +2\int_0^t \sum \operatorname{Re}\,(\overline{\psi}_k \widehat{r}[\psi_k] dW_k)\end{aligned} \tag{7.2}$$

and from this estimate, along with the Gronwall inequality, we obtain

$$\begin{aligned}\|\boldsymbol{\psi}(t)\|^2_{L^{2,h}} \leqslant \|\boldsymbol{\psi}_0\|^2_{L^{2,h}} e^{2t\sum\limits_j \mu_j} + C\int_0^t e^{2\sum\limits_j \mu_j (t-\tau)} \Big(\operatorname{Re}\sum_k (\overline{\psi}_k \widehat{r}[\psi_k] dW_k)(\tau) \\ + \sum_j \mu_j \tau\Big)\mu_j \, d\tau \,.\end{aligned} \tag{7.3}$$

Multiplying both sides of (7.3) by $e^{-2t\sum\limits_j \mu_j}$ and taking the maximum over $t \in [0,T]$, we obtain

$$\sup_{t\in[0,T]}\left(e^{-2t\sum\limits_j \mu_j}\|\boldsymbol{\psi}(t)\|^2_{L^{2,h}}\right)\leqslant \|\boldsymbol{\psi}_0\|^2_{L^{2,h}}+C_2+\sup_{t\in[0,T]}\|\mathbf{M}\|\,, \tag{7.4}$$

where $\mathbf{M}=(M_k(t), kh\in G_h)$, and

$$M_k(t)=C\int\limits_0^t e^{-2\tau\sum\limits_j \mu_j}\operatorname{Re}\big(\overline{\psi_k(\tau)}r[\psi_k]\big)dW_k(\tau)\,.$$

The process $M_k(t)$ is a martingale with respect to the flirtation Σ_t (see [44, p. 353]). This, due to the Birkholder-Gaudi inequality, implies

$$E\sup_{[0,T]}|M_k(t)|\leqslant [C(T)]^{\frac12};$$

see [44, p. 353]. Therefore, taking the mathematical expectation of both sides of (7.4), we obtain (7.1). □

Similarly to Proposition 7.1, using (5.56), (5.67), and (5.68) (instead of (5.8) and (5.9)), one can prove Proposition 7.1'. Let $\psi(t)$ be the strong solution of (4.29) and (2.12). Then

$$E\|\nabla_h^+\psi\|_{L^\infty(0,T;L^{2,h})}\leqslant C(T)<\infty \quad \text{for any } T>0\,, \tag{7.5}$$

where $C(T)$ does not depend on h.

7.2 Estimate of the auxiliary random process

We introduce the seminorm

$$\|\boldsymbol{\psi}\|_{C^\alpha_{T,h}}=\sup_{\substack{0\leqslant t_1<t_2\leqslant T\\ |t_1-t_2|\leqslant 1}}\frac{\|\boldsymbol{\psi}(t_1)-\boldsymbol{\psi}(t_2)\|_{L^{2,h}}}{|t_1-t_2|^\alpha}\quad \forall\, T>0\,. \tag{7.6}$$

Recall that the function $r(\lambda)$ is defined in (3.19).

Now define

$$S(\lambda)=\int\limits_0^\lambda\frac{d\mu}{r(\mu)}\,. \tag{7.7}$$

In accordance with the general definition (3.20) and (3.21), we denote

$$\begin{aligned} S[\psi_k(t)]&=S(\operatorname{Re}\psi_k(t))+iS(\operatorname{Im}\psi_k(t))\\ \widehat{S}[\psi_k]z&=S(\operatorname{Re}\psi_k)\operatorname{Re}z+iS(\operatorname{Im}(\psi_k))\operatorname{Im}z \end{aligned} \tag{7.8}$$

for each complex number z. Applying the Ito formula to $S[\psi(t)]$, i.e., applying the Ito formula to the function $S(\mathrm{Re}\,\psi_k)$ and to the function $S(\mathrm{Im}\,\psi_k)$, we obtain

$$\begin{aligned} dS[\psi_k(t)] = -\widehat{S'}[\psi_k(t)]\Big((i\nabla_h + A_k)^2\psi_k(t) - \psi_k(t) + |\psi_k|^2\psi_k\Big)dt \\ +\widehat{S'}[\psi_k]\widehat{r}[\psi_k]dW_k + \tfrac{1}{2}\widehat{S''}[\psi_k](\widehat{r^2}[\psi_k][dW_k]^2)\,. \end{aligned} \tag{7.9}$$

Here,

$$\begin{aligned} &\widehat{S'}[\psi_k]\widehat{r}[\psi_k]dW_k \\ &\quad = S'(\mathrm{Re}\,\psi_k)r(\mathrm{Re}\,\psi_k)d\mathrm{Re}\,W_k + iS'(\mathrm{Im}\,\psi_k)r(\mathrm{Im}\,\psi_k)d\mathrm{Im}\,W_k = dW_k \end{aligned} \tag{7.10}$$

and the last equality holds because of (7.7). Note that the first term on the right-hand side of (7.9) should be understood in the same sense as was indicated in the second relation of (7.8). Moreover, by virtue of (4.39) and (7.7),

$$\begin{aligned} \widehat{S''}[\psi_k]\widehat{r}[\psi_k]\widehat{r}[\psi_k](d\mathrm{Re}\,W_k)^2 &= \frac{1}{2}S''(\mathrm{Re}\,\psi_k)r^2(\mathrm{Re}\,\psi_k)(d\mathrm{Re}\,W_k)^2 \\ &\quad + \frac{i}{2}S''(\mathrm{Im}\,\psi_k)r^2(\mathrm{Im}\,\psi_k)(d\mathrm{Im}\,W_k)^2 \\ &= -\frac{1}{2}\Big(r'(\mathrm{Re}\,\psi_k) + ir'(\mathrm{Im}\,\psi_k)\Big)\sum_{jh\in G_h}|\Theta_{kj}|^2\mu_j dt \\ &= -\frac{1}{2}\sum_j|\Theta_{kj}|^2\mu_j r'[\psi_k]dt\,. \end{aligned} \tag{7.11}$$

As a result, we obtain from (7.9)–(7.11) and (7.7) that

$$\begin{aligned} dS[\psi_k(t)] = \Big\{&-\widehat{r^{-1}}[\psi_k]\big((i\nabla_h + A_k)^2\psi_k - \psi_k + |\psi_k|^2\psi_k\big) \\ &-\frac{1}{2}r'[\psi_k]\sum_j|\Theta_{kj}|^2\mu_j\Big\}dt + dW_k\,, \end{aligned} \tag{7.12}$$

where the equality is understood in the sense of (3.20) and (3.21). Now using the results from [44], we derive an estimate for $\|S(\boldsymbol{\psi})\|_{C^\alpha_{T,h}}$.

Denote $\mathbf{Z}(t)$ as

$$\mathbf{Z}(t) = S[\boldsymbol{\psi}(t)] - \mathbf{W}(t)\,. \tag{7.13}$$

Equalities (7.12) and (7.13) imply

$$\begin{aligned}\dot{\mathbf{Z}}_k(t) &= \frac{d}{dt}Z_k(t)\\ &= -\Big(\widehat{r^{-1}}[\psi_k]\big((i\nabla_h + A_k)^2\psi_k - \psi_k + |\psi_k|^2\psi_k\big) - r'[\psi_k]\sum_j |\Theta_{kj}|^2\mu_j\Big)\,.\end{aligned} \tag{7.14}$$

Lemma 7.2. *For any $T > 0$ the inequality*

$$\|\mathbf{Z}\|_{C^{\frac{1}{2}}_{T,h}} \leqslant C\Big\{1 + \Big(\int_0^T \big(\|\Delta_h\boldsymbol{\psi}(t)\|^2_{L^{2,h}} + \|\nabla_h^+\boldsymbol{\psi}(t)\|^2_{L^{2,h}} + \|\boldsymbol{\psi}(t)\|^2_{L^{2,h}} + \|\boldsymbol{\psi}(t)\|^6_{L^{6,h}}\big)dt\Big)^{1/2}\Big\} \tag{7.15}$$

holds, where C does not depend on h.

Proof. By virtue of (7.14) and (3.19), we have

$$\|\dot{\mathbf{Z}}(t)\|^2_{L^{2,h}} \leqslant C_1\left(\|\Delta_h\boldsymbol{\psi}(t)\|^2_{L^{2,h}} + \|\nabla_h^+\boldsymbol{\psi}(t)\|^2_{L^{2,h}} + \|\boldsymbol{\psi}(t)\|^2_{L^{2,h}} + \|\boldsymbol{\psi}\|^6_{L^{6,h}} + 1\right)$$

and therefore

$$\|\dot{\mathbf{Z}}(t)\|_{L^{2,h}} \leqslant C_1^{1/2}\big(\|\Delta_h\boldsymbol{\psi}(t)\|_{L^{2,h}} + \|\nabla_h^+\boldsymbol{\psi}(t)\|_{L^{2,h}} + \|\boldsymbol{\psi}(t)\|_{L^{2,h}} + \|\boldsymbol{\psi}(t)\|^3_{L^{6,h}} + 1\big)\,. \tag{7.16}$$

This inequality implies

$$\begin{aligned}\|\mathbf{Z}(t_2) - \mathbf{Z}(t_1)\|_{L^{2,h}} &\leqslant \int_{t_1}^{t_2} \|\dot{\mathbf{Z}}(t)\|_{L^{2,h}}dt\\ &\leqslant C_1^{1/2}\int_{t_1}^{t_2}\Big(\|\boldsymbol{\psi}(t)\|_{L^{2,h}} + \|\Delta_h\boldsymbol{\psi}(t)\|_{L^{2,h}} + \|\nabla_h^+\boldsymbol{\psi}(t)\|_{L^{2,h}}\\ &\qquad + \|\boldsymbol{\psi}\|^3_{L^{6,h}} + 1\Big)dt\\ &\leqslant C\Big[\Big(\int_{t_1}^{t_2}\big[\|\Delta_h\boldsymbol{\psi}(t)\|^2_{L^{2,h}} + \|\nabla_h^+\boldsymbol{\psi}(t)\|^2_{L^{2,h}}\\ &\qquad + \|\boldsymbol{\psi}(t)\|^2_{L^{2,h}} + \|\boldsymbol{\psi}(t)\|^6_{L^{6,h}}\big]dt\Big)^{\frac{1}{2}}(t_2 - t_1)^{1/2} + (t_2 - t_1)\Big]\,.\end{aligned}$$

By using the definition (7.6), we obtain the desired result (7.15). □

Recall that the Levi modulus is the function $\aleph(t) = |t \ln t|^{1/2}$ and the norm $\|\mathbf{W}\|_{C_{L,T,h}}$ is defined as

$$\|\mathbf{W}\|_{C_{L,T,h}} = \sup_{\substack{0 \leqslant t_1 < t_2 < T \\ |t_1 - t_2| < 1/e}} \frac{\|\mathbf{W}(t_1) - \mathbf{W}(t_2)\|_{L^{2,h}}}{\aleph(t_2 - t_1)} \,. \tag{7.17}$$

Recall that $\Lambda_h = P_h^* \Lambda$ is the distribution of the Wiener process $\mathbf{W}(t)$ from (4.11), where Λ is the distribution of the initial Wiener process (see (3.1)). The measure Λ_h is defined on $\mathcal{B}(\mathbb{C}_h)$, where $\mathbb{C}_h = C(0, \infty; L^{2,h}(G_h))$. In [44, p. 356] the following assertion was proved.

Lemma 7.3. *There exist positive constants C_1 and C_2 independent of h (and of Λ_h) such that for any $\alpha > 0$*

$$\Lambda_{T,\alpha}^h \equiv \Lambda_h(\{\mathbf{W} \in \mathbb{C}_h \,:\, \|\mathbf{W}\|_{C_{L,T}} > C_1 \alpha\}) \leqslant C_2 T \frac{\sqrt{Tr_h}}{\alpha} 2^{-\alpha^2/2\, Tr_h} \,, \tag{7.18}$$

where $Tr_h = \sum\limits_{jh \in G_h} \mu_j$ is the trace of the correlation operator $\widehat{\mathcal{K}}$ defined in (4.17) *and below* (4.18) *and corresponding to the Wiener process $\mathbf{W}(t)$.*

Lemma 7.4. *The process $S[\boldsymbol{\psi}(t)]$ with function S defined in* (7.7) *satisfies the bound*

$$\begin{aligned} \|S[\boldsymbol{\psi}]\|^2_{C_{L,T,h}} \leqslant 2C_1 \Big[1 + \int_0^T \Big(& \|\Delta_h \boldsymbol{\psi}(t)\|^2_{L^{2,h}} + \|\nabla_h^+ \boldsymbol{\psi}(t)\|^2_{L^{2,h}} \\ & + \|\boldsymbol{\psi}(t)\|^2_{L^{2,h}} + \|\boldsymbol{\psi}(t)\|^6_{L^{6,h}} \Big) dt \Big] + 2\|\mathbf{W}\|^2_{C_{L,T,h}} \,, \end{aligned} \tag{7.19}$$

where C_1 does not depend on h.

Proof. The bound (7.19) directly follows from (7.13) and Lemma 7.2 if we take into account that $|t_2 - t_1|^{1/2} \leqslant \aleph(t_2 - t_1) \equiv \big|(t_2 - t_1) \ln |t_2 - t_1|\big|^{\frac{1}{2}}$ for $|t_2 - t_1| < \frac{1}{e}$. □

Theorem 7.5. *Let $\boldsymbol{\psi}(t)$ be the solution of the stochastic problem* (4.30), (2.23), *and* (2.12). *Then the bound*

$$E\|S[\boldsymbol{\psi}]\|^2_{C_{L,T,h}} \leqslant C(T) \,, \tag{7.20}$$

holds, where $C(T)$ does not depend on h.

Proof. We take the mathematical expectation of both sides of (7.19) and, to estimate the right-hand side, we use Lemma 7.3 and the bounds (5.6), (5.13), and (5.52). As a result, we obtain (7.20). □

7.3 Estimate of the mean modulus of continuity

We define the norm

$$\|\boldsymbol{\psi}\|_{C^L(0,T;L^{1,h}(G_h))} = \sup_{\substack{0\leqslant t_1<t_2<T\\ |t_1-t_2|<1/e}} \frac{\|\boldsymbol{\psi}(t_1)-\boldsymbol{\psi}(t_2)\|_{L^{1,h}}}{\aleph(t_1-t_2)}\,, \tag{7.21}$$

where

$$\|\boldsymbol{\psi}\|_{L^{1,h}} = h^d \sum_{kh\in G_h} |\psi_k|\,. \tag{7.22}$$

Note that by virtue of definitions (7.7) and (3.19), the function $S(\lambda)$ possesses the inverse function $R(S)$

$$R(S(\lambda)) = \lambda\,. \tag{7.23}$$

Lemma 7.6. *There exists a constant $C>0$ such that*

$$|\lambda_1-\lambda_2| \leqslant C(1+|\lambda_1|+|\lambda_2|)\ |S(\lambda_1)-S(\lambda_2)| \quad \forall\ \lambda_1,\lambda_2\in\mathbb{R}^1\,. \tag{7.24}$$

Proof. Let $\lambda_1>\lambda_2$. Then $S(\lambda_1)>S(\lambda_2)$. By virtue of (7.23) and (7.7), $R'(S(\lambda))=r(\lambda)>0$. Therefore, using the Lagrange theorem, we obtain

$$\begin{aligned}\lambda_1-\lambda_2 &= R(S(\lambda_1))-R(S(\lambda_2))\\ &\leqslant \sup_{\mu\in[\lambda_2,\lambda_1]} R'(S(\mu))|S(\lambda_1)-S(\lambda_2)| \leqslant R'(S(\lambda_1))\ |S(\lambda_1)-S(\lambda_2)|\\ &\leqslant C(1+|\lambda_1|+|\lambda_2|)\ |S(\lambda_1)-S(\lambda_2)|\,.\end{aligned} \tag{7.25}$$

□

Theorem 7.7. *Let $\boldsymbol{\psi}(t)$ be the strong solution of the stochastic problem* (4.30)*,* (2.23)*, and* (2.12)*. Then the following estimate holds*:

$$E\|\boldsymbol{\psi}\|_{C^L(0,T;L^{1,h}(G_h))} \leqslant C(T)\,. \tag{7.26}$$

Proof. It is enough to prove the bound

$$\|\boldsymbol{\psi}\|_{C^L(0,T;L^{1,h}(G_h))} \leqslant C\left(1+\sup_{0\leqslant t\leqslant T}\|\boldsymbol{\psi}(t)\|^2+\|S(\boldsymbol{\psi})\|^2_{C_{L,T,h}}\right) \tag{7.27}$$

because, after taking the mathematical expectation of both sides of (7.27) and using (7.20) and (7.1), we obtain (7.26). Substituting $\lambda_i=\operatorname{Re}\psi_k(t_i)$, $i=1,2$, or $\lambda_i=\operatorname{Im}\psi_k(t_i)$, $i=1,2$, into (7.25) gives

$$h^d\sum_{kh\in G_h}|\psi_k(t_1)-\psi_k(t_2)|$$

$$\leqslant C(1+\|\boldsymbol{\psi}(t_1)\|_{L^{2,h}}+\|\boldsymbol{\psi}(t_2)\|_{L^{2,h}})\|S[\boldsymbol{\psi}(t_1)]-S[\boldsymbol{\psi}(t_2)]\|_{L^{2,h}}\,.$$

Dividing both parts of this bound by the Levi modulus and taking into account the definitions (7.17) and (7.21), we obtain

$$\|\boldsymbol{\psi}\|_{C^L(0,T;L^{1,h}(G_h))}\leqslant C(1+\sup_{t\in[0,T]}\|\boldsymbol{\psi}(t)\|_{L^{2,h}})\|S[\boldsymbol{\psi}(t)]\|_{C_{L,T,h}}\,.$$

This inequality clearly implies (7.27). □

8 Compactness Theorems

In order to pass to the limit in the stochastic equation (4.30), we need some compactness theorems which we present in this section.

8.1 On compact sets in $L^2(G)$

For almost all $\omega\in\Omega$ the strong solution $\boldsymbol{\psi}(t)$ of Equation (4.30) belongs to $L^2(0,T;L^{2,h}(G_h))$, where $L^{2,h}(G_h)=P_hL^2(G)$ is the space defined before (4.8). Let $1\leqslant p<\infty$. Similarly to the space $L^{2,h}(G_h)$, we can introduce the space $L^{p,h}(G_h)$ of vector-valued functions $\boldsymbol{\psi}=\{\psi_k\,:\,kh\in G_h\}$ supplied with the norm

$$\|\boldsymbol{\psi}\|^p_{L^{p,h}(G_h)}=h^d\sum_{kh\in G_h}|\psi_k|^p\,. \tag{8.1}$$

Clearly, $L^{p,h}(G_h)=P_hL^p(G)$, where the operator P_h is defined as well as the operator P_h from (4.9). As in (4.10), one can prove that the operator $P_h:L^p(G)\to L^{p,h}(G_h)$ is bounded. We define the space

$$H^1_{A,h}(G_h)=\{\boldsymbol{\psi}\in L^{2,h}(G_h),\ \psi\text{ is defined on }\partial G_h^+\text{ by (2.23)}\} \tag{8.2}$$

and the norm (see (5.5)):

$$\begin{aligned}\|\boldsymbol{\psi}\|^2_{H^1_{A,h}}&=h^d\widetilde{\sum_{j,k}}(|\partial^+_{j,h}\psi_k|^2+|\psi_k|^2)\\&\equiv h^d\sum_{j=1}^d\sum_{kh\in G_h\cup\partial G_h^+(-j)}(|\partial^+_{j,h}\psi_k|^2+|\psi_k|^2)\,.\end{aligned} \tag{8.3}$$

We can identify the space $L^{p,h}(G_h)$ (as well as the space (8.2)) with subspaces of functions belonging to $L^p(G)$ by the operator (4.14):

$$L^{p,h}\ni\boldsymbol{\psi}=\{\psi_k\}\to\psi_h(x)=\sum_{kh\in G_h}h^{-d}\psi_k\mathcal{X}_{Q_k}(x)\in L^p(G)\,, \tag{8.4}$$

where $\mathcal{X}_{Q_k}(x)$ is the characteristic function of the set Q_k (i.e., $\mathcal{X}_{Q_k}(x) = 1$, for $x \in Q_k$, $\mathcal{X}_{Q_k}(x) = 0$ for $x \neq Q_k$) and the sets Q_k are defined by (4.1)–(4.5). We denote by $\widehat{L}^{p,h}(G)$ the subspace of $L^p(G)$ formed by identifying (8.4). The following assertion follows from (4.6)–(4.7) and a bound similar to (4.10).

Proposition 8.1. *The spaces $L^{p,h}(G_h)$ and $\widehat{L}^{p,h}(G)$ are isomorphic (so the norm (8.1) is equivalent to the norm of $\widehat{L}^{p,h}(G) \subset L^p(G)$) and the isomorphism is defined by (8.4).*

In the space $\widehat{L}^{2,h}(G)$, the norm (8.3) generates the norm

$$\|\psi_h\|^2_{\widehat{H}^1_{A,h}} = \int\limits_G \left(\sum_{j=1}^d \frac{|\psi_h(x+e_jh) - \psi_h(x)|^2}{h^2} + |\psi_h(x)|^2 \right) dx \,. \tag{8.5}$$

To calculate the finite difference in (8.5), we assume that $\psi_h(x)$ is defined on $\bigcup\limits_{kh \in G_h \cup \partial G_h^+} Q_k$ and, on sets Q_k, $kh \in G_h^+$, $\psi_h(x)$ is defined with the help of (2.23).

More precisely, in order to determine the finite difference quotient $(\psi_h(x + e_jh) - \psi_h(x))/h$, we use the polyhedra

$$Q_k = \{x = (x_1, \dots, x_d) \in \mathbb{R}^d \,:\, x_j \in [h(k_j - \frac{1}{2})), h(k_j + \frac{1}{2})), j = 1, \dots, d\} \tag{8.6}$$

for each $kh \in G_h \cup \partial G_h^+$, defining $\psi_h(x)$ for $x \in Q_k$ with $kh \in \partial G_h^+(\pm m)$ by (2.23). Then for $kh \in \partial G_h^-(\pm m)$ we change the polyhedra (8.6) in the definition of the quotient $(\psi_h(x+e_mh) - \psi_h(x))/h$ on the appropriate set Q_k from (4.2)–(4.5).

We denote by $\widehat{H}^1_{A,h}(G) = P_h^* H^1_{A,h}(G_h)$, where P_h^* is the operator (4.14) and $H^1_{A,h}(G_h)$ is the space (8.2) with the norm given by (8.3). Similar to Proposition 8.1, the following assertion holds.

Proposition 8.2. *The operator (4.14) establishes an isomorphism between $H^1_{A,h}(G_h)$ and $\widehat{H}^1_{A,h}(G)$, i.e., the norms (8.3) and (8.5) are equivalent with constants independent of h.*

Proof. One can easily obtain the necessary estimates with the help of the explanation near (8.6) and the relations (4.6) and (4.7). □

Below, we assume that $h = h_n = 2^{-n}h_0 \to 0$ as $n \to \infty$. For each $R > 0$ we set

$$B_R(\widehat{H}^1_{A,h}) = \{\psi \in \widehat{L}^{2,h}(G) \,:\, \|\psi\|_{\widehat{H}^1_{A,h}} \leqslant R\} \tag{8.7}$$

and

$$B_R(H^1) = \{\psi \in H^1(G) \,:\, \|\psi\|_1 \leqslant R\} \,. \tag{8.8}$$

Lemma 8.3. *For each $R > 0$ the set*

$$\Theta_R \equiv \bigcup_{n=1}^{\infty} B_R(\widehat{H}^1_{A,h_n}) \cup B_R(H^1) \tag{8.9}$$

is compact in $L^2(G)$ and in $L^1(G)$.

Proof. We choose from an arbitrary sequence $\psi_m \in \Theta_R$ a subsequence converging in $L^2(G)$. Two cases are possible: (i) there exists $n_0 > 0$ such that $\psi_m \in \bigcup\limits_{n=1}^{n_0} B_R(\widehat{H}^1_{A,h_n}) \cup B_R(H^1)$ for each m; (ii) there exists a subsequence $\{m'\}$ of the sequence $\{m\}$ such that $\psi_{m'} \in B_R(\widehat{H}^1_{A,h_{n_{m'}}})$ and $n_{m'} \to \infty$ as $m' \to \infty$.

In the first case, we can choose a subsequence $\{m'\} \subset \{m\}$ such that (a) $\psi_{m'} \in \bigcup\limits_{n=1}^{n_0} B_R(\widehat{H}^1_{A,h_n})$ for all $m' \in \{m'\}$ or (b) $\psi_{m'} \in B_R(H^1)$ for all $m' \in \{m'\}$. For case (a), we can choose a converging subsequence $\{\psi_{m''}\}$ from $\{\psi_{m'}\}$ because $\bigcup\limits_{h=1}^{n_0} B_R(\widehat{H}^1_{A,h_n})$ is a finite dimensional closed bounded set. For case (b), we can choose a converging subsequence $\{\psi_{m''}\}$ because, as is well known, the embedding $H^1(G) \subset L_2(G)$ is compact.

In the second case, we can choose a subsequence $\{\psi_{m''}\} \subset \{\psi_{m'}\}$ weakly converging to $\widetilde{\psi}(x)$ in $L^2(G)$ as $m'' \to \infty$. Moreover, by virtue of the definitions (8.7) –(8.9), for each ε there exists $\delta > 0$ and $N > 0$ such that for all h satisfying $\|h\| < \delta$ and for all $n \geqslant N$,

$$\int |\psi_n(x) - \psi_n(x-h)|^2 < \varepsilon\,. \tag{8.10}$$

Then, by (8.10), we use standard arguments to choose a subsequence $\{\psi_q\} \subset \{\psi_m\}$ such that $\|\psi_q - \widetilde{\psi}\|_{L^2(G)} \to 0$ as $q \to \infty$ (see [40, Chapt. 1, Sect. 4]. □

8.2 Compact sets in the space of time-dependent functions

Let E_0, E, and E_1 denote reflexive Banach spaces such that the embeddings $E_0 \subset E \subset E_1$ are continuous and the embedding $E_0 \subset E$ is compact. Then the Dubinsky theorem (see [44, p. 131-132]) can be stated as follows.

Theorem 8.4. *Let $1 < q, q_1 < \infty$, and let M be a bounded set in $L_q(0,T;E_0)$ consisting of functions $u(t)$ equicontinuous in $C(0,T;E_1)$. Then M is relatively compact in $L_{q_1}(0,T;E)$ and $C(0,T;E_1)$.*

We establish some variants of this theorem which we will need. First, let us apply this theorem to the following situation. We introduce the space

$$\mathcal{W} = \{\psi(t,x) \in L^2(0,T;H^1(G)) \cap C^L(0,T;L^1(G))\}\,, \tag{8.11}$$

where

$$C^L(0,T;L^1(G)) = \Big\{\psi(t,x), (t,x) \in (0,T)\times G \,:\, \|\psi\|_{C_{L,T,1}} = \sup_{\substack{0\leqslant t_1<t_2\leqslant T\\ |t_1-t_2|<e^{-1}}} \frac{\|\psi(t_1,\cdot)-\psi(t_2,\cdot)\|_{L^1(G)}}{\aleph(t_2-t_1)} + \sup_{0\leqslant t\leqslant T}\|\psi(t,\cdot)\|_{L^1(G)} < \infty\Big\}\,, \tag{8.12}$$

where again $\aleph(t) = |t\ln t|^{\frac{1}{2}}$ for $t>0$.

Theorem 8.5. *The set*

$$B_R(\mathcal{W}) = \{\psi(t,x)\in\mathcal{W} \,:\, \|\psi\|_{\mathcal{W}} \leqslant R\} \tag{8.13}$$

is compact in the space $L^4((0,T)\times G)\cap C(0,T;L^1(G))$.

Proof. To apply Theorem 8.4, we take $E_0 = H^1(G)$, $E = L^4(G)$, $E_1 = L^1(G)$, and $M = B_R(\mathcal{W})$. Clearly, M consists of functions that are equicontinuous in $C(0,T;E_1)$. □

Let

$$\mathcal{W}_h = \Big\{\psi(t,x)\in L^2(0,T;\widehat{H}^1_{A,h}(G)) \,:\, \|\psi\|_{C_{L,T,1}} = \sup_{\substack{0\leqslant t_1<t_2<T\\ |t_1-t_2|<e^{-1}}} \frac{\|\psi(t_1,\cdot)-\psi(t_2,\cdot)\|_{L^1(G)}}{\aleph(t_2-t_1)} + \sup_{0\leqslant t\leqslant T}\|\psi(t,\cdot)\|_{L^1(G)} < \infty\Big\} \tag{8.14}$$

and

$$B_R(\mathcal{W}_h) = \{\psi\in\mathcal{W}_h \,:\, \|\psi\|_{C_{L,T,1}} + \|\psi\|_{L^2(0,T;\widehat{H}^1_{A,h}(G))} \leqslant R\}\,. \tag{8.15}$$

Since $\mathcal{W}_h$ consists of functions equicontinuous in $C(0,T;L^1(G))$, the following assertion holds.

Proposition 8.6. *The set* (8.15) *is compact in the space* $L^4((0,T)\times G)\cap C(0,T;L^1(G))$.

The following theorem then holds.

Theorem 8.7. *For each* $R>0$ *the set*

$$\Theta_R = \bigcup_{n=1}^{\infty} B_R(\mathcal{W}_{h_n}) \cup B_R(\mathcal{W}) \tag{8.16}$$

is compact in $Z_T \equiv L^2((0,T)\times G)\cap C(0,T;L^1(G))$*. Here,* $h_n = h_0 2^{-n}$ *and* $B_R(\mathcal{W}_h)$ *and* $B_R(\mathcal{W})$ *are the sets* (8.15) *and* (8.13) *respectively.*

Proof. By virtue of Theorem 8.5 and Proposition 8.6, the sets $B_R(\mathcal{W}_{h_n})$ and $B_R(\mathcal{W})$ are compact in $L^4((0,T)\times G)$. Now, to complete the theorem, we apply the proof sketched in Lemma 8.3. □

9 Weak Solution of the Discrete Stochastic Problem

Our aim here is to pass to the limit as $h \to 0$ in the problem (4.30), (2.12), and (2.23) in order to prove an existence theorem for the boundary value problem (3.22), (2.2), and (2.3) for the stochastic Ginzburg–Landau equation. For this purpose, we need the definition of a weak solution of (4.30), (2.12), and (2.23).

9.1 Definition of the weak solution for the discrete problem

Recall that we suppose that the initial condition from (2.3) is a random process, i.e., $\psi_0(x) = \psi_0(x,\omega)$, $x \in G$, $\omega \in \Omega$, and we suppose that the map $\psi_0 : \Omega \to L^2(G)$ is measurable, i.e., $\psi_0 : \Sigma \to \mathcal{B}(L^2(G))$ where (Ω, Σ, m) is the initial probability space. Moreover, we assume that the random value ψ_0 and the Wiener process $W(t,x,\omega)$ defined in Sect. 3 are independent, i.e., for each $B \in \mathcal{B}(C(0,\infty;L^2(G))$ and $b \in \mathcal{B}(L^2(G))$,

$$\begin{aligned} &m\big(\{\omega\,:\,W(\cdot,\cdot,\omega)\in B, \psi_0(\cdot,\omega)\in b\}\big) \\ &\qquad = m\big(\{\omega\,:\,W(\cdot,\cdot,\omega)\in B\}\big)m\big(\{\omega\,:\,\psi_0(\cdot,\omega)\in b\}\big)\,. \end{aligned} \tag{9.1}$$

Now we construct certain projections of $\psi_0(\cdot,\omega)$ and $W(\cdot,\cdot,\omega)$. Using the projection $P_h : L^2(G) \to L^{2,h}(G_h)$ defined in (4.9), we can define the projection $P_h\psi_0(\omega)$ and $P_hW(t,\omega)$ defined on (Ω,Σ,m) and taking the values $P_h\psi_0(\omega) \in L^{2,h}(G_h)$ and $P_hW(t,\omega) \in C(0,\infty;\widehat{L}^{2,h}(G))$ respectively. Moreover, using the projection $P_h^* : L^{2,h}(G_h) \to \widehat{L}^{2,h}(G) \subset L^2(G)$ defined in (4.14), we can define the projections $P_h^*P_h\psi((\cdot,\omega))$, $\omega \in \Omega$, with values belonging to $C(0,\infty;\widehat{L}^{2,h}(G)) \subset C(0,\infty;L^2(G))$. So, using the notation

$$\widehat{P}_h = P_h^* P_h\,, \tag{9.2}$$

where P_h is the operator (4.9) and P_h^* is the operator (4.14), we define the random value

$$\Omega \ni \omega \to \widehat{P}_h\psi_0(\cdot,\omega) \in \widehat{L}^{2,h}(G) \subset L^2(G) \tag{9.3}$$

and the Wiener random process

$$\Omega \ni \omega \to (\widehat{P}_h W)(\cdot,\cdot,\omega) \in C(0,\infty;\widehat{L}^{2,h}(G)) \subset C(0,\infty;L^2(G))\,. \tag{9.4}$$

The relationship (9.1) for $\psi_0(\cdot,\omega)$ and $W(\cdot,\cdot,\omega)$ implies the independence of $\widehat{P}_h(\psi_0(\cdot,\omega))$ and $\widehat{P}_h W(\cdot,\cdot,\omega)$.

Note that the increasing filtration Σ_t corresponding to the Wiener process $W(t,x,\omega)$ corresponds to the Wiener process $\widehat{P}_h W(t,x,\omega)$ as well.

We define the space of functions

$$\mathcal{U}_h = L^{2,\mathrm{loc}}(0,\infty;\widehat{H}^1_{A,h}(G)) \cap C^L(0,\infty;L^1(G)) \cap L^{6,\mathrm{loc}}(0,\infty;L^6(G))\,, \tag{9.5}$$

where the index L means the Levi modulus $|t\ln t|^{1/2}$ for $t \in (0,1/e)$. It is clear that $\mathcal{U}_h$ is a Frechet space with seminorms

$$\|\psi\|_{\mathcal{U}_{h,T}} = \|\psi\|_{L^2(0,T;\widehat{H}^1_{A,h}(G))} + \|\psi\|_{C^L(0,T;L^1(G))} + \|\psi\|_{L^6((0,T)\times G)}\,. \tag{9.6}$$

With the aid of the solution $\boldsymbol{\psi}(t,\omega)$ of the problem (4.30) and (2.12), we can define the random process

$$\Omega \ni \omega \to (P_h^*\boldsymbol{\psi})(\cdot,\cdot,\omega) \equiv \psi_h(\cdot,\cdot,\omega) \in \mathcal{U}_h\,. \tag{9.7}$$

The space $\mathcal{U}_h$ from (9.5) is well connected with the solution ψ_h but we will need also in the following a more extensive separable Frechet space for the solution; we have

$$Z = L^{2,\mathrm{loc}}(0,\infty;L^2(G)) \cap C(0,\infty;L^1(G)) \tag{9.8}$$

with finite seminorms given by

$$\|\psi\|_{Z_T} \equiv \|\psi\|_{L^2(0,T;L^2(G))} + \|\psi\|_{C(0,T;L^1(G))}\,, \quad T>0. \tag{9.9}$$

We will also use the spaces

$$\begin{aligned} Z_T &= L^2(0,T;L^2(G)) \cap C(0,T;L^1(G)),\\ \mathcal{U}_{h,T} &= L^2(0,T;\widehat{H}^1_{A,h}(G)) \cap C^L(0,T;L^1(G)) \cap L^6((0,T)\times G) \end{aligned} \tag{9.10}$$

supplied with the norms (9.9) and (9.6) correspondingly.

Recall that $B(Z)$ is a Borel σ-algebra of the space Z and $B_{\mathcal{U}_h}(Z) = B(Z)\cap \mathcal{U}_h$. By virtue of Theorem 2.1 from [44, Chapt. 2], $B_{\mathcal{U}_h}(Z) \subset \mathcal{B}(\mathcal{U}_h)$.

Definition 9.1. The weak statistical solution of (4.30), (2.12), and (2.23) is the probability distribution of the random process (9.7), i.e.,

$$\nu_h(B) = m\big(\{\omega\,:\,\psi_h(\cdot,\cdot,\omega) \in B\}\big) \quad \forall\, B \in \mathcal{B}_{\mathcal{U}_h}(Z)\,. \tag{9.11}$$

9.2 The equation for the weak solution of the discrete problem

Taking the integral form of the Ito equation (7.12) and applying the operator P_h^* from (4.14) we obtain

$$\begin{aligned} L_h(\psi_h) \equiv{}& S[\psi_h(t,x)] - S[\psi_{h,0}(\cdot)] \\ &+ \int_0^t \Big\{ \widehat{r^{-1}}[\psi_h(\tau,x)] \Big((i\nabla_h + \widehat{P}_h A(x))^2 \psi_h(\tau,x) - \psi_h + |\psi_h|^2 \psi_h \Big) \\ &- \frac{1}{2} \widehat{r'}[\psi_h] \sum_{j,k} |\Theta_{kj}|^2 \mu_j \mathcal{X}_{Q_k}(x) V(Q_k)^{-1} \Big\} d\tau = \widehat{P}_h W(t,x) \,. \end{aligned} \tag{9.12}$$

Let γ_0 be the restriction operator of functions $f(t,\cdot)$ at $t=0$, i.e., $\gamma_0 f = f(0,\cdot)$. We consider the operator

$$\mathfrak{A}_h \equiv (\gamma_0, L_h) \,:\, \mathcal{U}_h \to L^1(\Omega) \times Z \,, \tag{9.13}$$

where L_h is the operator given in (9.12).

Proposition 9.2. *The operator* (9.13) *is continuous.*

Proof. The proof of this assertion is obvious because the space $\widehat{H}^1_{A,h}(G)$ forming the space $\mathcal{U}_h$ is finite dimensional. □

We want to use the operator (9.13) to rewrite the weak solution (9.11) in some other form. Recall that the full preimage of the set $B \times B_0$, where $B \in Z$, $B_0 \in L^1(G)$, is defined as follows:

$$\mathfrak{A}_h^{-1}(B_0 \times B) = \{\psi \in \mathcal{U}_h \,:\, \mathfrak{A}_h \psi = (\gamma_0 \psi, L_h \psi) \in B_0 \times B\} \,. \tag{9.14}$$

By virtue of Proposition 9.2, $\mathfrak{A}_h^{-1}(B_0 \times B) \in \mathcal{B}(\mathcal{U}_h)$. This full preimage is strictly connected to the solution $\psi_h(t,x)$ of the problem (9.12). Indeed, we have

$$\begin{aligned} \psi_h(t,x,\omega) ={}& \psi_h(t,x,\psi_0(\cdot,\omega), W(\tau \in (0,t),\cdot,\omega)) \\ ={}& \mathfrak{A}_h^{-1}(t,x,\psi_0(\cdot,\omega), W(\tau \in (0,t),\cdot,\omega)) \,, \end{aligned} \tag{9.15}$$

where, in contrast to (9.14), $\mathfrak{A}_h^{-1}$ is the inverse (i.e., uniquely valued) operator of the operator $\mathfrak{A}_h$. The domain of the operator (9.15) is the set of initial conditions and right-hand sides, where the solution of (4.30), (2.12), and (2.23) exists and is unique and therefore the solution of (9.12) possesses the same property. This domain is given by

$$\mathcal{D}(\mathfrak{A}_h^{-1}) = (\widehat{P}_h L^1(G), \widehat{P}_h \widehat{W}) \,, \tag{9.16}$$

where $\widehat{W}$ is the image of the Wiener process defined in Sect. 3:

$$\widehat{W} = \{W(\cdot,\cdot,\omega), \omega \in \Omega\}, \quad W(\cdot,\cdot,\omega) \text{ is a Wiener process.} \tag{9.17}$$

Definition (9.17) implies that

$$\widehat{W} \text{ is a } \Lambda\text{-measurable set}\,. \tag{9.18}$$

Now for each $B_0 \in \mathcal{B}(L^1(G))$ and $B \in \mathcal{B}(Z)$, we can write (see [44, p. 343])

$$\begin{aligned}(\mathfrak{A}_h^* \nu_h)(B_0 \times B) &= \nu_h(\mathfrak{A}_h^{-1}(B_0 \times B)) \\ &= \nu_h(\{\psi_h \in \mathcal{U}_h \,:\, \mathfrak{A}_h \psi_h \in \widehat{P}_h B_0 \times \widehat{P}_h B\}) \\ &= m\big(\{\omega \,:\, \widehat{P}_h \psi_0(\cdot,\omega) \in \widehat{P}_h B_0, \widehat{P}_h W(\cdot,\cdot,\omega) \in \widehat{P}_h B\}\big) \\ &= \widehat{P}_h^* \mu(B_0) \times \widehat{P}_h^* \Lambda(B) = \mu_h(B_0)\Lambda_h(B)\,.\end{aligned} \tag{9.19}$$

The relation

$$(\mathfrak{A}_h^* \nu_h)(B_0 \times B) = \mu_h(B_0)\Lambda_h(B) \quad \forall\, B_0 \in \mathcal{B}(L^1(G)), B \in \mathcal{B}(Z) \tag{9.20}$$

is the desired equation for the weak statistical solution ν_h defined in (9.11).

10 Passage to the Limit in a Family of ν_{h_n}

To take this limit, we need certain additional compactness results which we present here.

10.1 Compactness of the family of measures ν_{h_n}

Recall that $h_n = h_0 2^{-n}$. First, we establish some estimates for ν_{h_n} We denote by Γ_T the restriction operator on the interval $(0,T)$, i.e.,

$$\Gamma_T \psi = \psi|_{(0,T)}\,. \tag{10.1}$$

Let $Z_T = \Gamma_T Z$ and

$$\nu_{hT}(C) = \nu_h(\Gamma_T^{-1} C) \quad \forall\, C \in \mathcal{B}(\mathcal{U}_T)\,. \tag{10.2}$$

Theorem 10.1. *Suppose that the distribution $\mu(d\psi_0)$ of the initial condition $\psi_0(x,\omega)$ satisfies the inequality*

$$\int \big(\|\psi_0\|^2_{L^2(G)} + \|\nabla\psi_0\|^2_{L^2(G)} + \|\psi_0\|^4_{L^4(G)}\big)\mu(d\psi_0) < \infty\,. \tag{10.3}$$

Then the measure ν_{hT} satisfies the estimates

$$\int\limits_{Z_T} \Big(\|\psi(t,\cdot)\|^2_{L^2(G)} + \int\limits_0^t (\|\nabla_h^+\psi(t,\cdot)\|^2_{L^2(G)} + \|\psi(t,\cdot)\|^4_{L^4(G)})dt\Big)\nu_{hT}(d\psi)$$
$$\leqslant C_1 e^{CT}\Big(1 + \int\limits_{L^2(G)} \|\psi_0\|^2\mu(d\psi_0)\Big)\,, \tag{10.4}$$

$$\int\limits_{Z_T} \Big(\|\psi(t,\cdot)\|^4_{L^4(G)} + \int\limits_0^t \|\psi(t,\cdot)\|^6_{L^6(G)})d\tau\Big)\,\nu_{hT}(d\psi)$$
$$\leqslant C_2 e^{Ct}(1 + \int\limits_{L^2(G)} \|\psi_0\|^4\mu(d\psi_0)\,, \tag{10.5}$$

and

$$\int\limits_{Z_T} \Big(\|\psi\|^2_{L^\infty((0,T);L^2(G))} + \|\psi\|_{C_{L,T,1}}\Big)\,\nu_{hT}dt \leqslant C(T)\,, \tag{10.6}$$

where the constants C_1, C_2, and C do not depend on h and T and $C(T)$ does not depend on h.

Proof. From the usual definition (10.2) and (9.11) of the measure ν_{hT} and Propositions 8.1 and 8.2, we can immediately derive (10.4) from (5.6), (10.5) from (5.13), and (10.6) from the bounds given in (7.1) and (7.26). □

Our goal is to prove the weak compactness of the measures ν_{h_n}. For this purpose, we use the following well-known theorem which is proved, for example, in [19] .

Theorem 10.2 (Prokhorov). *A family $\mathcal{M}$ of measures defined on the Borel σ-algebra $\mathcal{B}(Z)$ of a separable Banach space Z is weakly compact if*

(a) $\sup\{\mu(Z) : \mu \in \mathcal{M}\} < \infty$,

(b) *for any $\varepsilon > 0$ there exists a compact set $K \subset Z$ such that* $\sup\{\mu(Z\backslash K) : \mu \in \mathcal{M}\} < \varepsilon$.

Lemma 10.3. *The set of measures ν_{h_nT}, $n \in \mathbb{N}$, is weakly compact on $Z_T = L^2((0,T)\times G) \cap C(0,T;L^1(G))$.*

Proof. We use Theorem 10.2. Since $\nu_{h_n T}$ are probability measures, the condition (a) of the Prokhorov theorem is satisfied. We must check condition (b) of the theorem. For a compact set K we take the set Θ_R introduced in (8.16). By Theorem 8.7, Θ_R is compact in Z_T. Note that the measure $\nu_{h_k T}$ is concentrated in $L^2(0,T;\widehat{L}^{2,h_k}(G))$ and therefore

$$\operatorname{supp}\nu_{h_k}\cap\Theta_R = B_R(W_{h_k})\cap\operatorname{supp}\nu_{h_k T}\,. \tag{10.7}$$

Therefore, using (10.7) and the Chebyshev inequality as well as the bounds (10.4)–(10.6), we obtain

$$\begin{aligned}&\int\limits_{L^2(0,T\times G)\setminus\Theta_R}\nu_{h_k T}(d\psi) = \int\limits_{L^2(0,T;\widehat{L}^{2,h_k}(G))\setminus B_R(W_{h_k})}\nu_{h_k T}(d\psi)\\ &\quad\leqslant\frac{1}{R}\int\Big(\|\psi\|_{L^2(0,T;\widehat{H}^1_{A,h_k}(G)}+\|\psi\|_{C_{L,T,1}}\Big)\nu_{h_k,T}(d\psi)\leqslant\frac{C}{R}\,,\end{aligned} \tag{10.8}$$

where C does not depend on k. The inequality (10.8) implies that the measure ν_{h_m} satisfies condition (b). Therefore, the assertion of the lemma follows from Prokhorov's theorem. □

10.2 Passage to the limit

In this section, we demonstrate that the set of measures ν_{h_n}, $n\in\mathbb{N}$, is weakly compact on Z and thus we can choose a subsequence that converges weakly to ν in Z.

Theorem 10.4. *The set of measures ν_{h_n}, $n\in\mathbb{N}$, is weakly compact on Z.*

Proof. The proof is similar to the proof given in [44, p. 361]. □

By virtue of Theorem 10.4, we can choose from the sequence of measures $\{\nu_{h_n}\}$ the subsequences $\{\nu_{h_j}\}$ that converges weakly to ν on Z, i.e.,

$$\nu_{h_j}\to\nu\quad\text{as } j\to\infty \text{ weakly on } \mathcal{Z}. \tag{10.9}$$

We will show that the measure ν is the weak solution (see Definition 12.1 below) of the stochastic problem (3.22), (2.2), and (2.3).

11 Estimates for the Weak Solution

We first prove an estimate for ν_h.

11.1 An estimate for ν_h

In order to prove the analogue of the estimate given in (5.52), we have to define the second finite difference $\Delta_h\psi_h(x)$ for $\psi_h(x) \in \widehat{L}^{2,h}(G)$.

Assuming that the lattice function $\boldsymbol{\psi} = \{\psi_k\}$ satisfies (2.23), we can then define the norm

$$\|\boldsymbol{\psi}\|^2_{H^2_{A,h}(G_h)} = h^d \sum_{kh\in G_h} \left(|\Delta_h\psi_k|^2 + |\nabla_h^+\psi_k|^2 + |\psi_k|^2\right). \tag{11.1}$$

We set

$$\begin{aligned} H^2_{A,h}(G_h) = \{&\boldsymbol{\psi} \in L^{2,h}(G_h), \\ &\boldsymbol{\psi} \text{ satisfies (2.23), supplied with the norm (11.1) } \}. \end{aligned} \tag{11.2}$$

We also define the space $\widehat{H}^2_{A,h}(G)$ along with its norm as

$$\begin{aligned} &\widehat{H}^2_{A,h}(G) = P_h^* H^2_{A,h}(G_h)\,, \\ &\|\psi_h\|^2_{\widehat{H}^2_{A,h}} = \int_G \Big(|\Delta_h\psi_h(x)|^2 + |\nabla_h^+\psi_h(x)|^2 + |\psi_h(x)|^2\Big)dx\,. \end{aligned} \tag{11.3}$$

Note that in a neighborhood of ∂G, the finite difference $|\Delta_h\psi_h(x)|^2$ is calculated as was explained near (8.5). More precisely, to calculate the difference $|\Delta_h\psi_h(x)|^2$, we use the polyhedra Q_k from (8.6) and, after these calculations, we change these polyhedra in a neighborhood of $\partial\Omega$ on appropriate polyhedra; see (4.2)–(4.5). The value of $\psi_h(x)$ on this polyhedra Q_k is defined by (2.23).

The following assertion which is analogous to Propositions 8.1 and 8.2 can be proved .

Proposition 11.1. *The spaces $H^2_{A,h}(G_h)$ and $\widehat{H}^2_{A,h}(G)$ are isomorphic and the norms in* (11.3) *and* (11.1) *are equivalent.*

The following theorem easily results from the estimate (5.52) .

Theorem 11.2. *The measure ν_{hT} satisfies the estimate*

$$\begin{aligned} &\int_{Z_T} \Big(\|\nabla_h^+\psi(t)\|^2_{L^2(G)} + \int_0^t \|\Delta_h\psi(\tau,\cdot)\|^2_{L^2(G)}\, d\tau\Big)\nu_{h,T}(d\psi) \\ &\qquad\leqslant C_1 e^{Ct}\Big(1 + \int \big(\|\nabla_h^+\psi_0\|^2_{L^2(G)} + \|\psi_0\|^4_{L^4(G)}\big)\mu(d\psi_0)\big)\,. \end{aligned} \tag{11.4}$$

Recall that $h_n = 2^{-n}h_0$. Below we will need modifications of Theorems 10.1 and 11.2, where on the left-hand sides of the inequalities in these theorems we need to replace ∇_h^+ and Δ_h with $\nabla_{h_m}^+$ and Δ_{h_m} respectively. In addition, $\nu_{h,T}(d\psi)$ must be changed to $\nu_{h_n,T}(d\psi)$ for $n > m$. To establish such estimates, we prove some preliminary lemmas in the next section.

11.2 Preliminary lemmas

In this section, we provide several preliminary results which will be needed to prove estimates for the measure ν.

Lemma 11.3. *Let u_k, $k = 1, \dots, N$, $h > 0$, be a lattice function. Then*

$$\sum_{k=1}^{N-n} \left| \frac{u_{k+n} - u_k}{nh} \right|^2 \leqslant \sum_{k=1}^{N} \left| \frac{u_{k+1} - u_k}{h} \right|^2 . \tag{11.5}$$

Proof. Since $(a_1 + \dots + a_j)^2 \leqslant j(a_1^2 + \dots + a_j^2)$ for positive $a_1, \dots, a_j$, we have

$$\begin{aligned} \sum_{k=1}^{N-n} \left| \frac{u_{k+n} - u_k}{nh} \right|^2 &= \frac{1}{n^2} \sum_{k=1}^{N-n} \left| \sum_{j=1}^{n} \frac{u_{k+j} - u_{k+j-1}}{h} \right|^2 \\ &\leqslant \frac{1}{n} \sum_{k=1}^{N-n} \sum_{j=1}^{n} \left| \frac{u_{k+j} - u_{k+j-1}}{h} \right|^2 \\ &\leqslant \sum_{k=1}^{N} \left| \frac{u_{k+1} - u_k}{h} \right|^2 , \end{aligned}$$

where to obtain the last inequality we have taken into account that the previous sum can be represented as the sum of groups of identical summands and the number of identical summands in each group are not more than n. ◻

Lemma 11.4. *Let u_k, $k = 0, \dots, N$, $h > 0$, be a lattice function. Then*

$$\sum_{k=n}^{N-n} \left| \frac{u_{k+n} - 2u_k + u_{k-n}}{(nh)^2} \right|^2 \leqslant 4 \sum_{k=1}^{N-1} \left| \frac{u_{k+1} - 2u_k + u_{k-1}}{h^2} \right|^2 . \tag{11.6}$$

Proof. For $k = 1, \dots, N-1$ we set $\Delta_h u_k = u_{k+1} - 2u_k + u_{k-1}$. One can prove that

$$u_{k+n} - 2u_k + u_{k-n} = \sum_{j=1}^{n} j \Delta_h u_{k+n-j} + \sum_{j=1}^{n-1} (n-j) \Delta_h u_{k-j} .$$

Therefore,

$$\sum_{k=n}^{N-n}\left|\frac{u_{k+n}-2u_k+u_{k-n}}{(nh)^2}\right|^2$$

$$\leqslant \frac{2n}{(nh)^4}\sum_{k=n}^{N-n}\Big(\sum_{j=1}^{n} j^2\,|\Delta_h u_{k+n-j}|^2+\sum_{j=1}^{n-1}(n-j)^2\,|\Delta_h u_{k-j}|^2\Big)$$

$$\leqslant \frac{2}{nh^4}\sum_{k=n}^{N-n}\Big(\sum_{j=1}^{n}|\Delta_h u_{k+n-j}|^2+\sum_{j=1}^{n-1}|\Delta_h u_{k-j}|^2\Big)\ \leqslant\ 4\sum_{k=1}^{N-1}\left|\frac{\Delta_h u_k}{h^2}\right|^2$$

because the maximal number of elements in each group of identical summands in the penultimate sum is $2n$. □

For the approximate domain $G_{h_n}\cup\partial G^+_{h_n}$ we intend to define the first and second finite difference quotients $\nabla^+_{h_m}$ and Δ_{h_m} with $m<n$. For $j=1,\dots,d$ denote

$$G_{h_n}(+j;h_m)=\left\{k\in\mathbb{Z}^d\ :\ kh_n\in G_{h_n}\,,(k+2^{n-m}e_j)h_n\in G_{h_n}\cup\partial G^+_{h_n}\right\}\,.$$

Clearly, for each $kh_n\in G_{h_n}(+j;h_m)$, the difference quotient $\partial^+_{j,h_m}u_k=(u_{k+2^{n-m}e_j}-u_k)/h_m$ is well defined. In an analogous manner, we denote

$$G_{h_n}(-j;h_m)=\left\{k\in\mathbb{Z}^d\ :\ kh_n\in G_{h_n}\,,(k-2^{n-m}e_j)h_n\in G_{h_n}\cup\partial G^+_{h_n}\right\}\,.$$

Let

$$G_{h_n}(+;h_m)=\bigcap_{j=1}^{d}G_{h_n}(+j;h_m),\qquad G_{h_n}(-;h_m)=\bigcap_{j=1}^{d}G_{h_n}(-j;h_m)\quad(11.7)$$

and

$$G_{h_n}(h_m)=\bigcap_{j=1}^{d}\left(G_{h_n}(+j;h_m)\cap G_{h_n}(-j,h_m)\right)\,.\quad(11.8)$$

It is clear that the subsets (11.7) and (11.8) of $G_{h_n}\cap\partial G^+_{h_n}$ satisfy the following properties: for all $kh_n\in G_{h_n}(+;h_m)$, the operator $\nabla^+_{h_m}u_k$ is well defined and, for $kh_n\in G_{h_n}(h_m)$, the operator $\Delta_{h_m}u_k$ is well defined.

We are now in a position to prove the following lemma.

Lemma 11.5. *For each $\boldsymbol{\psi}\in L^{2,h_n}(G_{h_n})$*

$$h_n^d\sum_{kh_n\in G_{h_n}(+;h_m)}\left|\nabla^+_{h_m}\psi_k\right|^2\leqslant\|\nabla^+_{h_n}\boldsymbol{\psi}\|^2_{L^{2,h_n}(G_{h_n})}\quad(11.9)$$

and

$$h_n^d \sum_{kh_n \in G_{h_n}(h_m)} |\Delta_{h_m}\psi_k|^2 \leqslant 4\|\Delta_{h_n}\boldsymbol{\psi}\|^2_{L^{2,h_n}(G_{h_n})}\,. \tag{11.10}$$

Proof. The bound (11.9) is a direct corollary of Lemma 11.3 and the bound (11.10) follows directly from Lemma 11.4. □

Denote

$$G(h_m) = \bigcup_{kh_n \in G_{h_n}(h_m)} Q_k\,, \tag{11.11}$$

where the sets Q_k are defined by (4.1) with $h = h_n$. Then, using the operator $P^*_{h_n}$ defined in (4.14), we immediately obtain from Lemma 11.5 the following assertion.

Lemma 11.6. *For each $\psi(x) \in \widehat{H}^2_{A,h_n}(G)$*

$$\int_{G(h_m)} \left|\nabla^+_{h_m}\psi(x)\right|^2 dx \leqslant C \int_G \left|\nabla^+_{h_n}\psi(x)\right|^2 dx \tag{11.12}$$

and

$$\int_{G(h_n)} |\Delta_{h_m}\psi(x)|^2\, dx \leqslant C \int_G |\Delta_{h_n}\psi(x)|^2\, dx\,, \tag{11.13}$$

Recall that calculation of the functions from (11.12) and (11.13) near the boundaries of G and $G(h_m)$ should be made as was explained near (8.5) and (11.3) with $h = h_n$. where C does not depend on ψ, n, or m.

At last we are now able to prove the following corollary of Proposition 7.1 and Theorems 10.1 and 11.2.

Theorem 11.7. *Let the distribution $\mu(d\psi_0)$ of the initial condition $\psi_0(x,\omega)$ satisfy* (10.3). *Then for each $m < n$ the measures $\nu_{h_n,T}(d\psi)$ satisfy the estimates*

$$\begin{aligned}\int_{Z_T} \Big(\int_0^T \big(\|\Delta_{h_m}\psi(\tau,\cdot)\|^2_{L^2(G(h_m))} + \|\nabla^+_{h_m}\psi(\tau,\cdot)\|^2_{L^2(G(h_m))}\big)\, d\tau\Big)\nu_{h_n T}(d\psi)\\ \leqslant C_T\Big(1 + \int_{L^2(G)} \big(\|\psi_0\|^2_{L^2(G)} + \|\psi_0\|^4_{L^4(G)} + \|\nabla\psi_0\|^2_{L^2(G)}\Big)\mu_0(d\psi_0)\,,\end{aligned} \tag{11.14}$$

where the constant C_T depends only on T. Moreover,

$$\int_{Z_T} \sup_{t\in(0,T)} \|\nabla^+_{h_m}\psi(t,\cdot)\|_{L^2(G(h_m))}\nu_{h_n,T}(d\psi) \leqslant C(T) < \infty \quad \forall T > 0\,, \tag{11.15}$$

where the constant $C(T)$ does not depend on h_m or $h_{n,T}$.

Proof. The theorem follows immediately from Lemma 11.6, Proposition 7.1, and Theorems 10.1 and 11.2. □

11.3 Estimates for the measure ν

We are now in a position to prove the main theorem of this section. We set

$$H^1_\Delta(G) = \Big\{ u(x) \in H^1(G) \,:\, \Delta u(x) \in L^2(G), \\ \|u\|^2_{H^1_\Delta(G)} = \int_G \big(|\Delta u|^2 + |\nabla u|^2 + |u|^2\big)dx < \infty \Big\}. \tag{11.16}$$

Theorem 11.8. *Let the distribution $\mu(d\psi_0)$ of the initial condition ψ_0 satisfy* (10.3). *Then the statistical solution ν constructed in* (10.9) *is supported on the space*

$$supp\,\nu \subset L^{2,\mathrm{loc}}(0,\infty; H^1_\Delta(G)) \cap L^{6,\mathrm{loc}}(0,\infty; L^6(G)) \cap C^L(0,\infty; L^1(G))\,. \tag{11.17}$$

Moreover, the following estimates hold. For every $T > 0$ there exists a constant C_T depending only on T such that

$$\int_{\mathcal{U}_T} \Big(\int_0^T \|\Delta\psi\|^2_{L^2(G)} + \|\nabla\psi\|^2_{L^2(G)} + \|\psi\|^6_{L^6(G)} d\tau \Big) \nu_T(d\psi) \\ \leqslant C_T \Big[1 + \int_{L^2(G)} \big(\|\psi_0\|^2_{L^2(G)} + \|\psi_0\|^4_{L^4(G)} + \|\nabla\psi_0\|^2_{L^2(G)} \big) \mu(d\psi_0) \Big] \tag{11.18}$$

$$\int_{\mathcal{U}_T} (\|\psi\|^2_{L^\infty(0,T;L^2(G))} + \|\nabla\psi\|^2_{L^\infty(0,T;L^2(G))}) \nu_T(d\psi) \leqslant C(T) < \infty \quad \forall T > 0 \tag{11.19}$$

and

$$\int_{Z_T} \|\psi\|_{C^L(0,T;L^1(G))} \nu(d\psi) \leqslant C(T) < \infty \quad \forall T > 0\,. \tag{11.20}$$

Proof. Let $\phi_R(\lambda) \in C^\infty(\mathbb{R}_+)$, $\phi_R(\lambda) = \lambda$ for $\lambda < R$, and $\phi_R(\lambda) = R+1$ for $\lambda \geqslant R+1$. Then the bound (11.14) implies the inequality

$$\int \phi_R\Big(\int_0^T \big(\|\nabla^+_{h_m} \psi(\tau,\cdot)\|^2_{L^2(G(h_m))} + \|\Delta_{h_m}\psi(\tau,\cdot)\|^2_{L^2(G(h_m))}$$

$$+\|\psi(\tau,\cdot)\|^2_{L^2(G)}\big)\,d\tau\Big)\nu_{h_nT}(d\psi) \;\leqslant\; \widehat{C}_T$$

$$\equiv C_T(1+\int\limits_{L^2(G)}\big(\|\psi_0\|^2_{L^2(G)}+\|\psi_0\|^4_{L^4(G)}+\|\nabla\psi_0\|^2_{L^2(G)}\big)\mu(d\psi_0)\,. \tag{11.21}$$

Since the functional under the integral on the left-hand side of (11.21) is bounded and continuous on the space Z from (9.8), we can pass to the limit as $n\to\infty$ in (11.21). As a result, we obtain

$$\int\phi_R\Big(\int\limits_0^T\big(\|\nabla^+_{h_m}\phi(\tau,\cdot)\|^2_{L^2(G(h_m))}+\|\Delta_{h_m}\psi(\tau,\cdot)\|^2_{L^2(G(h_m))} \\ +\|\psi(\tau,\cdot)\|^2_{L^2(G)}\big)\,d\tau\Big)\nu_T(d\psi)\leqslant\widehat{C}_T\,. \tag{11.22}$$

Using the Beppo Levi theorem, we can pass to the limit in (11.22) as $R\to\infty$ to obtain

$$\int\Big(\int\limits_0^T\big(\|\nabla^+_{h_m}\psi(\tau,\cdot)\|^2_{L^2(G(h_m))}+\|\Delta_{h_m}\psi(\tau,\cdot)\|^2_{L^2(G(h_m))} \\ +\|\psi(\tau,\cdot)\|^2_{L^2(G)}\big)\,d\tau\Big)\nu_T(d\psi)\leqslant\widehat{C}_T\,. \tag{11.23}$$

It is easy to prove that

$$\begin{aligned}&\|\Delta_{h_m}u\|_{L^2(G(h_m))}\to\|\Delta u\|_{L^2(G)}<\infty\\&\|\nabla^+_{h_m}u\|_{L^2(G(h_m))}\to\|\nabla u\|_{L^2(G)}<\infty\end{aligned} \tag{11.24}$$

as $h_m\to 0$ if and only if $u\in H^1_\Delta(G)$. Passing to the limit in (11.23) as $h_m\to 0$, with the help of the Fatou theorem and taking into account (11.24), we find that the measure $\nu_T(d\psi)$ satisfies the inequality

$$\int\int\limits_0^T\|\psi(\tau,\cdot)\|^2_{H^1_\Delta(G)}d\tau\;\nu_T(du)\leqslant\widehat{C}_T \tag{11.25}$$

and therefore it is supported on the space $L^2(0,T;H^1_\Delta(G))$. Since the embeddings $H^1_\Delta(G)\subset H^1(G)\subset L^6(G)$ are continuous when the dimension of $G=d\leqslant 3$, the norm $\|u\|_{L^6}$ is continuous on $H^1_\Delta(G)$. Therefore, using as the above function $\phi_R(\lambda)$, we can pass to the limit as $n\to\infty$ in the term of the inequality (10.5) containing $\|\psi\|^6_{L^6(G)}$. As a result, we obtain

$$\int \int_0^T \|\psi(\tau,\cdot)\|^6_{L^6(G)} d\tau \; \nu_T(d\psi) \leqslant \widehat{C}_T \, . \tag{11.26}$$

The inequality (11.19), as well as the bound (11.20) can be obtained with the help of the method used in [44, p. 363]. □

12 The Equation for the Weak Solution of the Stochastic Ginzburg–Landau Problem

Roughly speaking, the weak solution is a measure satisfying a certain equation. We begin with the formal derivation of this equation.

12.1 Definition of the weak solution

The stochastic Ginzburg–Landau equation can be written as the Ito differential equation (3.22) with boundary and initial conditions (2.2) and (2.3) respectively. We let $dW(t,x)$ denote the white noise corresponding to the Wiener process defined in Sect. 3.1, $\psi_0(x) = \psi_0(x,\omega) \in L^4(G) \cap H^1(G)$ is a random initial condition with distribution $\mu(d\psi_0)$, and $\psi_0(x)$ and $W(t,x)$ are independent. Let $S(\lambda)$ be the function given in (7.7). Applying formally the Ito formula to the function $S(\psi(t,x))$ and writing the resulting Ito differential in integral form, we obtain

$$\begin{aligned} L(\psi) &\equiv S[\psi(t,x)] - S[\psi_0(x)] \\ &+ \int_0^t \Big(\widehat{r^{-1}}[\psi(\tau,x)] \left\{ (i\nabla + A(x))^2 \psi(\tau,x) - \psi(\tau,x) + |\psi|^2 \psi(\tau,x) \right\} \\ &\qquad + \frac{1}{2} \widehat{r'}[\psi] \mathcal{K}_{11}(x,x) \Big) \, d\tau = W(t,x) \, , \end{aligned} \tag{12.1}$$

where $\mathcal{K}_{11}(x,x)$ is defined in (3.14).

We introduce the spaces

$$\mathcal{U}_T = L^2(0,T; H^2_A(G)) \cap C^L(0,T; L^1(G)) \cap L^6((0,T) \times G), \quad T > 0 \, , \tag{12.2}$$

and

$$\mathcal{U} = L^{2,\mathrm{loc}}(0,\infty; H^2_A(G)) \cap C^L(0,\infty; L^1(G)) \cap L^{6,\mathrm{loc}}(0,\infty; L^6(G)) \tag{12.3}$$

with the norm for space (12.2)

$$\|\psi\|_{\mathcal{U},T} = \|\psi\|_{L^2(0,T;H^2_A(G))} + \|\psi\|_{C^L(0,T:L^1(G))} \tag{12.4}$$

and with the topology for the space (12.3) defined by the seminorms (12.4) with arbitrary $T > 0$.

Similarly, we consider the continuous operator

$$\mathfrak{A} = (\gamma_0, L) : \mathcal{U} \to L^1(G) \times Z\,. \tag{12.5}$$

Repeating formally the derivation of the equation for the weak statistical solution of the approximation for the Ginzburg–Landau equation, we obtain the following analogue of (9.20):

$$(\mathfrak{A}^*\nu)(B_0 \times B) = \mu(B_0)\Lambda(B) \quad \forall\, B_0 \in \mathcal{B}(L^1(G)),\ B \in \mathcal{B}(Z)\,. \tag{12.6}$$

Definition 12.1. The probability measure ν on $\mathcal{B}(\mathcal{U})$ is called the weak statistical solution of the stochastic Ginzburg–Landau equation (3.22) if it is concentrated on $\mathcal{U}$, satisfies the inequalities (11.18), (11.19), and (11.20), and satisfies (12.6), where $\mathfrak{A}$ is the operator from (12.5) and (12.1).

12.2 The first steps of the proof for ν to satisfy (12.6)

We will show that the measure ν defined in (10.9) satisfies (12.6). Since the other properties in Definition 12.1 are already proven for ν, this gives that ν is a weak statistical solution of the stochastic Ginzburg–Landau equation. We can show that (12.6) is equivalent to the equality

$$\int \eta(\gamma_0\psi)\phi(L(\psi))\nu(d\psi) = \int \eta(\psi_0)\mu(d\psi_0) \int \phi(W)\Lambda(dW) \tag{12.7}$$

for all $\eta \in C_b(L^2(G))$ and $\phi \in C_b(C(0,\infty;L^1(G))$ (recall that $C_b(H)$ is the space of bounded, continuous functions on the Banach space H) in the same way as the analogous assertion was proved in [44, p. 364].

We already proved that there exists a strong stochastic solution of the problem (9.12). Therefore, (9.12) implies (9.20) and (9.20) implies that

$$E(\eta(\gamma_0\psi_h)\phi(L_h(\psi_h)) = \int \eta(\widehat{P_h}\psi_0)\mu(d\psi_0) \int \phi(\widehat{P_h}W)\Lambda(dW)\,, \tag{12.8}$$

where $\widehat{P_h}$ is the operator defined in (9.2). Performing a change of variables on the left-hand side of (12.8), we obtain

$$\int \eta(\gamma_0\widehat{P_h}\psi)\phi(L_h(\widehat{P_h}\psi)\nu_h(d\psi)$$

$$= \int \eta(\widehat{P_h}\psi_0)\mu(d\psi_0) \int \phi(\widehat{P_h}W)\Lambda(dW)\,. \tag{12.9}$$

We derive (12.7) by passing to the limit in (12.9) as $h = h_j \to 0$.

Since for each $\psi_0 \in L^2(G)$ and $W \in C(0,\infty; L^1(G))$ we have $\widehat{P_h}\psi_0 \to \psi_0$ as $h \to 0$ in $L^2(G)$ and $\widehat{P_h}W \to W$ as $h \to 0$ in $C(0,\infty;L^1(G))$, we have the following formulas:

$$\begin{aligned} &\int \eta(\widehat{P_h}\psi_0)\mu(d\psi_0) \to \int \eta(\psi_0)\mu(d\psi_0) \\ &\int \phi(\widehat{P_h}W)\Lambda(dW) \to \int \phi(W)\Lambda(dW) \end{aligned} \tag{12.10}$$

as $h \to 0$.

We now pass to the limit on the left-hand side of (12.9). By virtue of the arguments in [44, p. 364], it is enough to prove (12.7) only for cylindrical functionals ϕ, i.e., for ϕ that actually depend only on a finite number of arguments and is constant with respect an infinite part of the arguments. But each such functional $\phi(u)$ can be approximated by a finite sum of the form

$$\phi(u) \approx \sum_k e^{i[u,v_k]},$$

where

$$[u, v_k] = \int_0^\infty \int_G u\overline{v}_k dxdt.$$

Consequently, we can modify $\phi(L_h(\widehat{P_h}\psi))$ in (12.9) using $e^{i[L_h(\widehat{P_h}\psi),v]}$. We can now write

$$\int \eta(\gamma_0\widehat{P_h}\psi)\phi(L_h(\widehat{P_h}\psi))\nu_h(d\psi) \rightsquigarrow \int \eta(\gamma_0\widehat{P_h}\psi)e^{i[L_h(\widehat{P_h}\psi),v]}\nu_h(d\psi)\,. \tag{12.11}$$

We pass to the limit as $h \to 0$ on the right-hand side of (12.11).

Taking $v \in L^2(0,\infty;H^2(G))$, $v(t,x) = 0$ for $t > t_v$, where $H^2(G)$ is the usual Sobolev space, we can rewrite (9.12) as follows:

$$[L_h(\psi), v] = f_{1,h}(\psi) + f_{2,h}(\psi) + f_{3,h}(\psi) \quad \text{with } \widehat{P_h}\psi \text{ changed on } \psi\,, \tag{12.12}$$

where

$$f_{1,h}(\psi) \;=\; \int_0^\infty \int_G \Big\{ S(\psi(t,x) - S(\gamma_0\widehat{P_h}\psi(\cdot,x))$$

$$+\int_0^t \left(\widehat{r^{-1}}[\psi(\tau,x)]|\psi|^2\psi(\tau,x)-\psi(\tau,x)\right)d\tau\Big\}\overline{v(t,x)}\,dxdt\,, \quad (12.13)$$

$$f_{2,h}(\psi)=\frac{1}{2}\int_0^\infty\int_G\int_0^t r'[\psi(\tau,x)] \Big(\sum_{kh,jh\in G_h}\mathcal{X}_{Q_j}(x)V(Q_k)^{-1}|\Theta_{jk}|^2\mu_k\Big)d\tau\,\overline{v(t,x)}\,dxdt\,, \quad (12.14)$$

and

$$f_{3,h}(\psi)= \int_0^\infty\int_G\int_0^t \widehat{r^{-1}}[\psi(\tau,x)]\Big((i\nabla_h+\widehat{P_h}A(x))^2\psi(\tau,x)\Big)\overline{v(t,x)}\,d\tau dxdt\,, \quad (12.15)$$

where recall that $\widehat{r^{-1}}[\psi(\tau,x)]z$, $z\in\mathbb{C}$, is understood in the meaning of (3.20) and (3.21). First of all, we rewrite $f_{3,h}(\psi)$ by summing by parts. We suppose that each $v(x)\in H^2(G)$ is extended onto $G(\varepsilon)=\{x\in\mathbb{R}^d:\rho(x,G)=\inf_{y\in G}|x-y|<\varepsilon\}$, where $\varepsilon>0$ is fixed, by a fixed extension operator $\mathcal{E}:H^2(G)\to H^2(G(\varepsilon))$ and we denote this extension $\mathcal{E}v(x)$ by $v(x)$. Thus, for small enough h, the difference quotients $\partial^+_{h_j}v(x)=\frac{1}{h}(v(x+e_jh)-v(x))$, $j=1,\dots,d$, are well defined for almost all $x\in G$.

Lemma 12.2. *The expression* (12.15) *is equivalent to*

$$\begin{aligned} f_{3,h}(\psi) &= \int_0^\infty\int_G\int_0^t \Big\{\widehat{r^{-1}}[\psi(\tau,x)]\big((\nabla_h^+-i\widehat{P_h}A(x)\big)\psi(\tau,x))\overline{\nabla_h^+v(t,x)} \\ &\quad+\widehat{r^{-1}}[\psi(\tau,x)]\big((i\nabla_h^++\widehat{P_h}A(x))\psi(\tau,x)\big)\overline{\widehat{P_h}A(x)v(t,x)} \\ &\quad+\sum_{j=1}^d\big(\partial^-_{h_j}\widehat{r^{-1}}[\psi(\tau,x)]\big) \\ &\qquad\Big(\nabla_h^+-i\widehat{P_h}A(x-he_j)\psi(\tau,x-he_j)\Big)\overline{v(t,x)}\Big\}\,d\tau dxdt \end{aligned} \quad (12.16)$$

for each $\psi(\tau,x)=\widehat{P_h}\psi(\tau,x)\in L^2(0,\infty;\widehat{H}^2_{A,h}(G)$ *with the space* $\widehat{H}^2_{A,h}(G)$ *defined in* (11.3)*,* $v(t,x)\in L^2(0,\infty;H^2(G(\varepsilon))$*, and* $v(t,x)=0$ *for* $t>t_0$*.*

Proof. We denote

$$\phi(\tau,x)=(\nabla_h^+-i\widehat{P}_hA(x))\psi(\tau,x)$$

$$\equiv\{\partial_{h_j}^+-i\widehat{P}_hA^j(x))\psi(\tau,x),j=1,\dots,d\}=\{\phi^j(\tau,x),j=1,\dots,d\}$$

and rewrite (12.15) as

$$f_{3,h}(\psi)=-\int_0^\infty\int_G\int_0^t\widehat{r^{-1}}[\psi(\tau,x)] \Big(\sum_{j=1}^d(\partial_{h_j}^--i\widehat{P}_hA^j(x))\phi^j(\tau,x)\Big)\overline{v(t,x)}\,d\tau dxdt\,. \tag{12.17}$$

Taking into account the identity

$$f(x)\partial_{h_j}^-g(x)=\partial_{h_j}^-(f(x)g(x))-(\partial_{h_j}^-f(x))g(x-he_j)$$

and summing by parts, we obtain

$$\begin{aligned}&-\sum_{j=1}^d\int_0^\infty\int_G\int_0^t\widehat{r^{-1}}[\psi(\tau,x)](\partial_{h_j}^-\phi^j(\tau,x))\overline{v(t,x)}\,d\tau dxdt\\ &=-\sum_{j=1}^d\int_0^\infty\int_G\int_0^t\Big\{\partial_{h_j}^-(\widehat{r^{-1}}[\psi(\tau,x)](\phi^j(\tau,x)))\overline{v(t,x)}\\ &\qquad-(\partial_{h_j}^-\widehat{r^{-1}}[\psi(\tau,x)])(\phi^j(x-he_j))\overline{v(t,x)}\Big\}\,d\tau dxdt\\ &=-\sum_{j=1}^d\int_0^\infty\int_G\int_0^t\Big\{\widehat{r^{-1}}[\psi(\tau,x)](\phi^j(\tau,x))\overline{\partial_{h_j}^+v(t,x)}\\ &\qquad+(\partial_{h_j}^-\widehat{r^{-1}}[\psi(\tau,x)])(\phi^j(x-he_j))\overline{v(t,x)}\,d\tau dxdt\,.\end{aligned} \tag{12.18}$$

Note that the term with the integral over ∂G is equal to zero because $\psi(\tau,x)\in\widehat{H}^2_{A,h}(G)$ and by virtue of Lemma 2.3. The relations (12.17) and (12.18) imply (12.16). □

Now we have to pass to the limit as $h\to 0$ in the integral

$$\int\eta(\gamma_0\psi)e^{i[L_h(\psi)v]}\nu_h(d\psi)=\int\eta(\gamma_0\psi)e^{i(f_1(\psi)+f_{2,h}(\psi)+f_{3,h}(\psi))}\nu_h(d\psi)\,. \tag{12.19}$$

To do this, we first have to study $f_{2,h}(\psi)$ and $f_{3,h}(\psi)$.

12.3 Investigation of $f_{2,h}(\psi)$

For $f_{2,h}(\psi)$ we prove the following result.

Lemma 12.3. *The following relation holds*:

$$\sum_{kh,jk\in G_h} \mathcal{X}_{Q_j}(x)|\Theta_{jk}|^2\mu_k = \sum_r \mathcal{K}_{rr}\mathcal{X}_{Q_r}(x) \to \mathcal{K}(x,x) \quad \text{as } h\to 0 \text{ a.e. } x\in G\,, \tag{12.20}$$

where $\mathcal{K}(x,y) = 2(\mathcal{K}_{11}(x,y) - i\mathcal{K}_{12}(x,y))$ *is the correlation function* (3.14) *of the Wiener process* $W(t,x)$ *and* $\mathcal{K}(x,x) = 2\mathcal{K}_{11}(x,x)$.

Proof. Recall that the matrix $\Theta_{\ell j}$ from (4.19) is unitary, i.e.,

$$\sum_k \Theta_{mk}\overline{\Theta}_{ik} = \delta_{mi} \qquad \text{and} \qquad \sum_k \Theta_{km}\overline{\Theta}_{ki} = \delta_{mi}\,. \tag{12.21}$$

We can rewrite (4.19) as follows:

$$\sum_{lr} \overline{\Theta}_{lj}\mathcal{K}_{lr}\Theta_{rk} = \delta_{jk}\mu_k\,. \tag{12.22}$$

Multiplying both parts of (12.22) by Θ_{mj}, summing over j, and using (12.21), we obtain

$$\sum_r \mathcal{K}_{mr}\Theta_{rk} = \Theta_{mk}\mu_k\,. \tag{12.23}$$

Multiplying both sides of (12.23) by $\overline{\Theta}_{jk}$, summing over k, and using (12.21), we obtain

$$\mathcal{K}_{mj} = \sum_k \Theta_{mk}\overline{\Theta}_{jk}\mu_k\,. \tag{12.24}$$

Multiplying both sides of (12.24) by $\mathcal{X}_{Q_m}(x)\mathcal{X}_{Q_j}(y)$ and summing on m,j such that $mh\in G_h$ and $jh\in G_h$, we obtain

$$\sum_{m,j} \mathcal{K}_{mj}\mathcal{X}_{Q_m}(x)\mathcal{X}_{Q_j}(y) = \sum_k \mu_k \sum_{m,j} \Theta_{mk}\overline{\Theta_{jk}}\mathcal{X}_{Q_m}(x)\mathcal{X}_{Q_j}(y)\,. \tag{12.25}$$

Setting $y=x$ in (12.25) and using (4.17), we obtain

$$\sum_k \mu_k \sum_m |\Theta_{mk}|^2\mathcal{X}_{Q_m}(x) = \sum_m \mathcal{K}_{mm}\mathcal{X}_{Q_m}(x) = \sum_m \mathcal{X}_{Q_m}(x)V^{-2}(Q_m)\int_{Q_m}\int_{Q_m} \mathcal{K}(x,y)\,dxdy\,. \tag{12.26}$$

Clearly, the right-hand side of (12.26) tends to $2\mathcal{K}_{11}(x,x)$ for almost all $x \in G$ as $h \to 0$. □

12.4 Subspaces of piecewise linear functions

The investigation of $f_{3,h}(\psi)$ is more difficult. First, we introduce the space of piecewise linear functions on G. For $kh \in G_h$ we consider the piecewise linear function

$$\varepsilon_k(x) = \begin{cases} 1, & x = kh, \\ 0, & x \notin \text{cube with tops } (k \pm e_j)h, j = 1, \dots, d, \\ \text{piecewise linear} & \text{otherwise.} \end{cases} \tag{12.27}$$

We define $PL_h(G)$ as the linear space of functions generated by the basis $\{\varepsilon_k(x), kh \in G_h\}$ and restricted to G. If this space is supplied with the norm of $L^2(G)$, we use the notation $PL_h(G)$ as well. If $PL_h(G)$ is supplied with the norm

$$\|u\|^2_{PL^1_h} = \|\nabla^+_h u\|^2_{L^2(G)} + \|u\|^2_{L^2(G)} \,,$$

we denote this space as $PL^1_h(G)$. If it is supplied with the norm

$$\|u\|^2_{PL^{2,h}} = \|\Delta_h u\|^2_{L^2(G)} + \|\nabla^+_h u\|^2_{L^2(G)} + \|u\|^2_{L^2(G)} \,,$$

then we denote this space as $PL^2_h(G)$. (For the calculation of $\nabla^+_h u$ and $\Delta_h u$ in these norms the functions $\varepsilon_k(x)$ with $kh \in \partial G^+_h$ and with coefficients from (2.23) should also be used.)

Theorem 12.4. *There exist constants C_1 and C_2, independent of h, such that for every $u \in PL^1_h(G)$*

$$C_1\|\partial_j u\|^2_{L^2(G)} \leqslant \|\partial^+_{j,h} u\|^2_{L^2(G)} \leqslant C_2\|\partial_j u\|^2_{L^2(G)}, \quad j = 1, \dots, d \,. \tag{12.28}$$

Proof. The estimates are established with the help of direct calculations. □

Note that the second estimate in (12.28) holds for each $u \in H^1(G)$, where, in the definition of $\partial^+_{j,h} u$, a certain extension operator $E_\delta : H^1(G) \to H^1(G(\delta)$ is used, where $G(\delta)$ is a neighborhood of G with $\text{dist}(\partial G, \partial G(\delta)) = \delta$ with $\delta > 0$ is fixed.

Theorem 12.5. *There exists a topological isomorphism*

$$R_h : \widehat{L}^{2,h}(G) \to PL_h(G) \,. \tag{12.29}$$

Moreover, the following estimates for the operator R_h hold:

$$\|R_h u\|_{PL_h^1(G)} \leqslant C_1\|u\|_{\widehat{H}^1_{A,h}(G)} \leqslant C_2\|R_h u\|_{PL_h^1(G)} \tag{12.30}$$

$$\|R_h u\|_{PL_h^2(G)} \leqslant C_1\|u\|_{\widehat{H}^2_{A,h}(G)} \leqslant C_2\|R_h u\|_{PL_h^2(G)}\,. \tag{12.31}$$

Proof. The isomorphism R_h is established as follows. For each $u(x) \in \widehat{L}^{2,h}(G)$ we take

$$Ru(kh) = u(kh) \quad \forall\ kh \in G_h \cup \partial G_h^+\,. \tag{12.32}$$

(For calculating $u(kh)$ for $kh \in \partial G_h^+$ we use the boundary conditions (2.23).) Since in both the spaces $\widehat{L}^{2,h}(G)$ and $PL_h(G)$ the values of the points $kh \in G_h \cup \partial G_h^+$ define the function uniquely for each $x \in G$, (12.32) establishes the isomorphism. The estimates (12.30) and (12.31) are proved by direct calculations. □

12.5 The measures $\widehat{\nu}_{h_n}$ and their weak compactness

We need the following analogue of the compactness lemma given in Lemma 8.3.

Lemma 12.6. *For each $R > 0$ the set*

$$\Theta_R = \bigcup_{n=1}^{\infty} B_R(PL^2_{h_n}(G)) \cup B_R(H^2_A(G)) \tag{12.33}$$

is compact in $H^1(G)$ if $B_R(H) = \{x \in H\ :\ \|x\|_H \leqslant R\}$ for each Hilbert space H.

Proof. Similarly to Lemma 8.3, it suffices to choose from the sequence $u_h \in B_R(PL^2_{h_n})$ a subsequence convergent in $H^1(G)$. Clearly, we can choose a subsequence $u_m \to \widehat{u}$ weakly in $H^1(G)$ because, by virtue of (12.30) and (12.31), $u_n \in B_R(PL^2_{h_n}) \subset B_R(H^1(G))$. The following bound holds:

$$\int_G \left|\partial^-_{jh}\partial_{\ell h}\psi(x)\right|^2 dx \leqslant C \int_G \left|\partial^-_{j,h}\partial^+_{\ell h}\psi(x)\right|^2 dx \leqslant C_1\|\psi\|_{PL_h^2(G)}\,, \tag{12.34}$$

where C and C_1 do not depend on h. Indeed, the first inequality follows clearly from (12.28) and the second is a corollary of the discrete analogue of the elliptic theory. Recall that by the definition of $\|\psi\|_{PL_h^2(G)}$, the boundary condition for ψ is fixed by (2.23). Since the right-hand side of (12.34) with $\psi = u_n$ is bounded by C_1R, (12.34) implies that for each $\varepsilon > 0$ there exists $\delta > 0$ such that for $h < \delta$

$$\int |\big(\nabla u_m(x - e_j h) - \nabla u_m(x)\big)^2 \, dx < \varepsilon.$$

By this inequality we can choose a subsequence $\{u_k\} \subset \{u_m\}$ strongly converging in $H^1(G)$. □

Using Lemma 12.6 analogously to Theorem 8.7, we can prove the following theorem.

Theorem 12.7. *For each $R > 0$ the set*

$$\widehat{\Theta}_R = \bigcup_{n=1}^{\infty} B_R(W_{h,T}) \cup B_R(W_T) \tag{12.35}$$

is compact in $L^2(0,T;H^1(G)) \cap L^4(0,T:L^4(G)) \cap C(0,T;L^1(G))$, where

$$\begin{aligned} W_{h,T} &= L^2(0,T;PL^2_{h_n}(G)) \cap C^L(0,T;L^1(\Omega)), \\ W_T &= L^2(0,T;H^1_\Delta(G)) \cap C^L(0,T;L^1(\Omega)) \cap L^6(0,T:L^6(G)) \end{aligned} \tag{12.36}$$

and where $H^1_\Delta(G)$ is the space defined in (11.16).

Clearly, the isomorphism (12.29) generates the isomorphism

$$R_h : L^2(0,T;\widehat{H}^1_{A,h}(G)) \to L^2(0,T;PL^1_h(G)) \,. \tag{12.37}$$

Using (12.37) and the weak solution $\nu_{h_n}(d\psi)$ defined in (9.11), we can define the following measure $\widehat{\nu}_{h,T}$ on $L^2(0,T;PL^1_h(G))$:

$$\widehat{\nu}_{hT}(B) = \nu_{hT}(R_h^{-1}B) \quad \forall\, B \in \mathcal{B}(L^2(0,T;PL^1_h(G))) \,. \tag{12.38}$$

The definition (12.38), the estimates (10.4) and (10.6) for ν_{hT}, and the inequalities (11.14) and (12.31) imply the following inequality for the measures $\widehat{\nu}_{h_nT}$:

$$\int \Big(\int_0^T \|\psi(t,\cdot)\|^2_{PL^2_h(G)} dt + \|\psi\|_{C^L(0,T;L^1(G))} \Big) \widehat{\nu}_{h_nT} \leqslant C_T \tag{12.39}$$

with C_T independent of h.

Using this estimate, the compactness result in Theorem 12.7, and the Prokhorov theorem (see Theorem 10.2), by following the proof of Lemma 10.3, we obtain the following result.

Theorem 12.8. *The measures $\widehat{\nu}_{h_nT}(\omega)$ are weakly compact on $L^2(0,T;H^1(G))$. Moreover,*

$$\widehat{\nu}_{h_k,T} \to \nu_T \quad \textit{as } k \to \infty \textit{ weakly on } L^2(0,T;H^1(G)) \,, \tag{12.40}$$

where h_k is a subsequence of the sequence h_j in (10.9)*,* $\nu_T = \Gamma *_T \nu$*, where* ν_T *is the measure* (12.40)*,* ν *is the measure* (10.9)*, and* Γ_T *is the operator* (10.1)*.*

Proof. It was already explained that $\widehat{\nu}_{h_k,T} \to \widehat{\nu}_T$ weakly on $L^2(0,T;H^1(G))$, where $\widehat{\nu}_T$ is a certain measure. To prove $\Gamma *_T \nu = \widehat{\nu}_T$, we have to take into account the fact that

$$R_h \widehat{P}_h u \to u \quad \text{as } h \to 0 \qquad \forall\, u \in L^2(G)\,. \tag{12.41}$$

Indeed, by virtue of (12.38),

$$\int f(u)\widehat{\nu}_{h,T}(du) = \int f(R_h\widehat{P}_h v)\nu_{h,T}(du)$$

if $f(u)$ is continuous on $L^2\big((0,T)\times G\big)$. Passing to the limit as $h \to 0$, with the help of (12.41), we obtain $\widehat{\nu}_T = \nu_T = \Gamma *_T \nu$. □

12.6 The final steps for passage to the limit

Now we are in a position to pass to the limit in (12.19). Let $N_h = R_h^{-1}$ be the operator inverse to (12.29). The equality (12.38) can be rewritten as

$$\nu_{hT}(B) = \widehat{\nu}_{h,T}(N_h^{-1}B) \quad \forall\, B \in \mathcal{B}(L^2(0,T;L^2(G)) \tag{12.42}$$

and using this, we can rewrite (12.19) in the form

$$\int \eta(\gamma_0\psi)e^{i[L_h(\psi),v]}\nu_h(d\psi) = \int \eta(\gamma_0 N_h u)e^{i[L_h(N_h u),v]}\widehat{\nu}_h(du)\,. \tag{12.43}$$

The most difficult term for passing to the limit in (12.19) as $h \to 0$ is the term $f_{3,h}(N_h u)$ from (12.16). In that integral, $u(\tau,x) \in L^2(0,T;PL_h^1(G))$. But as follows from the lemma formulated below, the operator N_h can be extended from $PL_h^1(G)$ to $H^1(G)$.

Lemma 12.9. *The operator N_h can be extended from $PL_h^1(G)$ to $H^1(G)$. Moreover, for each $u \in H^1(G)$*

$$\|\nabla_h^+ u - \nabla u\|_{L^2(G)} \to 0 \quad \textit{as } h \to 0\,. \tag{12.44}$$

Proof. In addition to the basis $\{\varepsilon_k(x), kh \in G_h \cup \partial G_h^+\}$, we introduce in PL_h an associated basis $\{\varepsilon_k^*(x), kh \in G_h \cup \partial G_h^+\}$ that is defined by the condition

$$\int\limits_{G(\delta)} \varepsilon_j(x)\overline{\varepsilon_k^*(x)}\,dx = \delta_{kj}\,, \tag{12.45}$$

where δ_{kj} is the Kronecker symbol and $G(\delta) = \{x \in \mathbb{R}^d : \rho(x,G) = \inf_{y\in G}|x-y| < \delta\}$ is a neighborhood of G containing the set $\bigcup\limits_{kh\in G_h\cup\partial G_h^+} Q_k$ with Q_k defined in (4.1)-(4.5) and (8.6). To construct $\{\varepsilon_k^*(x)\}$, we look for these functions in the form

$$\varepsilon_k^*(x) = \sum_{jh\in G_h\cup\partial G_h^+} \alpha_{kj}\varepsilon_j(x)\,, \tag{12.46}$$

where α_{k_j} is the solution of the system of linear algebraic equations obtained after substitution (12.46) into (12.45). By the definition of the operator N_h,

$$N_h f(x) = \sum_{jh\in G_h\cup\partial G_h^+} f_j \mathcal{X}_{Q_j}(x), \quad \text{where} \quad f(x) = \sum_{kh\in G_h\cup\partial G_h^+} f_k\varepsilon_k(x) \in PL_h$$

and $\mathcal{X}_{Q_j}(x)$ is the characteristic function of the set Q_j. The extension of this operator on $H^1(G(\delta))$ is defined as follows:

$$N_h f(x) = \sum_{kh\in G_h\cup\partial G_h^+} \mathcal{X}_{Q_j}(x) \int\limits_{G(\delta)} f(x)\overline{\varepsilon_j^*(x)}\,dx\,. \tag{12.47}$$

The relation (12.44) is verified by direction calculations. □

Using Lemma 12.9, it is easy to prove the following result.

Lemma 12.10. (a) *For each sufficiently small* h *the functional* $f_{3,h}(N_h u)$ *defined in* (12.16) *is continuous in* $u \in L^{2,\mathrm{loc}}(0,\infty;H^1(G))$.

(b) *For each* $u \in L^{2,\mathrm{loc}}(0,\infty;H^1(G))$

$$\begin{aligned} f_{3,h}(N_h u) \underset{h\to 0}{\longrightarrow} \int\limits_0^\infty \int\limits_G \int\limits_0^t \Big\{ & \widehat{r^{-1}}[u(\tau,x)]\big((\nabla - iA(x))u(\tau,x)\big)\overline{\nabla v(t,x)} \\ & -\widehat{r^{-1}}[u(\tau,x)]\big((i\nabla + A(x))u(\tau,x)\big)\overline{A(x)v(t,x)} \\ & + \sum_{j=1}^d \big(\partial_j \widehat{r^{-1}}[u(\tau,x)]\big)\big((\nabla - iA(x))u(\tau,x)\big)\overline{v(t,x)}\Big\}\, d\tau dx dt\,. \end{aligned} \tag{12.48}$$

Lemmas 12.3 and 12.10 imply the following assertion.

Lemma 12.11. (a) *For each sufficiently small* h *the functional* $[L_h(N_h u), v]$ *is continuous in* $u \in L^{2,\mathrm{loc}}(0,\infty;H^1(G)) \cap L^{4,\mathrm{loc}}(0,\infty;L^4(G))$.

(b) *For each* $u \in L^{2,\mathrm{loc}}(0,\infty;H^1(G)) \cap L^{4,\mathrm{loc}}(0,\infty;L^4(G))$

$$[L_h(N_h u), v] \to [L_w(u), v] \qquad \text{as } h \to 0\,, \tag{12.49}$$

where

$$
\begin{aligned}
[L_w(u),v] = \int_0^\infty \int_G \Big\{ & \big(S(u(t,x) - S(\gamma_0 u(\cdot,x))\\
& + \int_0^t \widehat{r^{-1}}[u(\tau,x)](|u|^2 u(\tau,x) - u(\tau,x))\, d\tau\Big\}\overline{v}(t,x)\\
& + \int_0^t \frac{1}{2} r'[u(\tau,x)]\, d\tau K(x,x)\overline{v(t,x)}\\
& + \int_0^t \Big\{\widehat{r^{-1}}[u(\tau,x)]\big((\nabla - iA(x))u(\tau,x)\big)\overline{\nabla v(t,x)}\\
& + \widehat{r^{-1}}[u(\tau,x)]\big((i\nabla + A(x))u(\tau,x)\big)\overline{A(x)v(t,x)}\\
& + \sum_{j=1}^d \big(\partial_j \widehat{r^{-1}}[u(\tau,x)]\big)\big((\nabla - iA(x))u(\tau,x)\big)\overline{v(t,x)}\Big\}d\tau\Big\} dxdt
\end{aligned}
\tag{12.50}
$$

and where the index w *in* $[L_w(u),v]$ *means that* (12.50) *is the weak form of the operator* L.

Now we are in a position to prove the main lemma.

Lemma 12.12. *The following relation holds*:

$$\int \eta(\gamma_0\psi)e^{i[L_h(\psi),v]}\nu_h(d\psi) \underset{h\to 0}{\longrightarrow} \int \eta(\gamma_0\psi)e^{i[L_w(\psi),v[}\nu(d\psi)\,, \tag{12.51}$$

where $\nu(d\psi)$ *is the measure from* (10.9) *and* $[L_w(\psi,v]$ *is defined in* (12.50).

Proof. By virtue of (12.42), it is sufficient to prove

$$\int \eta(\gamma_0 N_h u)e^{i[L_h(N_h u),v]}\widehat{\nu}_h(du) \underset{h\to 0}{\longrightarrow} \int \eta(\gamma_0 u)e^{i[L_w(u),v]}\nu(du)\,. \tag{12.52}$$

Theorem 12.8 and the continuity on $L^{2,\mathrm{loc}}(0,\infty;H^1(G))\cap C(0,\infty;L^1(G))\cap L^{4,\mathrm{loc}}(0,\infty;L^4(G))$ of the functional $u\to e^{i[L_w(u),v]}\gamma_0(u)$ imply

$$\int \eta(\gamma_0 u)e^{i[L_w(u),v]}\widehat{\nu}_h(du) \to \int \eta(\gamma_0,u)e^{iL_w(u),v]}\nu(du), \quad h\to 0\,. \tag{12.53}$$

By virtue of Theorem 12.7, for each R, the set $\widehat{\Theta}_R$ defined in (12.35) is compact in $L^2(0,T;H^1(G))\cap L^4((0,T)\times G)\cap C(0,T;L^1(G))$, where T is chosen in such a way that $v(t,x)\equiv 0$ for $t>T$. Thus, by Lemma 12.11, for each $R>0$,

$$\gamma_0(N_h u)e^{i[L_h(N_h u),v]} \to \gamma_0(u)e^{i[L_w(u),v]} \quad \text{as } h\to 0 \tag{12.54}$$

uniformly over $u \in \widehat{\Theta}_R$. In addition, for every $\varepsilon > 0$ there exists $R > 0$ such that

$$\int\limits_{Q_R} \left|\gamma_0(N_h u) e^{i[L_h(N_h u),v]}\right| \widehat{\nu}_h(du) < \varepsilon \quad \forall\, h\,, \tag{12.55}$$

where $Q_k = L^2(0,T;H^1(G))\backslash\widehat{\Theta}_R$. The relations (12.53)–(12.55) imply (12.52). □

Thus, we obtain from (12.9)–(12.11) and (12.51) the equality

$$\int \eta(\gamma_0\psi) e^{i[L_w(\psi),v]}\nu(d\psi) = \int \eta(\psi_0)\mu(d\psi_0) \int e^{i[W,v]} \Lambda(dW) \tag{12.56}$$

for each $v(t,x) \in L^2(0,\infty;H^1(G))$, $v(t,x) \equiv 0$ for $t > t_v$. Now we are in a position to prove (12.7).

12.7 Proof of the equality (12.7)

By virtue of Theorem 11.8, the statistical solution $\nu(d\psi)$ (more precisely, its restriction $\nu_T(d\psi)$ on the time interval $(0,T)$) is supported on the space W_T defined in (12.36).

Theorem 12.13. *The weak statistical solution $\nu(d\psi)$ satisfies Equation* (12.7) *for each $\eta \in C_b(L^2(G))$ and $\phi \in C_b(0,\infty;L^2(G))$.*

Proof. The main step of the proof is to show that, besides (12.56), the weak statistical solution $\nu(d\psi)$ satisfies the equality

$$\int \eta(\gamma_0\psi) e^{i[L(\psi),v]}\nu(d\psi) = \int \eta(\psi_0)\mu(d\psi_0) \int e^{i[w,v]} \Lambda(dW) \tag{12.57}$$

for each $v(t,x) \in L^2(0,\infty;H^1(G))$ with $v(t,x) = 0$ for $t > t_v$, where $L(\psi)$ is the strong form of the operator L defined in (12.1). Recall that $H_0^1(G) = \{u(x) \in H^1(G) \,:\, u|_{\partial G} = 0\}$; we must prove that

$$[L_w(\psi),v] = [L(\psi),v] \quad \forall\, v \in L^2(0,\infty;H_0^1(G)), v = 0 \text{ for } t > t_v. \tag{12.58}$$

By virtue of definitions (12.1) and (12.50) of $L(\psi)$ and $[L_w(\psi),v]$, to prove (12.58) we have to establish the equality

$$\int\limits_0^\infty \int\limits_G \int\limits_0^t \widehat{r^{-1}}[\psi(\tau,x)]\big((i+\nabla A(x))^2\psi(\tau,x)\; d\tau\; \overline{v(t,x)}\; dxdt$$

$$
\begin{aligned}
= \int_0^\infty \int_G \int_0^t \Big\{ & \widehat{r^{-1}}[\psi(\tau,x)]\big((\nabla - iA(x))\psi(\tau,x)\overline{\nabla v(t,x)} \\
& + \widehat{r^{-1}}[\psi(\tau,x)]\big(i\nabla + A(x))\psi(\tau,x)\big)\overline{A(x)v(t,x)} \\
& + \sum_{j=1}^d \Big(\partial_j \widehat{r^{-1}}[\psi(\tau,x)]\big((\partial_j - iA(x))\psi(\tau,x)\big)\overline{v(t,x)}\Big\} \, d\tau dx dt \, .
\end{aligned} \tag{12.59}
$$

To prove this equality, one has to integrate by parts in the first term on the right-hand side and take into account that $v|_{\partial G} = 0$. This integration by parts is well justified because $\nu(d\psi) = \nu_{t_v}(d\psi)$ is supported on W_{t_v} and therefore, in (12.58), $\psi \in W_{t_v}$.

Consequently, (12.56) with $v \in L^2(0,\infty; H_0^1(G))$ and (12.58) imply (12.57). Since both parts of equality (12.57) are continuous functionals with respect to $v \in L^2((0,T) \times G)$ with $v = 0$ for $t > T$ for arbitrary $T > 0$, (12.57) can be extended by continuity of $v \in L^2((0,T) \times G)$ ($v = 0$ for $t > T$) for each $T > 0$. Now (12.7) follows from (12.57) for each cylindrical η and ϕ and, after that, for arbitrary $\eta \in C_b(L^2(G))$ and $\phi \in C_b(0,\infty; L^2(G))$. □

13 Certain Properties of the Weak Statistical Solution ν

In this section, we show that the statistical solution $\nu(d\psi)$ is supported on solutions ψ of Equation (12.1) and these solutions ψ satisfy the boundary condition (2.2) on ∂G.

13.1 Boundary conditions

The following easy assertion is true.

Lemma 13.1. *Let $H^1_\Delta(G)$ and $H^2_A(G)$ denote the spaces defined in* (11.16) *and* (2.5) *respectively. Then*

$$
H^2_A(G) = \{\psi \in H^1_\Delta(G) \, : \, (i\nabla + A)\psi \cdot n|_{\partial G} = 0\} \equiv \widetilde{H} \, , \tag{13.1}
$$

where n is the unit outer normal to ∂G and the last identity is the definition of $\widetilde{H}$.

Proof. It is enough to prove the inclusion $\widetilde{H} \subset H^2_A(G)$ because the inverse inclusion is evident. If $\psi \in \widetilde{H}$, then

$$\Delta\psi = f \in L^2(G), \quad (i\nabla + A)\psi \cdot n|_{\partial G} = 0\,. \tag{13.2}$$

This boundary value problem is elliptic because its boundary condition satisfies the Lopatinsky condition. That is why the inequality

$$\|\psi\|_{H^2(G)} \leqslant C\|f\|_{L^2(G)} = C\|\Delta\psi\|_{L^2(G)}$$

holds, where C does not depend on ψ. This inequality implies $\widetilde{H} \subset H^2_A(G)$. □

Recall that the space $\mathcal{U}_T$ is defined in (12.2).

Theorem 13.2. *For each $T > 0$ the restriction $\nu_T(d\psi)$ of the statistical solution $\nu(d\psi)$ on the time interval $(0,T)$ is supported on the space $\mathcal{U}_T$.*

Proof. Since $\nu_T(d\psi)$ is supported on the space W_T defined in (12.36), we have to prove, by virtue of Lemma 13.1, that there exists a $\nu_T(d\psi)$-measurable set $\mathcal{F} \subset W_T$ such that $\nu_T(\mathcal{F}) = 1$ and $(i\nabla+A)\psi\cdot n|_{\partial G} = 0$ for each $\psi \in \mathcal{F}$. Taking $\eta \equiv 1$ in (12.56), we differentiate this equality twice on $v \in L^2(0,\infty; H^1(G))$ such that $v(t,x) \equiv 0$ for $t \geqslant T$. As a result, we obtain

$$\int [L_w(\psi)u]^2 e^{i[L_w(\psi),v]}\nu_T(d\psi) = \int [W,u]^2 e^{i[w,v]}\Lambda_T(dW)\,, \tag{13.3}$$

where $u \in L^2(0,T; H^1(G))$ is arbitrary. We take $v \equiv 0$ in (13.3) and then integrate by parts on the left-hand side of this equality as we did in (12.59). This integration by parts is well-justified because the inclusion $u \in L^2(0,T; H^1(G))$ implies that $u|_{\partial G} \in L^2(0,T; H^{1/2}(\partial G))$ and, as is well-known (see [17, 31]), the inclusion $\psi \in L^2(0,T; H^1_\Delta(G))$ implies $(i\nabla + A)\psi \cdot n|_{\partial G} \in L^2(0,T : H^{-1/2}(\partial G))$.

Since $u|_{\partial G} \neq 0$, in contrast to (12.58), after integration by parts we obtain

$$\begin{aligned}\int \Big(\int_0^T \int_{\partial G} \int_0^t \widehat{r^{-1}}[\psi(\tau,x)](\nabla - iA(x))\psi(\tau,x)) \cdot n\, \overline{u(t,x)}\, d\tau dx dt \\ +[L(\psi),u]\Big)^2 \nu_T(du) = \int [W,u]^2 \Lambda_T(dW)\,.\end{aligned} \tag{13.4}$$

Instead of $u(t,x)$in (13.4), we now take the sequence $u_n(t,x)$ that satisfies the properties:

a. $u_n(t,x) \to 0$ in $L^2((0,T) \times G)$;
b. for each n, $u_n(t,x)|_{\partial G} = \partial_t v(t,x)$, where $v(t,x) \in H^1_0(0,T; H^{1/2}(\partial G)$ is fixed.

Passing to the limit in (13.4) as $n \to \infty$ and after that integrating by parts on the left-hand side of the resulting equality, we obtain

$$\int \Big(\int\limits_0^T \int\limits_{\partial G} r^{-1}[\psi(t,x)]\big((\nabla - iA(x))\psi(t,x)\cdot n\big)\overline{v(t,x)}\, dxdt\Big)^2 \nu_T(du) = 0\,. \tag{13.5}$$

Now we choose a countable dense set $\{v_n\}$ in $L^2(0,T;H^{1/2}(G))$ and, for each n, put v_n in (13.5). As a result, for each n we obtain the measurable set $\mathcal{F}_n \subset W_T$ such that

$$\nu_T(\mathcal{F}_n) = 1\,,$$

$$\int\limits_0^T \int\limits_{\partial G} r^{-1}[\psi(t,x)]\big((\nabla - iA)\psi(t,x)\cdot n\big)\overline{v_n(t,x)}\, dxdt = 0 \quad \forall\, \psi \in \mathcal{F}_n\,. \tag{13.6}$$

We take $\mathcal{F} = \bigcap\limits_n \mathcal{F}_n$. Clearly, $\nu_T(\mathcal{F}) = 1$ and

$$\widehat{r^{-1}}[\psi(t,x)]\big((\nabla - iA(x))\psi(t,x)\cdot n\big)|_{(0,T)\times\partial G} = 0 \quad \forall\, \psi \in \mathcal{F}\,. \tag{13.7}$$

Since $r^{-1}(\mathrm{Re}\,\psi(t,x)) > 0$ and $r^{-1}(\mathrm{Im}\psi(t,x)) > 0$ for all $(t,x) \in (0,T)\times\overline{G}$, (13.7) implies

$$\nu_T(\mathcal{F}) = 1, \quad (i\nabla + A)\psi(t,x)\cdot n|_{(0,T)\times\partial G} = 0 \quad \forall\, \psi \in \mathcal{F}\,.$$

These equalities complete the proof of the theorem. □

13.2 Solvability for almost all data

Recall that the initial measure μ is supported on the space $H^1(G)$ and the Wiener measure Λ is supported on the set $\widehat{W}$ defined in (9.17).

Theorem 13.3. (a) *For $\mu\times\Lambda$-almost all data (ψ_0, W) there exists a solution $\psi \in \mathcal{U}$ of the problem* (12.1)*.*

(b) *The weak statistical solution ν is supported on solutions of the problem* (12.1) *and* (2.2)*.*

Proof. Since $\mathcal{U}$ defined in (12.3) is a separable Frechet space, by the Riesz theorem (see [19]), for any $N > 0$ there exists a compact set $K_N \subset \mathcal{U}$ such that

$$\nu(K_N) \geqslant 1 - \frac{1}{N}\,. \tag{13.8}$$

The continuity of the operator (12.5) implies that $\mathfrak{A}K_N$ is compact in $L^1(G)\times Z$ and therefore $\mathfrak{A}K_N \in \mathcal{B}(L^1(G)\times Z)$. We set

$$F_N = \mathfrak{A} K_N \cap \{H^1(G) \times \widehat{W}\}, \quad F = \bigcup_{N=1}^{\infty} F_N \,. \tag{13.9}$$

Since $H^1(G) \in \mathcal{B}(L^1(G))$ (see [44, Theorem 2.1]) and the set $\widehat{W}$ is Λ-measurable, each set from (13.9) is $\mu \times \Lambda$-measurable. By virtue of (12.6),

$$\nu(\mathfrak{A}^{-1}(H^1(G) \times \widehat{W})) = \mu(H^1(G)) \cdot \Lambda(\widehat{W}) = 1 \,. \tag{13.10}$$

Thus, taking into account (13.8)-(13.10), we obtain

$$\begin{aligned} \mu \times \Lambda(F) = \nu(\mathfrak{A}^{-1} F) \geqslant \nu(\mathfrak{A}^{-1}(H^1(G) \times \widehat{W}) \cap \bigcup_{N=1}^{\infty} K_N) \\ = \nu(\bigcup_{N=1}^{\infty} K_N) \geqslant \lim_{N \to \infty} \nu(K_N) = 1 \,. \end{aligned} \tag{13.11}$$

Directly from the definition $\mathfrak{A}^{-1} F = \{\psi \in \mathcal{U} \, : \, \mathfrak{A}\psi \in F\}$, we obtain

$$F \in \mathfrak{A}\mathcal{U} \,. \tag{13.12}$$

The relations (13.11) and (13.12) prove statement (a) of the theorem. We set

$$K = (\bigcup_{N=1}^{\infty} K_N) \cap \mathfrak{A}^{-1}(H^1(G) \times \widehat{W}) \,. \tag{13.13}$$

The relations (13.8), (13.10), and (13.13) imply $\nu(K) = 1$, and the relations (13.9) and (13.12) imply that $\mathfrak{A} K = F$. The last two relations prove statement (b) of the theorem. □

14 Uniqueness of the Weak Statistical Solution

The main step in proving the uniqueness of a weak statistical solution for the stochastic Ginzburg–Landau problem is a proof of uniqueness for (12.1) with fixed (non-stochastic) data $(\psi_0(x), W(t,x))$.

14.1 Reduction of uniqueness for statistical solution ν to uniqueness of the solution for (12.1)

Let F and K be the sets (13.9) and (13.13) respectively. In Theorem 13.3, we proved that the set F is $\mu \times \Lambda$-measurable, K is ν-measurable,

$$(\mu \times \Lambda)(F) = 1, \quad \nu(K) = 1, \quad \text{and} \quad \mathfrak{A}K = F\,, \tag{14.1}$$

where $\mathfrak{A}$ is the operator (12.5), ν is a weak statistical solution, μ is the initial measure, and Λ is the Wiener measure.

Lemma 14.1. *If, for each initial datum* $(\psi_0, W) \in F$*, an individual solution* ψ *of the problem* (12.1) *and* (2.2) *is unique in* K*, then the statistical solution* ν *of the stochastic Ginzburg–Landau problem* (3.22)*,* (2.2)*, and* (2.3) *is unique.*

Proof. In Theorem 13.3, we proved that each weak statistical solution ν corresponding to the given initial measure μ and the Wiener measure Λ is supported on the set K defined in (13.13). Since for each datum $(\psi_0, W) \in F$, the solution ψ of (12.1) and (2.2) is unique in K, the full preimage

$$\mathfrak{A}^{-1}F = \{\psi \in K \,:\, \mathfrak{A}\psi \in F\} \tag{14.2}$$

consists of the unique element $\psi \in K$ for each given datum $(\psi_0, W) \in F$. Therefore, a weak statistical solution $\nu(d\psi)$ is defined uniquely by the formula

$$\nu(B) = \nu(B \cap K) = \mu(\gamma_0 B)\Lambda(LB) \quad \forall\, B \in \mathcal{B}(\mathcal{U})\,. \tag{14.3}$$

□

14.2 Proof of the uniqueness of the solution of (12.1) *and* (2.2)*: the first step*

Suppose that for a given datum $(\psi_0, W) \in F$ there exist two solutions $\psi_i(t, x) \in K$, $i = 1, 2$, of the problem (12.1) and (2.2). Then

$$L(\psi_1) - L(\psi_2) = 0, \qquad (\psi_1 - \psi_2)|_{t=0} = 0\,, \tag{14.4}$$

where L is the operator defined in (12.1). Denote

$$\sigma(t, x) = S[\psi_1(t, x)] - S[\psi_2(t, x)]\,. \tag{14.5}$$

Since $\psi_i \in K \subset \mathcal{U}$, $i = 1, 2$, where $\mathcal{U}$ is the space (12.3), the relations (12.1) and (14.4) imply that for each $T > 0$, $\sigma(t, x) \in H^1(0, T; L^2(G))$, i.e., σ is differentiable in t. Thus, we can differentiate both parts of (14.4) with respect to t. Doing this, we obtain by (12.1):

$$\begin{aligned}
&\partial_t \sigma(t, x) + \widehat{r^{-1}}[\psi_1]\{(i\nabla + A)^2\psi_1 - \psi_1 + |\psi_1|^2\psi_1\} \\
&-\widehat{r^{-1}}[\psi_2]\{(i\nabla + A)^2\psi_2 - \psi_2 + |\psi_2|^2\psi_2)\} \\
&+(r'[\psi_1] - r'[\psi_2])\mathcal{K}_{11}(x, x) = 0\,.
\end{aligned} \tag{14.6}$$

Multiplying (14.6) by $\overline{\sigma(t,x)}$ and integrating over G, we obtain

$$\frac{1}{2}\partial_t\|\sigma(t,\cdot)\|^2_{L^2(G)} + T_1 + T_2 + T_3 + T_4 = 0\,, \tag{14.7}$$

where

$$T_1 = \int\limits_G \Big(\widehat{r^{-1}}[\psi_1]\{(i\nabla + A)^2\psi_1\} - \widehat{r^{-1}}[\psi_2]\{(i\nabla + A)^2\psi_2\}\Big)\overline{\sigma}\;dx\,, \tag{14.8}$$

$$T_2 = -\int\limits_G \Big(\widehat{r^{-1}}[\psi_1]\psi_1 - \widehat{r^{-1}}[\psi_2]\psi_2\Big)\overline{\sigma}\;dx\,, \tag{14.9}$$

$$T_3 = \int\limits_G \Big(r'[\psi_1] - r'[\psi_2]\Big)\{\mathcal{K}_{11}(x,x)\}\overline{\sigma}\;dx\,, \tag{14.10}$$

and

$$T_4 = \int\limits_G \Big(\widehat{r^{-1}}[\psi_1]\{|\psi_1|^2\psi_1\} - \widehat{r^{-1}}[\psi_2]\{|\psi_2|^2\psi_2\}\Big)\overline{\sigma}\;dx\,. \tag{14.11}$$

Taking into account

$$\nabla_x S[\psi(t,x)] = \widehat{r^{-1}}[\psi(t,x)]\nabla_x\psi(t,x) \tag{14.12}$$

and performing a transformation analogous to the one in (12.59), we obtain

$$T_1 = \int\limits_G |\nabla_x\sigma(t,x)|^2\;dx + T_5 + T_6 + T_7 + T_8 + T_9\,, \tag{14.13}$$

where

$$T_5 = \int\limits_G \Big(\widehat{r^{-1}}[\psi_2]\{iA\psi_2\} - \widehat{r^{-1}}[\psi_1]\{iA\psi_1\}\Big)\cdot\overline{\nabla\sigma}\;dx\,, \tag{14.14}$$

$$T_6 = \int\limits_G \Big(\widehat{r^{-1}}[\psi_1]\{i\nabla\psi_1\} - \widehat{r^{-1}}[\psi_2]\{i\nabla\psi_2\}\cdot A(x)\overline{\sigma}\;dx\,, \tag{14.15}$$

$$T_7 = \int\limits_G \Big(\widehat{r^{-1}}[\psi_1]\{A\psi_1\} - \widehat{r^{-1}}[\psi_2]\{A(x)\psi_2\}\cdot A(x)\overline{\sigma}\;dx\,, \tag{14.16}$$

$$T_8 = \int\limits_G \Big(\sum_{j=1}^d \widehat{\partial_j r^{-1}}[\psi_1]\{\partial_j\psi_1\} - \sum_{j=1}^d \widehat{\partial_j r^{-1}}[\psi_2]\{\partial_j\psi_2\}\Big)\overline{\sigma}\;dx\,, \tag{14.17}$$

and

$$T_9 = \int\limits_G \Big(\sum_{j=1}^d \widehat{\partial_j r^{-1}}[\psi_1]\{iA^j\psi_1\} - \sum_{j=1}^d \widehat{\partial_j r^{-1}}[\psi_2]\{iA^j\psi_2\}\Big)\overline{\sigma}\, dx\,. \tag{14.18}$$

We estimate these terms in the following three subsections.

14.3 Estimation of the terms T_2 to T_5, T_7, and T_9

We begin with a generalization of the bound (7.25). Let $r(\lambda)$, $S(\lambda)$, and $R(\lambda)$ be the functions (3.19), (7.7), and (7.23) respectively. Since by (7.23) we have $\lambda = R(S(\lambda))$, we obtain

$$1 = R'(S(\lambda))S'(\lambda) = \frac{R'(S(\lambda))}{r(\lambda)} \Rightarrow R'(S(\lambda)) = r(\lambda)\,, \tag{14.19}$$

where we have used (7.7). Therefore, for a real-valued function $f(\lambda) \in C^1(\mathbb{R}^1)$, we obtain, by the Lagrange theorem and (14.19),

$$f(\lambda_2) - f(\lambda_1) = f(R(S_2)) - f(R(S_1)) \leqslant \sup_{\lambda\in[\lambda_1,\lambda_2]} |f'(\lambda)r(\lambda)|\,|S_2 - S_1|\,, \tag{14.20}$$

where we have used the notation $S_i = S(\lambda_i)$, $i = 1, 2$. For $f(\lambda) = \lambda/r(\lambda)$ the function $f'(\lambda)r(\lambda)$ is bounded and therefore, by (14.20), (3.20), and (3.21), the term (14.9) admits the bound

$$|T_2| \leqslant C\int\limits_G |\sigma(t,x)|^2\, dx\,. \tag{14.21}$$

Since $A(x) \in C^2(\overline{G})$, we obtain in an analogous manner that

$$|T_7| \leqslant C\int\limits_G |\sigma(t,x)|^2\, dx \tag{14.22}$$

and

$$|T_3| \leqslant C\int |\mathcal{K}_{11}(x,x)||\sigma(t,x)|^2\, dx\,. \tag{14.23}$$

We impose on the correlation function $\mathcal{K}_{11}(x,x)$ the following additional condition:[4]

---

[4] Note that when dim$G = 2$, condition (14.24) follows from condition (3.17). Indeed, using the well-known representation $\mathcal{K}(x,y) = \sum_{j=1}^{\infty} \lambda_j e_j(x)\overline{e_j(y)}$ of the trace class kernel, one can easily derive from (3.17) that $\mathcal{K}(x,x) \in W_1^1(G) \subset L^2(G)$ (the last enclosure follows from Sobolev embedding theorem).

$$\mathcal{K}_{11}(x,x) \in L^p(G) \text{ with } p > 1 \text{ if dim } G = 2 \text{ and with } p > \tfrac{3}{2} \text{ if dim } G = 3. \tag{14.24}$$

Suppose that $d = \dim\, G = 2$. Using the Sobolev embedding theorem ($H^s(G) \subset L^q(G)$ for $s \geqslant d(\frac{1}{2} - \frac{1}{q})$), the interpolation inequality $\|u\|_{H^s} \leqslant C\|u\|_{L^2}^{1-s}\|u\|_{H^1}^s$ for $0 < s < 1$, and the notation $\frac{1}{q} = 1 - \frac{1}{p}$, we obtain from (14.23) that

$$\begin{aligned} |T_3| &\leqslant C\|\mathcal{K}_{11}\|_{L^p}\|\sigma\|_{L^{2q}}^2 \leqslant C\|\mathcal{K}_{11}\|_{L^p}\|\sigma\|_{H^{1-1/q}}^2 \\ &\leqslant C\|\mathcal{K}_{11}\|_{L^p}\|\sigma\|_{L^2}^{2/q}\|\sigma\|_{H^1}^{2-2/q} \leqslant \varepsilon\|\sigma\|_{H^1}^2 + C_\varepsilon\|\mathcal{K}_{11}\|_{L^p}^q\|\sigma\|_{L^2}^2 \,. \end{aligned} \tag{14.25}$$

The proof of (14.25) in the case $d = \dim\, G = 3$ is absolutely the same. Doing elementary algebraic transformations and using (14.20) and the Sobolev embedding theorem ($C(\overline{G}) \subset H^2(G)$ for $d \leqslant 3$), we obtain

$$\begin{aligned} |T_4| &\leqslant \int \Big(\big|\widehat{r^{-1}}[\psi_1]\psi_1 - \widehat{r^{-1}}[\psi_2]\psi_2\big|\, |\psi_1|^2 \\ &\qquad + \big|\widehat{r^{-1}}[\psi_2]\psi_2\big| (|\psi_1|^2 - |\psi_2|^2) \Big) |\sigma|\, dx \\ &\leqslant C \int \big(|\psi_1|^2|\sigma|^2 + (|\psi_1|^2 + |\psi_2|^2)|\sigma|^2 \big)\, dx \\ &\leqslant C\big(1 + \|\psi_1\|_{H^2}^2 + \|\psi_2\|_{H^2}^2\big) \int |\sigma|^2\, dx \end{aligned} \tag{14.26}$$

if $d = \dim\, G \leqslant 3$.

After elementary transformations, we obtain by (14.20) and the Sobolev embedding theorem

$$\begin{aligned} |T_5| \leqslant C \int\limits_G \Bigg(&\left| \frac{\operatorname{Im}\psi_2}{r(\operatorname{Re}\psi_2)} - \frac{\operatorname{Im}\psi_1}{r(\operatorname{Re}\psi_1)} \right| \\ &+ \left| \frac{\operatorname{Re}\psi_2}{r(\operatorname{Im}\psi_2)} - \frac{\operatorname{Re}\psi_1}{r(\operatorname{Im}\psi_1)} \right| \Bigg) |A \cdot \nabla\sigma|\, dx \\ \leqslant C \int\limits_G \Bigg(&\frac{|\operatorname{Im}\psi_2 - \operatorname{Im}\psi_1|}{r(\operatorname{Re}\psi_2)} + |\operatorname{Im}\psi_1| \left| \frac{1}{r(\operatorname{Re}\psi_1)} - \frac{1}{r(\operatorname{Re}\psi_2)} \right| \\ &+ \frac{|\operatorname{Re}\psi_2 - \operatorname{Re}\psi_1|}{r(\operatorname{Im}\psi_1)} + |\operatorname{Re}\psi_1| \left| \frac{1}{r(\operatorname{Im}\psi_1)} - \frac{1}{r(\operatorname{Im}\psi_2)} \right| \Bigg) |\nabla\sigma|\, dx \end{aligned}$$

$$\begin{aligned}
&\leqslant C\int_G (1+|\psi_1|+|\psi_2|)\big(|S(\mathrm{Re}\,\psi_1)-S(\mathrm{Re}\,(\psi_2)|\\
&\qquad +|S(\mathrm{Im}\,\psi_1)-S(\mathrm{Im}\,(\psi_2)|\big)\,|\nabla\sigma|\;dx\\
&\leqslant C(1+\|\psi_1\|_{H^2(G)}+\|\psi_2\|_{H^2(G)})\int|\sigma|\,|\nabla\sigma|\;dx\\
&\leqslant \varepsilon\|\nabla\sigma\|^2_{L^2(G)}+C_\varepsilon(1+\|\psi_1\|_{H^2(G)}+\|\psi_2\|_{H^2(G)})\int|\sigma|^2\;dx\,.
\end{aligned}\tag{14.27}$$

To bound T_9, we first do some simple transformations using (14.12) to obtain

$$\begin{aligned}
T_9=-\int_G\Big\{\Big(&\frac{r'(\mathrm{Re}\,\psi_1)}{r(\mathrm{Re}\,\psi_1)}\big(\nabla S(\mathrm{Re}\,\psi_1)\cdot A\big)\mathrm{Im}\,\psi_1\\
&-\frac{r'(\mathrm{Re}\,\psi_2)}{r(\mathrm{Re}\,\psi_2)}\big(\nabla S(\mathrm{Re}\,\psi_2)\cdot A\big)\mathrm{Im}\,\psi_2\Big)\\
&-i\frac{r'(\mathrm{Im}\,\psi_1)}{r(\mathrm{Im}\,\psi_1)}\big(\nabla S(\mathrm{Im}\,\psi_1)\cdot A\big)\mathrm{Re}\,\psi_1\\
&-i\frac{r'(\mathrm{Im}\,\psi_2)}{r(\mathrm{Im}\,\psi_2)}\big(\nabla S(\mathrm{Im}\,\psi_2)\cdot A\big)\mathrm{Re}\,\psi_2\Big)\Big\}\overline{\sigma}\;dx
\end{aligned}$$

so that

$$\begin{aligned}
T_9=-\int_G\Big(\Big\{\Big(&\frac{r'(\mathrm{Re}\,\psi_1)}{r(\mathrm{Re}\,\psi_1)}\nabla\mathrm{Re}\,\sigma\cdot A\,\mathrm{Im}\,\psi_1+\nabla S(\mathrm{Re}\,\psi_2)\cdot\\
&A\Big[\Big(\frac{r'(\mathrm{Re}\,\psi_1)}{r(\mathrm{Re}\,\psi_1)}-\frac{r'(\mathrm{Re}\,\psi_2)}{r(\mathrm{Re}\,\psi_2)}\Big)\mathrm{Im}\,\psi_1+\frac{r'(\mathrm{Re}\,\psi_2)}{r(\mathrm{Re}\,\psi 2)}(\mathrm{Im}\,\psi_1-\mathrm{Im}\,\psi_2)\Big]\Big\}\\
&-i\Big\{\frac{r'(\mathrm{Im}\,\psi_1)}{r(\mathrm{Im}\,\psi_1)}\nabla\mathrm{Im}\,\sigma\cdot A\,\mathrm{Re}\,\psi_1\\
&+\nabla S(\mathrm{Im}\,\psi)\cdot A\Big[\Big(\frac{r'(\mathrm{Im}\,\psi_1)}{r(\mathrm{Im}\,\psi_1)}-\frac{r'(\mathrm{Im}\,\psi_2)}{r(\mathrm{Im}\,\psi_2)}\Big)\mathrm{Re}\,\psi_1\\
&+\frac{r'(\mathrm{Im}\,\psi_2)}{r(\mathrm{Im}\,\psi_2)}(\mathrm{Re}\,\psi_1-\mathrm{Re}\,\psi_2)\Big]\Big\}\Big)\overline{\sigma}\;dx\,.
\end{aligned}\tag{14.28}$$

A simple bound of the right-hand side of (14.28) and the use of (14.20) gives

$$|T_9| \leqslant C \int_G |\nabla\sigma|\, |\psi_1|\, |\sigma| + |\nabla\psi_2|\, (1 + |\psi_1| + |\psi_2|)\, |\sigma|^2\, dx\,. \tag{14.29}$$

Using the same tools as in (14.25), we have (when dim $G \leqslant 3$)

$$\begin{aligned}
&\int (1 + |\psi_1| + |\psi_2|)|\nabla\psi_2|\, |\sigma|^2\, dx \\
&\leqslant C\big(1 + \|\psi_1\|_{L^6} + \|\psi_2\|_{L^6}\big)\|\nabla\psi_2\|_{L^3}\|\sigma\|_{L^4}^2 \\
&\leqslant C\big(1 + \|\nabla\psi_1\|_{L^2} + \|\nabla\psi_2\|_{L^2}\big)\|\nabla\psi_2\|_{H^{1/2}}\|\sigma\|_{H^{3/4}}^2 \\
&\leqslant C\big(1 + \|\nabla\psi_1\|_{L^2} + \|\nabla\psi_2\|_{L^2}\big)\|\nabla\psi_2\|_{L^2}^{1/2}\|\psi_2\|_{H^2}^{1/2}\|\sigma\|_{L^2}^{1/2}\|\sigma\|_{H^1}^{3/2} \\
&\leqslant \varepsilon\|\nabla\sigma\|_{L^2}^2 \\
&\quad + C_\varepsilon\big(1 + \|\nabla\psi_1\|_{L^2} + \|\nabla\psi_2\|_{L^2}\big)^4\|\nabla\psi_2\|_{L^2}^2\|\psi_2\|_{H^2}^2\|\sigma\|_{L^2}^2\,.
\end{aligned} \tag{14.30}$$

Using (14.29) and (14.30), we obtain

$$|T_9| \leqslant \varepsilon\|\nabla\sigma\|_{L^2}^2 + C_\varepsilon\big(1 + \|\nabla\psi_1\|_{L^2} + \|\nabla\psi_2\|_{L^2}\big)^4\|\nabla\psi_2\|_{L^2}^2\|\psi_2\|_{H^2}^2\|\sigma\|_{L^2}^2\,. \tag{14.31}$$

14.4 Estimation of T_6 and T_8

Using (14.12), we obtain

$$\begin{aligned}
T_6 = \int_G \Bigg(&\bigg(\frac{\nabla \mathrm{Im}\,\psi_2}{r(\mathrm{Re}\,\psi_2)} - \frac{\nabla \mathrm{Im}\,\psi_1}{r(\mathrm{Re}\,\psi_1)}\bigg) \\
&+ i\left(\frac{\nabla \mathrm{Re}\,\psi_1}{r(\mathrm{Im}\,\psi_1)} - \frac{\nabla \mathrm{Re}\,\psi_2}{r(\mathrm{Im}\,\psi_2)}\right)\Bigg) \cdot A\overline{\sigma}\, dx
\end{aligned}$$

so that

$$T_6 = \int_G \Bigg\{ \Big(\frac{r(\mathrm{Im}\,\psi_2)}{r(\mathrm{Re}\,\psi_2)}\nabla S(\mathrm{Im}\,\psi_2) - \frac{r(\mathrm{Im}\,\psi_1)}{r(\mathrm{Re}\,\psi_1)}\nabla S(\mathrm{Im}\,\psi_1)\Big)$$

$$-i\Big(\frac{r(\mathrm{Re}\,\psi_2)}{r(\mathrm{Im}\,\psi_2)}\nabla S(\mathrm{Re}\,\psi_2)-\frac{r(\mathrm{Re}\,\psi_1)}{r(\mathrm{Im}\,\psi_1)}\nabla S(\mathrm{Re}\,\psi_1)\Big)\Bigg\}\cdot A\overline{\sigma}\;dx$$

$$\begin{aligned}=\int\limits_G\Bigg\{&\Big(\frac{-r(\mathrm{Im}\,\psi_2)}{r(\mathrm{Re}\,\psi_2)}\nabla\mathrm{Im}\,\sigma+\Big(\frac{r(\mathrm{Im}\,\psi_2)}{r(\mathrm{Re}\,\psi_2)}-\frac{r(\mathrm{Im}\,\psi_1)}{r(\mathrm{Re}\,\psi_1)}\Big)\nabla S(\mathrm{Im}\,\psi_1)\Big)\\ &-i\Big(\frac{-r(\mathrm{Re}\,\psi_2)}{r(\mathrm{Im}\,\psi_2)}\nabla\mathrm{Re}\,\sigma\\ &+\Big(\frac{r(\mathrm{Re}\,\psi_2)}{r(\mathrm{Im}\,\psi_2)}-\frac{r(\mathrm{Re}\,\psi_1)}{r(\mathrm{Im}\,\psi_1)}\Big)\nabla S(\mathrm{Re}\,\psi_1)\Big)\cdot A\overline{\sigma}\;dx\,.\end{aligned}\tag{14.32}$$

Estimating with the help of (14.20), we obtain the bound

$$\begin{aligned}|T_6|\leqslant&\int\limits_G\big\{(1+|\psi_2|)|\nabla\sigma|+|r(\mathrm{Im}\,\psi_2)r(\mathrm{Re}\,\psi_1)\\ &\quad-r(\mathrm{Re}\,\psi_2)r(\mathrm{Im}\,\psi_1)|\;|\nabla\psi_1|\big\}|\sigma|\;dx\\ \leqslant&\int\limits_G\big\{(1+|\psi_2|)|\nabla\sigma|+\Big(|r(\mathrm{Im}\,\psi_2)-r(\mathrm{Im}\,\psi_1)|r(\mathrm{Re}\,\psi_1)\\ &\quad+r(\mathrm{Im}\,(\psi_1)|r(\mathrm{Re}\,\psi_1)-r(\mathrm{Re}\,(\psi_2)|\Big)|\nabla\psi_1|\big\}|\sigma|\;dx\\ \leqslant&\, C\int\limits_G(1+|\psi_2|)|\nabla\sigma|\;|\sigma|+(1+|\psi_1|^2+|\psi_2|^2)|\nabla\psi_1|\;|\sigma|^2\;dx\,.\end{aligned}\tag{14.33}$$

We assume now that $d=\mathrm{dim}G\leqslant 2$. Then

$$\begin{aligned}&\int(1+|\psi_1|^2+|\psi_2|^2)|\nabla\psi_1|\;|\sigma|^2\;dx\\ &\leqslant C\big(1+\|\psi_1\|^2_{L^{12}}+\|\psi_2\|^2_{L^{12}}\big)\|\nabla\psi_1\|_{L^3}\|\sigma\|^2_{L^4}\\ &\leqslant C\big(1+\|\psi_1\|^2_{H^1}+\|\psi_2\|^2_{H^1}\big)\|\psi_1\|_{H^{4/3}}\|\sigma\|^2_{H^{1/2}}\\ &\leqslant C\big(1+\|\psi_1\|^2_{H^1}+\|\psi_2\|^2_{H^1}\big)\|\psi_1\|^{2/3}_{H^1}\|\psi_1\|^{1/3}_{H^2}\|\sigma\|_{L^2}(\|\nabla\sigma\|_{L^2}+\|\sigma\|_{L^2})\\ &\leqslant\varepsilon\|\nabla\sigma\|^2_{L^2}+C_\varepsilon\big(1+\|\psi_1\|_{H^1}+\|\psi_2\|_{H^1}\big)^{16/3}\|\psi_1\|^{2/3}_{H^2}\|\sigma\|^2_{L^2}\,.\end{aligned}\tag{14.34}$$

Now (14.33) and (14.34) imply

$$|T_6| \leqslant \varepsilon \|\nabla\sigma\|_{L^2}^2$$
$$+C_\varepsilon\Big\{\big(1+\|\psi_1\|_{H^1}+\|\psi_2\|_{H^1}\big)^{16/3}\|\psi_2\|_{H^2})^{2/3}\Big\}\|\sigma\|_{L^2}^2 \,. \tag{14.35}$$

Finally we estimate the term T_8. By virtue of (3.20), (3.21), and (14.5) and through the use of the notation $[\nabla S]^2 = |\nabla \mathrm{Re}\, S|^2 + i|\nabla \mathrm{Im}\, S|^2$, we can write

$$T_8 = \int\limits_G \Big(\widehat{r'}[\psi_1][\nabla S[\psi_1]]^2 - \widehat{r'}[\psi_2][\nabla S[\psi_2]]^2\Big)\overline{\sigma}\; dx$$
$$= \int\limits_G \Big((\widehat{r'[\psi_1] - r'}[\psi_2][\nabla S[\psi_1]]^2 + \widehat{r'}[\psi_2]([\nabla S[\psi_1]]^2 - [\nabla S[\psi_2]]^2)\Big)\overline{\sigma}\; dx \,. \tag{14.36}$$

Bounding (14.36) with the help of (14.20) and (14.5), we obtain

$$|T_8| \leqslant \int\limits_G \Big(|\sigma|^2|\nabla\psi_1|^2 + |\nabla\sigma|\; |\sigma|(|\nabla\psi_1| + |\nabla\psi_2|)\Big)\; dx \,. \tag{14.37}$$

Assume that $d = \dim\, G \leqslant 2$. Then

$$\int\limits_G |\sigma|^2|\nabla\psi_1|^2\; dx \leqslant \|\sigma\|_{L^4}^2\|\nabla\psi_1\|_{L^4}^2 \leqslant \|\sigma\|_{H^{1/2}}^2\|\nabla\psi_1\|_{H^{1/2}}^2$$
$$\leqslant \|\sigma\|_{L^2}\|\sigma\|_{H^1}\|\nabla\psi_1\|_{L^2}\|\psi_1\|_{H^2}$$
$$\leqslant \varepsilon(\|\nabla\sigma\|_{L^2}^2 + \|\sigma\|_{L^2}^2) + C_\varepsilon\|\sigma\|_{L^2}^2\|\nabla\psi_1\|_{L^2}^2\|\psi_1\|_{H^2}^2$$

and

$$\int\limits_G |\nabla\sigma|\; |\sigma|(|\nabla\psi_1| + |\nabla\psi_2|)\; dx \leqslant \|\sigma\|_{L^6}\|\nabla\sigma\|_{L^2}(\|\nabla\psi_1\|_{L^3} + \|\nabla\psi_2\|_{L^3})$$
$$\leqslant \|\sigma\|_{H^{2/3}}\|\nabla\sigma\|_{L^2}\big(\|\nabla\psi_1\|_{H^{1/3}} + \|\nabla\psi_2\|_{H^{1/3}}\big)$$
$$\leqslant \Big(\|\sigma\|_{L^2}^{1/3}\|\nabla\sigma\|_{L^2}^{5/3} + \|\nabla\sigma\|_{L^2}\|\sigma\|_{L^2}\Big)$$
$$\Big(\|\nabla\psi_1\|_{L^2}^{2/3}\|\psi_1\|_{H^2}^{1/3} + \|\nabla\psi_2\|_{L^2}^{2/3}\|\psi_2\|_{H^2}^{1/3}\Big)$$
$$\leqslant \varepsilon\|\nabla\sigma\|_{L^2}^2 + C_\varepsilon\|\sigma\|_{L^2}^2\Big(\|\nabla\psi_1\|_{L^2}^4\|\psi_1\|_{H^2}^2 + \|\nabla\psi_2\|_{L^2}^4\|\psi_2\|_{H^2}^2 + 1\Big) \,.$$

The last two inequalities and (14.37) imply

$$|T_8| \leqslant \varepsilon \|\nabla\sigma\|_{L^2}^2 + C_\varepsilon \|\sigma\|_{L^2}^2 \Big(\|\nabla\psi_1\|_{L^2}^4 \|\psi_1\|_{H^2}^2 + \|\nabla\psi_2\|_{L^2}^4 \|\psi_2\|_{H^2}^2 + 1 \Big) . \tag{14.38}$$

Remark 14.1. We estimated all the terms except T_6 and T_8 under the assumption that $d = \dim\, G \leqslant 3$. We cannot estimate the terms T_6 and T_8 under this assumption. We are forced to assume that $d = \dim\, G \leqslant 2$ when we bound T_6 and T_8.

14.5 Uniqueness theorems

We are now in a position to prove a uniqueness theorem for individual solutions of the problem (12.1) and (2.2).

Theorem 14.2. *Let $d = \dim G = 2$, and let the correlation function $\mathcal{K}_{11}(x,x)$ for the Wiener measure Λ belong to $L^p(G)$ with a certain $p > 1$.*[5] *Then for each datum $(\psi_0, W) \in F$ a solution $\psi \in K$ of the problem* (12.1) *and* (2.2) *is unique.* (*Here F and K are the sets defined in* (13.9) *and* (13.13) *respectively.*)

Proof. Assume that, for a datum $(\psi_0, W) \in F$ there exist two solutions ψ_1 and ψ_2. Then for the function σ defined in (14.5) the following estimate is derived from (14.7) and (14.13):

$$\frac{1}{2}\partial_t \|\sigma(t,\cdot)\|_{L^2}^2 + \int_G |\nabla_x \sigma(t,x)|^2 \, dx \leqslant |T_2| + \cdots + |T_9| \, . \tag{14.39}$$

Using the estimates (14.21), (14.22), (14.25)–(14.27), (14.31), (14.35), and (14.38), we obtain

$$\partial_t \|\sigma(t,\cdot)\|_{L^2}^2 + \|\nabla\sigma(t,\cdot)\|_{L^2}^2 \leqslant \varepsilon\big(\|\nabla\sigma(t,\cdot)\|_{L^2}^2 + \|\sigma\|_{L^2}^2\big)$$
$$+ \Big(C_\varepsilon + C\big(1 + \|\psi_1\|_{H^2}^2 + \|\psi_2\|_{H^2}^2\big)\big(1 + \|\nabla\psi_1\|_{L^2}^6 + \|\nabla\psi_2\|_{L^2}^6\big)\Big) \|\sigma\|_{L^2}^2 \, . \tag{14.40}$$

By virtue of (11.18) and (11.19), for each $T > 0$ the following inclusions hold:

$$\psi_1 \in L^\infty(0,T;L^2(G)), \quad \nabla\psi_i \in L^\infty(0,T;L^2(G)), \quad \Delta\psi_i \in L^2(0,T;L^2(G)) \tag{14.41}$$

for $i = 1, 2$. Since the ψ_i satisfy the boundary condition (2.2), we have, by virtue of the estimates for the solution of the elliptic boundary value problem,

[5] The last condition follows from the assumptions (3.16) and (3.17).

$$\|\psi_i\|^2_{H^2(G)} \leqslant C\big(\|\Delta\psi_i\|^2_{L^2(G)} + \|\nabla\psi_i\|^2_{L^2(G)} + \|\psi_i\|^2_{L^2(G)}\big) \quad \text{for } i = 1, 2\,. \tag{14.42}$$

The bounds (14.41) and (14.42) imply that for each $T > 0$ the following estimate for the expression from the right-hand side of (14.40) holds:

$$\int_0^T \Big(C_\varepsilon + C\big(1+\|\psi_1\|^2_{H^2}+\|\psi_2\|^2_{H^2}\big)\big(1+\|\nabla\psi_1\|^6_{L^2}+\|\nabla\psi_2\|^6_{L^2}\big)\Big)\, dt < \infty\,. \tag{14.43}$$

Therefore, moving the term $\varepsilon\|\nabla\sigma\|^2_{L^2}$ from the right-hand side of (14.40) to the left-hand side and applying to the result the Gronwall inequality, we find that $\sigma(t, x) \equiv 0$. □

Lemma 14.1 and Theorem 14.2 imply the following result.

Theorem 14.3. *Let the assumptions of Theorem* 14.2 *hold. Then the weak statistical solution* ν *of the Ginzburg–Landau equation* (3.22) *is uniquely defined by the initial measure* μ *and the Wiener measure* Λ*.*

We consider now the case of additive white noise when $d = \dim G = 3$.

Theorem 14.4. *Let* $d = \dim G = 3$ *and* $r(\lambda) \equiv \rho_1$*, and let* $\mathcal{K}_{11}(x, x) \in L^p(G)$ *with* $p > \frac{3}{2}$*, where* $\mathcal{K}_{11}(x, y)$ *is the correlation f unction for the Wiener measure* Λ*. Then for each datum* $(\psi_0, W) \in F$ *a solution* $\psi \in K$ *of the problem* (12.1) *and* (2.2) *is unique.* (*Here,* F *and* K *are the sets defined in* (13.9) *and* (13.13) *respectively.*)

Proof. Taking into account the proof of Theorem 14.2, it is enough to establish the bound (14.40) that follows from (14.39) and the estimates for $|T_j|$, $j = 2, \ldots, 9$. Recall that, except for $j = 6$ and 8, estimates for all $|T_j|$ were obtained for $d = \dim G \leqslant 3$. So we have to estimate $|T_6|$ and $|T_8|$. Since $r(\lambda) \equiv$ constant, the equality $\partial_j r^{-1} \equiv 0$ holds and therefore, by (14.17), $T_8 = 0$. By virtue of (14.5), (14.12), and (14.15), we obtain for $r(\lambda) \equiv \rho_1$:

$$|T_6| = \Big|\int_G i\nabla\sigma\cdot A(x)\overline{\sigma}\, dx\Big| \leqslant \varepsilon\|\nabla\sigma\|^2_{L^2(G)} + C_\varepsilon\|\sigma\|^2_{L^2(G)}\,.$$

This complete the proof of estimate (14.40) and the proof of the theorem. □

Lemma 14.1 and Theorem 14.4 imply the following result.

Theorem 14.5. *Let the assumptions of Theorem* 14.4 *hold. Then the weak statistical solution* ν *of the Ginzburg–Landau equation* (3.22) *is uniquely defined by the initial measure* μ *and the Wiener measure* Λ*.*

15 The Strong Statistical Solution of the Stochastic Ginzburg–Landau Equation

Here, we construct the strong statistical solution, prove its uniqueness, and show that it satisfies not only Equation (12.1), but the problem (3.22), (2.2), and (2.3) as well.

15.1 Existence and uniqueness of a strong statistical solution

Recall that, in Sect. 3, an abstract probability space $(\Omega, \Sigma, m(d\omega))$, a random Wiener process $W : \Omega \to C(0,\infty; L^2(G))$, and a random initial condition $\psi_0 : \Omega \to L^1(G)$ were introduced such that $\psi_0(t,\omega)$ and $W(t,x,\omega)$ are independent. In addition, the Wiener measure $\Lambda(dW)$ is a probability distribution of $W(t,x,\omega)$ and $\mu(d\psi_0)$ is a probability distribution of the initial condition $\psi_0(t,\omega)$. Above, we proved the existence of a weak statistical solution $\nu(\Gamma)$, $\Gamma \in \mathcal{B}(\mathcal{U})$, that satisfies (12.6) with the operator $\mathfrak{A}$ defined in (12.1) and (12.5). Based on this existence theorem, we proved in Theorem 13.3 that there exists an $\mu \times \Lambda$-measurable set F, defined in (13.9), such that for each datum $(\psi_0, W) \in F$ there exists a solution $\psi \in K$ of (12.1) (the set K is defined in (13.13)). Moreover, in Theorem 14.2, we proved that this solution ψ is unique in K. This means that the operator

$$\mathfrak{A}^{-1} \equiv (L, \gamma_0)^{-1} : F \to K\,, \tag{15.1}$$

where L is defined in (12.1), is uniquely defined. We introduce the set

$$\Omega_0 = \{\omega \in \Omega \,:\, (\psi_0(\cdot,\omega); W(\cdot,\cdot,\omega) \in F\}\,. \tag{15.2}$$

Since, by (13.11), $\mu \times \Lambda(F) = 1$ we obtain

$$m(\Omega_0) = 1\,. \tag{15.3}$$

We define the random function

$$\psi(t,x,\omega) = \begin{cases} (L,\gamma_0)^{-1}\big(\psi_0(\cdot,\omega), W(\cdot,\cdot,\omega)(t,x), & \omega \in \Omega_0, \\ 0, & \omega \in \Omega \setminus \Omega_0\,. \end{cases} \tag{15.4}$$

Analogous to the approach in [44, Chapt. 10, Proposition 4.3], one can prove the measurability of the map

$$\psi : (\Omega, \Sigma) \to (\mathcal{U}, \mathcal{B}(\mathcal{U}))\,. \tag{15.5}$$

The relations (15.4) and (12.6) imply that the weak statistical solution $\nu(d\psi)$ is a probability distribution of the random map (15.4). By definition, the random map (15.4) satisfies (12.1) for m-almost all $\omega \in \Omega$. Theorem 14.2 implies that the solution (15.4) and (15.5) is defined uniquely by the random datum $(\psi_0(\cdot,\omega), W(\cdot,\cdot,\omega))$.

Note that the assumption (10.3) on the initial measure $\mu(d\psi_0)$ implies that the initial random value $\psi_0(t,\omega)$ satisfies

$$\int \Big(\|\psi_0\|^2_{L^2(G)} + \|\nabla\psi_0\|_{L^2(G)} + \|\psi_0\|^4_{L^4(G)}\Big) m(d\omega) < \infty\,. \tag{15.6}$$

Moreover, Theorem 11.8, (2.5), and (13.1) imply that the following inequalities hold:

$$\begin{aligned} \int\limits_{\mathcal{U}_T} \Big(\|\psi\|^2_{L^\infty(0,T;H^1(G))} + \int\limits_0^T \big(\|\psi\|^2_{H^2(G)} + \|\psi\|^6_{L^6(G)}\big)dt\Big) m(d\omega) \\ \leqslant C_T\Big(1 + \int \big(\|\psi_0\|^2_{H^1(G)} + \|\psi_0\|^4_{L^4(G)}\big)\Big) m(d\omega) \end{aligned} \tag{15.7}$$

and

$$\int\limits_{\mathcal{U}_T} \|\psi\|_{C^L(0,T;L^1(G))} m(d\omega) \leqslant C_T\Big(1 + \int\big(\|\psi_0\|^2_{H^1(G)} + \|\psi_0\|^4_{L^4(G)}\big) m(d\omega)\Big)\,. \tag{15.8}$$

Thus, we have proved the following result.

Theorem 15.1. *Assume that the random initial value $\psi_0(x,\omega)$ and the Wiener process $W(t,x,\omega)$ are independent and ψ_0 satisfies* (15.6). *Then the definition* (15.2) *and* (15.4) *of the strong statistical solution $\psi(t,x,\omega)$ is correct. $\psi(x,\omega)$ satisfies* (12.1) *for m-almost all ω and, by virtue of this equation, ψ is defined uniquely by the datum $(\psi_0(\cdot,\omega), W(\cdot,\cdot,\omega))$. Moreover, ψ satisfies the bounds* (15.7) *and* (15.8).

15.2 On one family of scalar Wiener processes

In order to complete our investigation, we have to prove that the random process (15.4) satisfies the stochastic Ginzburg–Landau equation (3.22) or (what is equivalent) (3.24). To do this, we have to provide some preliminary results.

Since the function $\mathcal{K}(x,y)$ from (3.14) is the kernel of the correlation operator for the complex Wiener process $W(t,x,\omega)$ and this operator is self-adjoint non-negative and of trace-class one, the set of all eigenfunctions $\{e_j(x), j = 1,2,\ldots\}$ of this operator composes an orthonormal basis in the complex space $L^2(G)$. Moreover, if $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_k \geqslant \cdots \geqslant 0$ are the corresponding eigenvalues, then the following identity holds:

$$\mathcal{K}(x,y) = \sum_{j=1}^{\infty} \lambda_j e_j(x)\overline{e_j(y)}\,. \tag{15.9}$$

We introduce the following family of scalar (complex-valued) Wiener processes:

$$W_j(t,\omega) = \int_G W(t,x,\omega)\overline{e_j(x)}\,dx \quad \text{for } j = 1,2,\ldots. \tag{15.10}$$

Then, evidently,

$$W(t,x,\omega) = \sum_{j=1}^{\infty} W_j(t,\omega)e_j(x)\,. \tag{15.11}$$

Recall that for each random function $f(\omega)$ the following notation is used:

$$Ef = \int_\Omega f(\omega)m(d\omega)\,. \tag{15.12}$$

Lemma 15.2. *For the Wiener processes* (15.10) *the following identities hold*:

$$EW_j(t)W_m(s) = 0 \quad \forall\, j,m \in N \tag{15.13}$$

and

$$EW_j(t)\overline{W_m(s)} = t \wedge s\ \lambda_m \delta_{jm}\,, \tag{15.14}$$

where δ_{jm} is the Kronecker delta symbol.

Proof. To prove (15.13), we substitute (15.11) into (3.11), multiply the resulting inequality by $\overline{e_j(x)e_m(y)}$, and integrate with respect to x and y. To prove (15.14), we substitute (15.11) into (3.14) and repeat the steps indicated above. □

As is well-known, (15.13) and (15.14) are equivalent to the independence of $W_j(t)$ and $W_m(s)$ for each j and m and to $W_j(t)$ and $\overline{W_m(s)}$ for $j \neq m$.

Consider now the question of the independence of $\mathrm{Re}\, W_j(t)$ and $\mathrm{Im}\, W_m(s)$ that are defined by

$$W_j(t) = \mathrm{Re}\, W_j(t) + i\mathrm{Im}\, W_j(t)\,. \tag{15.15}$$

Lemma 15.3. *For the Wiener processes* $\mathrm{Re}\, W_j(t)$ *and* $\mathrm{Im}\, W_m(s)$ *the following identities hold*:

$$E\ \mathrm{Re}\, W_j(t)\mathrm{Re}\, W_m(s) = E\ \mathrm{Im}\, W_j(t)\mathrm{Im}\, W_n(s) = \frac{1}{2} t \wedge s\ \delta_{jm}\lambda_m \tag{15.16}$$

and

$$E\ \mathrm{Re}\, W_j(t)\mathrm{Im}\, W_m(s) = 0\,. \tag{15.17}$$

Proof. Substitution of (15.15) into (15.13) and (15.14) gives

$$\begin{aligned} &E\ \mathrm{Re}\, W_j(t)\mathrm{Re}\, W_m(s) - E\ \mathrm{Im}\, W_j(t)\mathrm{Im}\, W_m(s) \\ &\quad + iE\ \mathrm{Re}\, W_j(t)\mathrm{Im}\, W_m(s) + iE\ \mathrm{Im}\, W_j(t)\mathrm{Re}\, W_m(s) = 0 \end{aligned} \tag{15.18}$$

and

$$\begin{aligned} &E\ \mathrm{Re}\, W_j(t)\mathrm{Re}\, W_m(s) + E\ \mathrm{Im}\, W_j(t)\mathrm{Im}\, W_m(s) \\ &\quad - iE\ \mathrm{Re}\, W_j(t)\mathrm{Im}\, W_m(s) + iE\ \mathrm{Im}\, W_j(t)\mathrm{Re}\, W_m(s) = \lambda_j\, t \wedge s\ \delta_{jm}\,, \end{aligned} \tag{15.19}$$

respectively. In fact, (15.18) and (15.13) are four linear algebraic equations in terms of four unknown quantities. Solving these equations, we obtain (15.16) and (15.17). □

15.3 Equation for a strong statistical solution

We are now in a position to prove that the strong statistical solution $\psi(t,x)$, constructed in Sect. 15.1, satisfies the Ginzburg–Landau equation (3.22) or, what is equivalent, (3.24). Here, we understand the Ito integral in (3.24) using the decomposition (15.11):

$$\begin{aligned} &\psi(t,x) + \int\limits_0^t \Big((i\nabla + A)^2\psi(s,x) - \psi(s,x) + |\psi(s,x)|^2\psi(s,x)\Big)ds \\ &\qquad = \sum_{j=1}^{\infty} \int\limits_0^t \widehat{r}[\psi(s,x)]\{e_j(x)dW_j(t)\}\Big) + \psi_0(x)\,. \end{aligned} \tag{15.20}$$

The integral on ds in (15.20) is understood as a Bochner integral for a function with values in $L^2(G)$. To explain the meaning of the stochastic integral in (15.20), we first write, using (3.20) and (3.21), the identity

$$\int\limits_0^t \widehat{r}[\psi(s,x)]\big\{e_j(x)\ dW_j(t)\big\} = \mathrm{Re}\, e_j(x) \int\limits_0^t r(\mathrm{Re}\,\psi(s,x))\ d\mathrm{Re}\, W_j(s)$$

$$- \mathrm{Im}\, e_j(x) \int\limits_0^t r(\mathrm{Re}\,\psi(s,x)\ d\mathrm{Im}\, W_j(s)$$

$$
\begin{aligned}
&+i\Big\{\mathrm{Re}\, e_j(s)\int_0^t r(\mathrm{Im}\,\psi(s,x))\; d\mathrm{Im}\, W_j(s) \\
&+\mathrm{Im}\, e_j(s)\int_0^t r(\mathrm{Im}\,\psi(s,x))\; d\mathrm{Re}\, W_j(s)\Big\}\,.
\end{aligned}
\tag{15.21}
$$

The stochastic integrals on the right-hand side of (15.21) are understood in the usual classical sense (see, for example, [26]) for each fixed $x \in G$ because $\psi(s,x) \in L^2(0,T;H^2(G)) \subset L^2(0,T;C(\overline{G}))$.

Multiplying both parts of (15.21) by an arbitrary $\overline{v(x,\omega)} \in L^2(G \times \Omega)$, integrating on x over G, squaring, applying Doob's inequality (see [26, p. 174], and taking into account (15.16), we obtain, for each $T > 0$,

$$
\begin{aligned}
&E \sup_{t\in[0,T]} \left| \int_G \overline{v(x)} \int_0^t \widehat{r}[\psi(s,x)]\big\{e_j(x) dW_j(s)\big\} dx \right|^2 \\
&= E \sup_{t\in[0,T]} \left| \int_0^t \int_G \overline{v}(x)\widehat{r}[\psi(s,x)]\big\{e_j(x) dx dW_j(s)\big\} \right|^2 \\
&\leqslant C\lambda_j E \int_0^T \Big(\int_G |v(x)|\, |e_j(x)|\, |r[\psi(s,x)]|\, dx \Big)^2 ds \\
&\leqslant C\lambda_j E\|v\|^2_{L^2}\big(1 + E\|\psi\|^2_{L^2(0,T;H^2(G))}\big)\,,
\end{aligned}
\tag{15.22}
$$

where C does not depend on j. Since $\sum_j \lambda_j \leqslant C$, the inequality (15.22) proves that the series on the right-hand side of (15.20) converges weakly in $L^2(G \times \Omega)$. Thus, all terms in (15.20) are well-defined.

Theorem 15.4. *Let the conditions of Theorem* 15.1 *be fulfilled. Then the random process* $\psi(t,x,\omega)$ *defined in* (15.4) *satisfies Equation* (15.20).

Proof. We apply the Ito formula (see [26, Chapt. 6, Sect. 5]) [6] to the stochastic integral $S[\psi(t,x)]$ that is defined by (12.1). Note that, by (15.11), the stochastic integral from (12.1) can be rewritten as follows

[6] In [26], the Ito formula has been proved for a finite-dimensional vector-valued Wiener process $\mathbf{W}(t) = (W_j(t)), j = 1,\ldots,n$. In order to extend this proof to a stochastic integral with an infinite-dimensional vector-valued Wiener processes as in (15.20), it is enough to apply the arguments that were used above to explain the meaning of the stochastic integral in (15.20).

$$dS[\psi(t,x)] + \widehat{r^{-1}}[\psi(t,x)]\big\{(i\nabla + A)^2\psi(t,x) - \psi(t,x) + |\psi|^2\psi(t,x)\big\} \\ + \frac{1}{2}\widehat{r'}[\psi]\mathcal{K}_{11}(x,x) = \sum_{j=1}^{\infty} e_j(x)dW_j(t)\,. \tag{15.23}$$

By virtue of (15.16) and (15.17), for the calculation of $dW_j(t)dW_m(t)$ in the Ito formula, we can use the identities

$$d\mathrm{Re}\,W_j d\mathrm{Re}\,W_m = d\mathrm{Im}\,W_j d\mathrm{Im}\,W_m = \frac{1}{2}\lambda_j\delta_{jm}dt \tag{15.24}$$

and

$$d\mathrm{Re}\,W_j d\mathrm{Im}\,W_m = d\mathrm{Re}\,W_j dt = d\mathrm{Im}\,W_m dt = 0 \quad \forall\, m,j\,. \tag{15.25}$$

Recall that the functions $r(\lambda)$, $S(\lambda)$, and $R(\lambda)$ are defined in (3.19), (7.7), and (7.23) respectively. We apply Ito's formula to the functional $\int\limits_G R[S(t,x)]\cdot \overline{v(x)}\,dx$, where $v(x)\in L^2(G)$. We have

$$d\int R[S[\psi(t,x)]]\cdot\overline{v(x)}\,dx = \int \widehat{R'}[S[\psi]]\{dS\}\cdot\overline{v(x)}\,dx \\ + \frac{1}{2}\widehat{R''}[S[\psi]]\{dS,dS\}\overline{v(x)}\,dx\,. \tag{15.26}$$

By (7.7) and (7.23), $R'(S(\lambda)) = r(\lambda)$. Using this and (15.23), we obtain

$$\int \widehat{R'}[S[\psi]]\{dS\}\cdot\overline{v(x)}\,dx = \int \widehat{r}[\psi]\{dS\}\cdot\overline{v}\,dx \\ = -\int\limits_G \widehat{r}[\psi]\big\{\widehat{r^{-1}}[\psi]\big((i\nabla + A)^2\psi(t,x) - \psi + |\psi|^2\psi\big) \\ + \frac{1}{2}r'[\psi]\mathcal{K}_{11}(x,x)\big\}\cdot\overline{v(x)}\,dxdt + \sum_{j=1}^{\infty}\int \widehat{r}[\psi]\{e_j(x)dW_j(t)\} \tag{15.27} \\ = -\int\limits_G \big((i\nabla + A)^2\psi(t,x) - \psi + |\psi|^2\psi \\ + \frac{1}{2}\widehat{r}[\psi]\{r'[\psi]\}\mathcal{K}_{11}(x,x)\big)\overline{v(x)}\,dt + \sum_{j=1}^{\infty}\int \widehat{r}[\psi]\{e_j(x)dW_j(t)\}\,.$$

This term can be rewritten by using (3.20) and (3.21) as

$$\int R'[S[\psi]]\{dS\}\overline{v(x)}\,dx = \int \widehat{r}[\psi]\{dS\}\overline{v}\,dx$$
$$= \int \Big(r(\operatorname{Re}\psi) d\operatorname{Re} S + i r(\operatorname{Im}\psi) d\operatorname{Im} S \Big) \overline{v}\,dx\,. \tag{15.28}$$

By virtue of (15.23)–(15.25) and (15.28), we can rewrite the second term on the right-hand side of (15.26) as

$$\frac{1}{2}\int \widehat{R''}[S[\psi]]\{dS, dS\}\overline{v(x)}\,dx$$
$$= \frac{1}{2}\int\Big(\partial_{\operatorname{Re} S} r\big(R(\operatorname{Re} S)\big) d\operatorname{Re} S d\operatorname{Re} S + i\partial_{\operatorname{Im} S} r\big(R(\operatorname{Im} S)\big)\ d\operatorname{Im} S\ d\operatorname{Im} S\Big)\overline{v}\,dx$$
$$= \frac{1}{2}\int\big(r'(\operatorname{Re}\psi) r(\operatorname{Re}\psi) d\operatorname{Re} W\ d\operatorname{Re} W$$
$$+ i r'(\operatorname{Im}\psi) r(\operatorname{Im}\psi) d\operatorname{Im} W\ d\operatorname{Im} W\Big)\overline{v}\,dx$$
$$= \frac{1}{2}\int\Big(r'(\operatorname{Re}\psi) r(\operatorname{Re}\psi)\sum_j \big(d\operatorname{Re} W_j \operatorname{Re} e_j - d\operatorname{Im} W_J \operatorname{Im} e_j\big)$$
$$\cdot\Big(\sum_m \big(d\operatorname{Re} W_m \operatorname{Re} e_m - d\operatorname{Im} W_m \operatorname{Im} e_m\big)\Big)$$
$$+ i r'(\operatorname{Im}\psi) r(\operatorname{Im}\psi)\Big(\sum_j d\operatorname{Im} W_j \operatorname{Re} e_j + d\operatorname{Re} W_j \operatorname{Im} e_j\Big)$$
$$\cdot\Big(\sum_m d\operatorname{Im} W_m \operatorname{Re} e_m + d\operatorname{Re} W_m \operatorname{Im} e_m\Big)\Big)\overline{v(x)}\,dx$$
$$= \frac{1}{2}\int\Big(r'(\operatorname{Re}\psi) r(\operatorname{Re}\psi)\frac{1}{2}\sum_j \lambda_j |e_j(x)|^2$$
$$+ i r'(\operatorname{Im}\psi) r(\operatorname{Im}\psi)\frac{1}{2}\sum_j \lambda_j |e_j(x)|^2\Big)\overline{v(x)}\,dxdt \equiv T\,. \tag{15.29}$$

By (15.9) and (3.14), $\sum\limits_j \lambda_j |e_j(x)|^2 = 2\mathcal{K}_{11}(x,x)$ and therefore the right-hand side of (15.29) is equal to the expression

$$T = \int \widehat{r'}[\psi]\{r[\psi]\mathcal{K}_{11}(x,x)\}\overline{v(x)}\,dxdt\,. \tag{15.30}$$

Taking into account that, on the left-hand side of (15.26), $R[S[\psi(t,x)]] = \psi(t,x)$, we obtain from (15.26), (15.27), (15.29), and (15.30) the final formula

$$d\int_G \psi(t,x)\overline{v(x)}\,dx + \int_G \big((i\nabla + A)^2\psi(t,x) - \psi + |\psi|^2\psi\big)\overline{v(x)}\,dx = \sum_{j=1}^{\infty}\int \widehat{r}[\psi]\{e_j(x)dW_j(t)\}\overline{v}\,dx\,. \tag{15.31}$$

This equality holds for each $v(x) \in L^2(G)$. Clearly, this equality is equivalent to

$$d\psi(t,x) + \big\{(i\nabla + A)^2\psi(t,x) - \psi(t,x) + |\psi(t,x)|^2\psi(t,x)\big\}\,dt = \widehat{r}[\psi]\Big\{\sum_{j=1}^{\infty} e_j(x)dW_j(t)\Big\} \tag{15.32}$$

and (15.32) is equivalent to (15.20). □

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