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Robin-Robin domain decomposition methods for the steady-state Stokes-Darcy system with the Beavers-Joseph interface condition

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Abstract Domain decomposition methods for solving the coupled Stokes-Darcy system with the Beavers-Joseph interface condition are proposed and analyzed. Robin boundary conditions are used to decouple the Stokes and Darcy parts of the system. Then, parallel and serial domain decomposition methods are constructed based on the two decoupled sub-problems. Convergence of the two methods is demonstrated and the results of computational experiments are presented to illustrate the convergence.

Keywords Stokes-Darcy flow · Beavers-Joseph interface condition · domain decomposition method · finite element

1 Introduction

The Stokes-Darcy model arises in many applications such as surface water flows, groundwater flows in karst aquifers, and petroleum extraction. This model describes the free flow of a liquid by the Stokes equation and the confined flow in a porous media by the Darcy equation; the two flows are coupled through interface conditions. For the problems mentioned, the resulting coupled Stokes-Darcy model has higher fidelity than

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either the Darcy or Stokes systems on their own. However, coupling the two constituent models leads to a very complex system. Most of previous works on the Stokes-Darcy system use the Beavers-Joseph-Saffman-Jones (BJSJ) [22, 24, 33] interface conditions or even further simplification because well-posedness can be demonstrated in a fairly straightforward manner. However, the BJSJ condition ignores certain contributions made by the flow in the porous media flow to the coupling of the two models; the ignored contributions may be important in some applications such as karst aquifers. The more physically faithful Beavers-Joseph (BJ) [4] interface condition is more accurate because it fully accounts for the contributions of the two flows in the coupling of the two models. The major reason why most previous works did not use the BJ condition is that the well posedness of the Stokes-Darcy model with BJ condition had not been demonstrated. However, recently, this problem was resolved in [8, 9, 11, 21]. For the steady Stokes-Darcy model with the BJ condition, it is proved that the model is well posed if the exchange cofficient α introduced in (2.6) is sufficiently small.

Several methods have been developed to numerically solve the Stokes-Darcy problem, including coupled finite element methods [2, 8, 10, 25, 32], domain decomposition methods [12–18, 23], Lagrange multiplier methods [20, 26], two grid methods [29], discontinuous Galerkin methods [19,31], and boundary integral methods [35]. Many other methods have been developed to solve the Stokes-Brinkman and other similar models; see [1,3,5–7,28,30,34,36,37] and the reference cited therein. Among these methods, domain decomposition is more natural than others because the problem domain naturally consists of two different subdomains and because parallel computation is facilitated; see, e.g., [12] for the BJSJ condition and [13–18, 23] for simplified BJSJ conditions. In this article, we will extend the previous work in [12] to Robin-type domain decomposition methods for the steady Stokes-Darcy system with BJ interface condition, which is more accurate than BJSJ condition [11].

The rest of paper is organized as follows. In Section 2, we introduce the Stokes-Darcy system with the Beavers-Joseph interface condition. In Section 3, the system is decoupled by using Robin interface conditions. In Section 4, parallel and serial domain decomposition methods are proposed. In Sections 5 and 6, we analyze the convergence of the domain decomposition methods. Finally, in Section 7, we present some numerical results that illustrate the convergence of the domain decomposition methods and show their features.

2 Steady Stokes-Darcy model with Beavers-Joseph interface condition

We consider the coupled Stokes-Darcy system on a bounded domain $\Omega = \Omega_D \cup \Omega_S \subset \mathbb{R}^{\mathrm{d}}$, $(d = 2, 3)$; see Figure 1.

In the porous media region Ω_D, the flow is governed by the Darcy system

$$\overrightarrow{u}_D = -\mathbb{K}\nabla\phi_D, \tag{2.1}$$

$$\nabla \cdot \overrightarrow{u}_D = f_D. \tag{2.2}$$

Here, $\overrightarrow{u}_D$ is the fluid discharge rate in the porous media, $\mathbb{K}$ is the hydraulic conductivity tensor, f_D is a sink/source term, and ϕ_D is the hydraulic head defined as $z + \frac{p_D}{\rho g}$, where p_D denotes the dynamic pressure, z the height, ρ the density, and g the gravitational acceleration. In this article, we assume the media in Ω_D is homogeneous

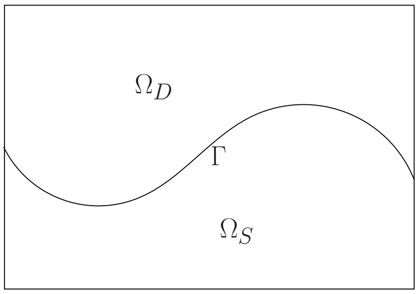

Fig. 1 A sketch of the porous media domain Ω_D,the free-flow domain Ω_S, and the interface Γ.

isotropic so that $\mathbb{K}$ is a constant scalar matrix. In the fluid region Ω_S, the fluid flow is assumed to satisfy the Stokes system

$$-\nabla \cdot \mathbb{T}(\vec{u}_S, p_S) = \vec{f}_S, \quad (2.3)$$
$$\nabla \cdot \vec{u}_S = 0, \quad (2.4)$$

where $\vec{u}_S$ is the fluid velocity, p_S is the kinematic pressure, $\vec{f}_S$ is the external body force, ν is the kinematic viscosity of the fluid, $\mathbb{T}(\vec{u}_S, p_S) = 2\nu\mathbb{D}(\vec{u}_S) - p_S\mathbb{I}$ is the stress tensor, and $\mathbb{D}(\vec{u}_S) = 1/2(\nabla\vec{u}_S + \nabla^T\vec{u}_S)$ is the deformation tensor.

Let $\Gamma = \overline{\Omega}_D \cap \overline{\Omega}_S$ denote the interface between the fluid and porous media regions. On the interface Γ, we impose the following three interface conditions:

$$\vec{u}_S \cdot \vec{n}_S = -\vec{u}_D \cdot \vec{n}_D, \quad (2.5)$$
$$-\boldsymbol{\tau}_j \cdot (\mathbb{T}(\vec{u}_S, p_S) \cdot \vec{n}_S) = \frac{\alpha\nu\sqrt{\mathbf{d}}}{\sqrt{\mathrm{trace}(\prod)}}\boldsymbol{\tau}_j \cdot (\vec{u}_S - \vec{u}_D), \quad (2.6)$$
$$-\vec{n}_S \cdot (\mathbb{T}(\vec{u}_S, p_S) \cdot \vec{n}_S) = g(\phi_D - z), \quad (2.7)$$

where $\vec{n}_S$ and $\vec{n}_D$ denote the unit outer normal to the fluid and the porous media regions at the interface Γ, respectively; $\boldsymbol{\tau}_j$ $(j = 1, \ldots, d-1)$ denote mutually orthogonal unit tangential vectors to the interface Γ, and $\prod = \frac{\mathbb{K}\nu}{g}$. The second condition (2.6) is referred to as the Beavers-Joseph (BJ) interface condition [4].

We assume that the hydraulic head ϕ_D and the fluid velocity $\vec{u}_S$ satisfy homogeneous Dirichlet boundary condition except on Γ , i.e., $\phi_D = 0$ on the boundary $\partial\Omega_D\backslash\Gamma$ and $\vec{u}_S = 0$ on the boundary $\partial\Omega_S\backslash\Gamma$.

The spaces that we utilize are

$$X_S = \{\vec{v} \in [H^1(\Omega_S)]^d \mid \vec{v} = 0 \text{ on } \partial\Omega_S\backslash\Gamma\},$$
$$Q_S = L^2(\Omega_S),$$
$$X_D = \{\psi \in H^1(\Omega_D) \mid \psi = 0 \text{ on } \partial\Omega_D\backslash\Gamma\}.$$

For the domain D ($D = \Omega_S$ or Ω_D), $(\cdot,\cdot)_D$ denotes the L^2 inner product on the domain D, and $\langle\cdot,\cdot\rangle$ denotes the L^2 inner product on the interface Γ or the duality pairing between $(H_{00}^{1/2}(\Gamma))'$ and $H_{00}^{1/2}(\Gamma)$.

With these notations, the weak formulation of the coupled Stokes-Darcy problem is given as follows [8,9]: find $(\overrightarrow{u}_S, p_S) \in X_S \times Q_S$ and $\phi_D \in X_D$ such that

$$\begin{aligned} & a_S(\overrightarrow{u}_S, \overrightarrow{v}) + b_S(\overrightarrow{v}, p_S) + a_D(\phi_D, \psi) + \langle g\phi_D, \overrightarrow{v} \cdot \overrightarrow{n}_S \rangle - \langle \overrightarrow{u}_S \cdot \overrightarrow{n}_S, \psi \rangle \\ & \quad + \frac{\alpha\nu\sqrt{\mathbf{d}}}{\sqrt{\text{trace}(\prod)}} \langle P_\tau(\overrightarrow{u}_S + \mathbb{K}\nabla\phi_D), P_\tau \overrightarrow{v} \rangle \\ & \quad = (f_D, \psi)_{\Omega_D} + (\overrightarrow{f}_S, \overrightarrow{v})_{\Omega_S} + \langle gz, \overrightarrow{v} \cdot \overrightarrow{n}_S \rangle \qquad (2.8) \\ & \qquad \forall\ \overrightarrow{v} \in X_S, \ \ \psi \in X_D, \\ & b_S(\overrightarrow{u}_S, q) = 0, \qquad \forall\ q \in Q_S, \qquad (2.9) \end{aligned}$$

where the bilinear forms are defined as

$$\begin{aligned} a_D(\phi_D, \psi) &= (\mathbb{K}\nabla\phi_D, \nabla\psi)_{\Omega_D}, \\ a_S(\overrightarrow{u}_S, \overrightarrow{v}) &= 2\nu(\mathbb{D}(\overrightarrow{u}_S), \mathbb{D}(\overrightarrow{v}))_{\Omega_S}, \\ b_S(\overrightarrow{v}, q) &= -(\nabla \cdot \overrightarrow{v}, q)_{\Omega_S}, \end{aligned}$$

and P_τ denotes the projection onto the tangent space on Γ, i.e.,

$$P_\tau \overrightarrow{u} = \sum_{j=1}^{d-1} (\overrightarrow{u} \cdot \boldsymbol{\tau}_j)\boldsymbol{\tau}_j.$$

In this article, we assume that $\mathbb{K}$ isotropic and α is small enough. In [9], it is shown that the system of (2.8) and (2.9) is well posed under these two assumptions.

3 Robin boundary conditions

In order to solve the coupled Stokes-Darcy problem utilizing a domain decomposition approach, we naturally consider (partial) Robin boundary conditions for the Stokes and the Darcy equations by following the idea in [12].

Let us consider the following Robin condition for the Darcy system: for a given constant $\gamma_p > 0$ and a given function η_p defined on Γ,

$$\gamma_p \mathbb{K}\nabla\widehat{\phi}_D \cdot \overrightarrow{n}_D + g\widehat{\phi}_D = \eta_p \qquad \text{on } \Gamma. \qquad (3.10)$$

Then, the corresponding weak formulation for the Darcy system is given by: for $\eta_p \in L^2(\Gamma)$, find $\widehat{\phi}_D \in X_D$ such that

$$a_D(\widehat{\phi}_D, \psi) + \langle \frac{g\widehat{\phi}_D}{\gamma_p}, \psi \rangle = (f_D, \psi)_{\Omega_D} + \langle \frac{\eta_p}{\gamma_p}, \psi \rangle \qquad \forall\, \psi \in X_D. \qquad (3.11)$$

Similarly, we propose the following two Robin type conditions for the Stokes equations: for a given constant $\gamma_f > 0$ and given functions η_f and $\overrightarrow{\eta}_{f\tau}$ defined on Γ,

$$\overrightarrow{n}_S \cdot (\mathbb{T}(\widehat{\overrightarrow{u}}_S, \widehat{p}_S) \cdot \overrightarrow{n}_S) + \gamma_f \widehat{\overrightarrow{u}}_S \cdot \overrightarrow{n}_S = \eta_f \qquad \text{on } \Gamma, \qquad (3.12)$$

$$-P_\tau(\mathbb{T}(\overrightarrow{u}_S, p_S) \cdot \overrightarrow{n}_S) - \frac{\alpha\nu\sqrt{\mathbf{d}}}{\sqrt{\text{trace}(\prod)}} P_\tau \overrightarrow{u}_S = \overrightarrow{\eta}_{f\tau} \qquad \text{on } \Gamma. \qquad (3.13)$$

Then, the corresponding weak formulation for the Stokes system is given by: for $\eta_f, \overrightarrow{\eta}_{f\tau} \in L^2(\Gamma)$, find $\widehat{\overrightarrow{u}}_S \in X_S$ and $\widehat{p}_S \in Q_S$ such that

$$a_S(\widehat{\overrightarrow{u}}_S, \overrightarrow{v}) + b_S(\overrightarrow{v}, \widehat{p}_S) + \gamma_f \langle \widehat{\overrightarrow{u}}_S \cdot \overrightarrow{n}_S, \overrightarrow{v} \cdot \overrightarrow{n}_S \rangle + \frac{\alpha\nu\sqrt{\mathbf{d}}}{\sqrt{\mathrm{trace}(\prod)}} \langle P_\tau \widehat{\overrightarrow{u}}_S, P_\tau \overrightarrow{v} \rangle$$
$$= (\overrightarrow{f}_S, \overrightarrow{v})_{\Omega_S} + \langle \eta_f, \overrightarrow{v} \cdot \overrightarrow{n}_S \rangle - \langle \overrightarrow{\eta}_{f\tau}, P_\tau \overrightarrow{v} \rangle \qquad \forall \overrightarrow{v} \in X_S \tag{3.14}$$
$$b_S(\widehat{\overrightarrow{u}}_S, q) = 0 \qquad \forall q \in Q_S. \tag{3.15}$$

We can combine the Stokes and Darcy systems with Robin boundary conditions into one system. Indeed, it is easy to see that if $\eta_p, \eta_f, \overrightarrow{\eta}_{f\tau} \in L^2(\Gamma)$ are given, then, there exists a unique solution $(\widehat{\phi}_D, \widehat{u}_f, \widehat{p}_S) \in X_D \times X_S \times Q_S$ such that

$$a_S(\widehat{\overrightarrow{u}}_S, \overrightarrow{v}) + b_S(\overrightarrow{v}, \widehat{p}_S) + a_D(\widehat{\phi}_D, \psi) + \gamma_f \langle \widehat{\overrightarrow{u}}_S \cdot \overrightarrow{n}_S, \overrightarrow{v} \cdot \overrightarrow{n}_S \rangle + \langle \frac{g\widehat{\phi}_D}{\gamma_p}, \psi \rangle$$
$$+ \frac{\alpha\nu\sqrt{\mathbf{d}}}{\sqrt{\mathrm{trace}(\prod)}} \langle P_\tau \widehat{\overrightarrow{u}}_S, P_\tau \overrightarrow{v} \rangle = (f_D, \psi)_{\Omega_D} + (\overrightarrow{f}_S, \overrightarrow{v})_{\Omega_S} + \langle \eta_f, \overrightarrow{v} \cdot \overrightarrow{n}_S \rangle$$
$$+ \langle \frac{\eta_p}{\gamma_p}, \psi \rangle - \langle \overrightarrow{\eta}_{f\tau}, P_\tau \overrightarrow{v} \rangle \qquad \forall \psi \in X_D, \overrightarrow{v} \in X_S, \tag{3.16}$$
$$b_S(\widehat{\overrightarrow{u}}_S, q) = 0 \qquad \forall q \in Q_S. \tag{3.17}$$

Our next step is to show that, for appropriate choices of γ_f, γ_p, η_f, η_p, and $\overrightarrow{\eta}_{f\tau}$, (smooth) solutions of the Stokes-Darcy system are equivalent to solutions of (3.16), and hence we may solve the latter system instead of the former.

Lemma 1 *Let* $(\phi_D, \overrightarrow{u}_S, p_S)$ *be the solution of the coupled Stokes-Darcy system* (2.8)–(2.9) *and let* $(\widehat{\phi}_D, \widehat{\overrightarrow{u}}_S, \widehat{p}_S)$ *be the solution of the decoupled Stokes and Darcy system with Robin boundary conditions* (3.16)–(3.17) *at the interface. Then,* $(\widehat{\phi}_D, \widehat{\overrightarrow{u}}_S, \widehat{p}_S) = (\phi_D, \overrightarrow{u}_S, p_S)$ *if and only if* γ_f, γ_p, η_f, $\overrightarrow{\eta}_{f\tau}$, *and* η_p *satisfy the following compatibility conditions:*

$$\eta_p = \gamma_p \widehat{\overrightarrow{u}}_S \cdot \overrightarrow{n}_S + g\widehat{\phi}_D, \tag{3.18}$$
$$\eta_f = \gamma_f \widehat{\overrightarrow{u}}_S \cdot \overrightarrow{n}_S - g\widehat{\phi}_D + gz, \tag{3.19}$$
$$\overrightarrow{\eta}_{f\tau} = \frac{\alpha\nu\sqrt{\boldsymbol{d}}}{\sqrt{\mathit{trace}(\prod)}} P_\tau(\mathbb{K}\nabla\phi_D). \tag{3.20}$$

Proof. For the necessity, we pick $\psi = 0$ and $\overrightarrow{v}$ such that $P_\tau \overrightarrow{v} = 0$ in (2.8)–(2.9) and (3.16)–(3.17), then by subtracting (3.16) from (2.8), we get

$$< \eta_f - \gamma_f \overrightarrow{u}_f \cdot \overrightarrow{n}_S + g\phi_D - gz, \overrightarrow{v} \cdot \overrightarrow{n}_S > = 0, \forall \overrightarrow{v} \in X_S \text{ with } P_\tau \overrightarrow{v} = 0$$

which implies (3.19). The necessity of (3.18) and (3.20) can be derived in a similar fashion.

As for the sufficiency, by substituting the compatibility conditions (3.18)–(3.20), we easily see that $(\widehat{\phi}_D, \widehat{\overrightarrow{u}}_S, \widehat{p}_S)$ solves the coupled Stokes-Darcy system (2.8)–(2.9). Since the solution to the Stokes-Darcy system is unique under the assumption that α is small enough [9, 21], we have $(\widehat{\phi}_D, \widehat{\overrightarrow{u}}_S, \widehat{p}_S) = (\phi_D, \overrightarrow{u}_S, p_S)$.

4 Robin-Robin domain decomposition methods

4.1 The parallel Robin-Robin domain decomposition algorithm

We propose the following parallel Robin-Robin domain decomposition method for solving the coupled Stokes-Darcy system with the BJ interface condition.

1. Initial values of η_p^0, η_f^0 and $\overrightarrow{\eta}_{f\tau}^0$ are guessed. They may be taken to be zero.
2. For $k = 0, 1, 2, \ldots$, independently solve the Stokes and Darcy systems with Robin boundary conditions. More precisely, $\phi_D^k \in X_D$ is computed from

$$a_D(\phi_D^k, \psi) + \langle \frac{g\phi_D^k}{\gamma_p}, \psi \rangle = \langle \frac{\eta_p^k}{\gamma_p}, \psi \rangle + (f_D, \psi)_{\Omega_D} \qquad \forall\, \psi \in X_D$$

and $\overrightarrow{u}_S^k \in X_S$ and $p_S^k \in Q_S$ are computed from

$$\begin{aligned} &a_S(\overrightarrow{u}_S^k, \overrightarrow{v}) + b_S(\overrightarrow{v}, p_S^k) + \gamma_f \langle \overrightarrow{u}_S^k \cdot \overrightarrow{n}_S, \overrightarrow{v} \cdot \overrightarrow{n}_S \rangle + \frac{\alpha\nu\sqrt{\mathbf{d}}}{\sqrt{\mathrm{trace}(\prod)}} \langle P_\tau \overrightarrow{u}_S^k, P_\tau \overrightarrow{v} \rangle \\ &\quad = \langle \eta_f^k, \overrightarrow{v} \cdot \overrightarrow{n}_S \rangle + (\overrightarrow{f}_S, \overrightarrow{v})_{\Omega_S} - \langle \overrightarrow{\eta}_{f\tau}^k, P_\tau \overrightarrow{v} \rangle \qquad \forall\, \overrightarrow{v} \in X_S, \\ &b_S(\overrightarrow{u}_S^k, q) = 0 \qquad \forall\, q \in Q_S. \end{aligned}$$

3. η_p^{k+1}, η_f^{k+1} and $\overrightarrow{\eta}_{f\tau}^{k+1}$ are updated in the following manner:

$$\begin{aligned} \eta_f^{k+1} &= a\eta_p^k + bg\phi_D^k + gz \\ \eta_p^{k+1} &= c\eta_f^k + d\overrightarrow{u}_S^k \cdot \overrightarrow{n}_S + gz \\ \overrightarrow{\eta}_{f\tau}^{k+1} &= \frac{\alpha\nu\sqrt{\mathbf{d}}}{\sqrt{\mathrm{trace}(\prod)}} P_\tau(\mathbb{K}\nabla\phi_D^k), \end{aligned}$$

where the coefficients a, b, c, d are chosen as follows:

$$a = \frac{\gamma_f}{\gamma_p}, \quad b = -1 - a, \quad c = -1, \quad d = \gamma_f + \gamma_p. \tag{4.21}$$

In the special case for which $\gamma_f = \gamma_p = \gamma$, we have

$$a = 1 \qquad b = -2 \qquad c = -1 \qquad d = 2\gamma.$$

4.2 The serial Robin-Robin domain decomposition algorithm

Similarly, we have the following serial Robin-Robin domain decomposition method for solving the coupled Stokes-Darcy system with the BJ interface condition.

1. Initial values of η_p^0, η_f^0 are guessed. They may be taken to be zero.
2. For $k = 0, 1, 2, \ldots$, solve the Darcy system with Robin boundary condition. More precisely, $\phi_D^k \in X_D$ is computed from

$$a_D(\phi_D^k, \psi) + \langle \frac{g\phi_D^k}{\gamma_p}, \psi \rangle = \langle \frac{\eta_p^k}{\gamma_p}, \psi \rangle + (f_D, \psi)_{\Omega_D} \qquad \forall\, \psi \in X_D;$$

3. $\overrightarrow{\eta}^k_{f\tau}$ is updated in the following manner:

$$\overrightarrow{\eta}^k_{f\tau} = \frac{\alpha\nu\sqrt{\mathbf{d}}}{\sqrt{\text{trace}(\prod)}} P_\tau(\mathbb{K}\nabla\phi^k_D).$$

4. For $k = 0, 1, 2, \ldots$, independently solve the Stokes and Darcy systems with Robin boundary conditions. More precisely, $\overrightarrow{u}^k_S \in X_S$ and $p^k_S \in Q_S$ are computed from

$$\begin{aligned} &a_S(\overrightarrow{u}^k_S, \overrightarrow{v}) + b_S(\overrightarrow{v}, p^k_S) + \gamma_f\langle \overrightarrow{u}^k_S \cdot \overrightarrow{n}_S, \overrightarrow{v} \cdot \overrightarrow{n}_S\rangle + \frac{\alpha\nu\sqrt{\mathbf{d}}}{\sqrt{\text{trace}(\prod)}}\langle P_\tau \overrightarrow{u}^k_S, P_\tau \overrightarrow{v}\rangle \\ &\quad = \langle \eta^k_f, \overrightarrow{v} \cdot \overrightarrow{n}_S\rangle + (\overrightarrow{f}_S, \overrightarrow{v})_{\Omega_S} - \langle \overrightarrow{\eta}^k_{f\tau}, P_\tau \overrightarrow{v}\rangle \qquad \forall\, \overrightarrow{v} \in X_S, \\ &b_S(\overrightarrow{u}^k_S, q) = 0 \qquad \forall\, q \in Q_S. \end{aligned}$$

5. η^{k+1}_p and η^{k+1}_f are updated in the following manner:

$$\begin{aligned} \eta^{k+1}_f &= a\eta^k_p + bg\phi^k_D + gz, \\ \eta^{k+1}_p &= c\eta^k_f + d\overrightarrow{u}^k_S \cdot \overrightarrow{n}_S + gz. \end{aligned}$$

5 Convergence of the parallel Robin-Robin DDM

We follow the elegant energy method proposed in [27] and the arguments in [12] to demonstrate the convergence of the parallel Robin-Robin domain decomposition method for appropriate choice of parameters γ_p and γ_f.

To this end, let $(\phi_D, \overrightarrow{u}_S, p_S)$ denote the solution of the coupled Stokes-Darcy system (2.8)–(2.9). Then, we have that $(\phi_D, \overrightarrow{u}_S, p_S)$ solves the equivalent decoupled system (3.16) with $\gamma_f, \gamma_p, \eta_p, \eta_f, \overrightarrow{\eta}_{f\tau}$ satisfying the compatibility conditions (3.18)–(3.20) with the hats removed.

Next, we define the error functions

$$\begin{aligned} &\varepsilon^k_D = \eta_p - \eta^k_p \qquad \varepsilon^k_S = \eta_f - \eta^k_f \qquad \overrightarrow{\varepsilon}^k_{S\tau} = \overrightarrow{\eta}_{f\tau} - \overrightarrow{\eta}^k_{f\tau} \\ &e^k_\phi = \phi_D - \phi^k_D \qquad \overrightarrow{e}^k_u = \overrightarrow{u}_S - \overrightarrow{u}^k_S \qquad e^k_p = p_S - p^k_S. \end{aligned}$$

Then, the error functions satisfy the following error equations:

$$\gamma_p a_D(e^k_\phi, \psi) + \langle g e^k_\phi, \psi\rangle = \langle \varepsilon^k_D, \psi\rangle \qquad \forall\, \psi \in X_D, \tag{5.22}$$

$$\begin{aligned} &a_S(\overrightarrow{e}^k_u, \overrightarrow{v}) + b_S(\overrightarrow{v}, e^k_p) + \gamma_f\langle \overrightarrow{e}^k_u \cdot \overrightarrow{n}_S, \overrightarrow{v} \cdot \overrightarrow{n}_S\rangle + \frac{\alpha\nu\sqrt{\mathbf{d}}}{\sqrt{\text{trace}(\prod)}}\langle P_\tau \overrightarrow{e}^k_u, P_\tau \overrightarrow{v}\rangle \\ &\quad = \langle \varepsilon^k_S, \overrightarrow{v} \cdot \overrightarrow{n}_S\rangle - \langle \overrightarrow{\varepsilon}^k_{S\tau}, P_\tau \overrightarrow{v}\rangle \qquad \forall\, \overrightarrow{v} \in X_S, \end{aligned} \tag{5.23}$$

$$b_S(\overrightarrow{e}^k_u, q) = 0 \qquad \forall\, q \in Q_S, \tag{5.24}$$

and, along the interface Γ,

$$\varepsilon^{k+1}_S = a\varepsilon^k_D + bge^k_\phi \tag{5.25}$$

$$\varepsilon^{k+1}_D = c\varepsilon^k_S + d\overrightarrow{e}^k_u \cdot \overrightarrow{n}_S. \tag{5.26}$$

$$\overrightarrow{\varepsilon}^{k+1}_{S\tau} = \frac{\alpha\nu\sqrt{\mathbf{d}}}{\sqrt{\text{trace}(\prod)}} P_\tau(\mathbb{K}\nabla e^k_\phi). \tag{5.27}$$

Equation (5.26) leads to

$$\|\varepsilon_D^{k+1}\|_\Gamma^2 = c^2\|\varepsilon_S^k\|_\Gamma^2 + d^2\|\overrightarrow{e}_u^k \cdot \overrightarrow{n}_S\|_\Gamma^2 + 2cd\langle\varepsilon_S^k, \overrightarrow{e}_u^k \cdot \overrightarrow{n}_S\rangle. \tag{5.28}$$

Setting $\overrightarrow{v} = \overrightarrow{e}_u^k$ in (5.23), we get

$$a_S(\overrightarrow{e}_u^k, \overrightarrow{e}_u^k) + b_S(\overrightarrow{e}_u^k, e_p^k) + \gamma_f\langle\overrightarrow{e}_u^k \cdot \overrightarrow{n}_S, \overrightarrow{e}_u^k \cdot \overrightarrow{n}_S\rangle + \frac{\alpha\nu\sqrt{\mathbf{d}}}{\sqrt{\text{trace}(\prod)}}\langle P_\tau\overrightarrow{e}_u^k, P_\tau\overrightarrow{e}_u^k\rangle$$
$$= \langle\varepsilon_S^k, \overrightarrow{e}_u^k \cdot \overrightarrow{n}_S\rangle - \langle\overrightarrow{\varepsilon}_{S\tau}^k, P_\tau\overrightarrow{e}_u^k\rangle \qquad .$$

Using (5.24) we have

$$b_S(\overrightarrow{e}_u^k, e_p^k) = 0.$$

Hence, by (5.27), we have

$$\langle\varepsilon_S^k, \overrightarrow{e}_u^k \cdot \overrightarrow{n}_S\rangle = a_S(\overrightarrow{e}_u^k, \overrightarrow{e}_u^k) + \gamma_f\|\overrightarrow{e}_u^k \cdot \overrightarrow{n}_S\|_\Gamma^2 + \frac{\alpha\nu\sqrt{\mathbf{d}}}{\sqrt{\text{trace}(\prod)}}\langle P_\tau(\overrightarrow{e}_u^k + \mathbb{K}\nabla e_\phi^{k-1}), P_\tau\overrightarrow{e}_u^k\rangle, \tag{5.29}$$

Combining (5.28) and (5.29), we have

$$\|\varepsilon_D^{k+1}\|_\Gamma^2 = c^2\|\varepsilon_S^k\|_\Gamma^2 + (d^2 + 2cd\gamma_f)\|\overrightarrow{e}_u^k \cdot \overrightarrow{n}_S\|_\Gamma^2 + 2cd\, a_S(\overrightarrow{e}_u^k, \overrightarrow{e}_u^k) + 2cd\,\frac{\alpha\nu\sqrt{\mathbf{d}}}{\sqrt{\text{trace}(\prod)}}\langle P_\tau(\overrightarrow{e}_u^k + \mathbb{K}\nabla e_\phi^{k-1}), P_\tau\overrightarrow{e}_u^k\rangle. \tag{5.30}$$

Similarly, (5.25) implies

$$\|\varepsilon_S^{k+1}\|_\Gamma^2 = a^2\|\varepsilon_D^k\|_\Gamma^2 + b^2\|ge_\phi^k\|_\Gamma^2 + 2ab\langle\varepsilon_D^k, ge_\phi^k\rangle.$$

Setting $\psi = ge_\phi^k$ in (5.22), we have

$$\langle\varepsilon_D^k, ge_\phi^k\rangle = \gamma_p a_D(e_\phi^k, ge_\phi^k) + \langle ge_\phi^k, ge_\phi^k\rangle.$$

Combining the last two equations, we deduce

$$\|\varepsilon_S^{k+1}\|_\Gamma^2 = a^2\|\varepsilon_D^k\|_\Gamma^2 + (b^2 + 2ab)\|ge_\phi^k\|_\Gamma^2 + 2ab\gamma_p g\, a_D(e_\phi^k, e_\phi^k). \tag{5.31}$$

Substituting (4.21) into (5.30) and (5.31), we have the following result.

Lemma 2 *The error functions satisfy*

$$\|\varepsilon_D^{k+1}\|_\Gamma^2 = \|\varepsilon_S^k\|_\Gamma^2 + (\gamma_p^2 - \gamma_f^2)\|\overrightarrow{e}_u^k \cdot \overrightarrow{n}_S\|_\Gamma^2 - 2(\gamma_f + \gamma_p)a_S(\overrightarrow{e}_u^k, \overrightarrow{e}_u^k) - 2(\gamma_f + \gamma_p)\,\frac{\alpha\nu\sqrt{d}}{\sqrt{trace(\prod)}}\langle P_\tau(\overrightarrow{e}_u^k + \mathbb{K}\nabla e_\phi^{k-1}), P_\tau\overrightarrow{e}_u^k\rangle, \tag{5.32}$$

$$\|\varepsilon_S^{k+1}\|_\Gamma^2 = \left(\frac{\gamma_f}{\gamma_p}\right)^2\|\varepsilon_D^k\|_\Gamma^2 + \left(1 - \left(\frac{\gamma_f}{\gamma_p}\right)^2\right)\|ge_\phi^k\|_\Gamma^2 - 2\gamma_f\left(1 + \frac{\gamma_f}{\gamma_p}\right)g\, a_D(e_\phi^k, e_\phi^k). \tag{5.33}$$

We are now ready to demonstrate the convergence of our parallel Robin-Robin domain decomposition method. The convergence analysis for $\gamma_f = \gamma_p$ and $\gamma_f < \gamma_p$ are different and will be treated separately.

Case 1: $\gamma_f = \gamma_p = \gamma$. In this case, we have

$$
\begin{aligned}
&\|\varepsilon_D^{k+1}\|_\Gamma^2 = \|\varepsilon_S^k\|_\Gamma^2 - 4\gamma a_S(\overrightarrow{e}_u^k, \overrightarrow{e}_u^k) - 4\gamma \frac{\alpha\nu\sqrt{\mathbf{d}}}{\sqrt{\mathrm{trace}(\prod)}}\langle P_\tau(\overrightarrow{e}_u^k + \mathbb{K}\nabla e_\phi^{k-1}), P_\tau \overrightarrow{e}_u^k\rangle \\
&\|\varepsilon_S^{k+1}\|_\Gamma^2 = \|\varepsilon_D^k\|_\Gamma^2 - 4\gamma g a_D(e_\phi^k, e_\phi^k).
\end{aligned}
$$

Adding the two equations and summing over k from $k = 1$ to N, we deduce

$$
\begin{aligned}
&\|\varepsilon_D^{N+1}\|_\Gamma^2 + \|\varepsilon_S^{N+1}\|_\Gamma^2 \\
&= \|\varepsilon_D^1\|_\Gamma^2 + \|\varepsilon_S^1\|_\Gamma^2 - 4\gamma \sum_{k=1}^N (a_S(\overrightarrow{e}_u^k, \overrightarrow{e}_u^k) + g\, a_D(e_\phi^k, e_\phi^k) \\
&\quad + \frac{\alpha\nu\sqrt{\mathbf{d}}}{\sqrt{\mathrm{trace}(\prod)}}\langle P_\tau(\overrightarrow{e}_u^k + \mathbb{K}\nabla e_\phi^{k-1}), P_\tau \overrightarrow{e}_u^k\rangle) \\
&= \|\varepsilon_D^1\|_\Gamma^2 + \|\varepsilon_S^1\|_\Gamma^2 - 4\gamma \sum_{k=1}^N (a_S(\overrightarrow{e}_u^k, \overrightarrow{e}_u^k) + g\, a_D(e_\phi^k, e_\phi^k) \\
&\quad + \frac{\alpha\nu\sqrt{\mathbf{d}}}{\sqrt{\mathrm{trace}(\prod)}}\langle P_\tau(\overrightarrow{e}_u^k + \mathbb{K}\nabla e_\phi^{k}), P_\tau \overrightarrow{e}_u^k\rangle) \\
&\quad -4\gamma \sum_{k=1}^N \frac{\alpha\nu\sqrt{\mathbf{d}}}{\sqrt{\mathrm{trace}(\prod)}}\langle P_\tau(\mathbb{K}\nabla e_\phi^{k-1} - \mathbb{K}\nabla e_\phi^k), P_\tau \overrightarrow{e}_u^k\rangle. \qquad (5.34)
\end{aligned}
$$

By the coercivity in Lemma 3.2 of [9], when α is small enough, we have

$$
\begin{aligned}
&a_S(\overrightarrow{e}_u^k, \overrightarrow{e}_u^k) + g\, a_D(e_\phi^k, e_\phi^k) + \frac{\alpha\nu\sqrt{\mathbf{d}}}{\sqrt{\mathrm{trace}(\prod)}}\langle P_\tau(\overrightarrow{e}_u^k + \mathbb{K}\nabla e_\phi^k), P_\tau \overrightarrow{e}_u^k\rangle \\
&\geq C_1 \left(\|\overrightarrow{e}_u^k\|_1^2 + \|e_\phi^k\|_1^2\right), \qquad (5.35)
\end{aligned}
$$

where C_1 depends on K and ν. Since we suppose $\mathbb{K}$ is isotropic, then $\mathbb{K} = K\mathbb{I}$ where K is an constant and $\mathbb{I}$ is the identity matrix. Since the tangential projection of the gradient to the tangential plane is the tangential derivative, then $(P_\tau(\mathbb{K}\nabla e_\phi^k))|_\Gamma = K\nabla_\tau(e_\phi^k|_\Gamma)$ where $\nabla_\tau(e_\phi^k|_\Gamma)$ is the gradient derivative of $e_\phi^k|_\Gamma$. Hence, we have $P_\tau \overrightarrow{e}_u^k|_\Gamma \in H_{00}^{1/2}(\Gamma)$ and $P_\tau(\mathbb{K}\nabla e_\phi^k|_\Gamma \in \left(H_{00}^{1/2}(\Gamma)\right)'$. Using the Cauchy-Schwarz inequality, the triangle inequality, and trace theorems, we have

$$
\begin{aligned}
&\sum_{k=1}^{N} \langle P_\tau(\mathbb{K}\nabla e_\phi^{k-1} - \mathbb{K}\nabla e_\phi^{k}), P_\tau \overrightarrow{e}_u^{k}\rangle \\
&\geq -\sum_{k=1}^{N} \|P_\tau(\mathbb{K}\nabla e_\phi^{k-1} - \mathbb{K}\nabla e_\phi^{k})\|_{-1/2,\Gamma}\|P_\tau \overrightarrow{e}_u^{k}\|_{1/2,\Gamma} \\
&\geq -\sum_{k=1}^{N} \left(\|P_\tau(\mathbb{K}\nabla e_\phi^{k-1})\|_{-1/2,\Gamma} + \|P_\tau(\mathbb{K}\nabla e_\phi^{k})\|_{-1/2,\Gamma}\right)\|P_\tau \overrightarrow{e}_u^{k}\|_{1/2,\Gamma} \\
&\geq -\sum_{k=1}^{N} \left(\frac{1}{2}\|P_\tau(\mathbb{K}\nabla e_\phi^{k-1})\|_{-1/2,\Gamma}^2 + \frac{1}{2}\|P_\tau(\mathbb{K}\nabla e_\phi^{k})\|_{-1/2,\Gamma}^2 + \|P_\tau \overrightarrow{e}_u^{k}\|_{1/2,\Gamma}^2\right) \\
&= -\sum_{k=1}^{N} \left(\frac{1}{2}\|K\nabla_\tau(e_\phi^{k-1}|_\Gamma)\|_{-1/2,\Gamma}^2 + \frac{1}{2}\|K\nabla_\tau(e_\phi^{k}|_\Gamma)\|_{-1/2,\Gamma}^2 + \|P_\tau \overrightarrow{e}_u^{k}\|_{1/2,\Gamma}^2\right) \\
&\geq -C_2\sum_{k=0}^{N}\left(\|\overrightarrow{e}_u^{k}\|_1^2 + \|e_\phi^k\|_1^2\right), \qquad (5.36)
\end{aligned}
$$

where C_2 depends on K. Hence, plugging (5.35) and (5.36) into (5.34), we get

$$
\begin{aligned}
0 &\leq \|\varepsilon_D^{N+1}\|_\Gamma^2 + \|\varepsilon_S^{N+1}\|_\Gamma^2 \\
&\leq \|\varepsilon_D^{1}\|_\Gamma^2 + \|\varepsilon_S^{1}\|_\Gamma^2 + 4\gamma C_1(\left\|\overrightarrow{e}_u^{0}\right\|_1^2 + \left\|e_\phi^0\right\|_1^2) \\
&\quad -4\gamma\left(C_1 - C_2\frac{\alpha\nu\sqrt{\mathbf{d}}}{\sqrt{\mathrm{trace}(\prod)}}\right)\sum_{k=0}^{N}(\left\|\overrightarrow{e}_u^{k}\right\|_1^2 + \left\|e_\phi^k\right\|_1^2)
\end{aligned}
$$

Hence, for any positive integer N,

$$
\begin{aligned}
&4\gamma\left(C_1 - C_2\frac{\alpha\nu\sqrt{\mathbf{d}}}{\sqrt{\mathrm{trace}(\prod)}}\right)\sum_{k=0}^{N}(\left\|\overrightarrow{e}_u^{k}\right\|_1^2 + \left\|e_\phi^k\right\|_1^2) \\
&\leq \left(\|\varepsilon_D^{1}\|_\Gamma^2 + \|\varepsilon_S^{1}\|_\Gamma^2\right) + 4\gamma C_1(\left\|\overrightarrow{e}_u^{0}\right\|_1^2 + \left\|e_\phi^0\right\|_1^2).
\end{aligned}
$$

If α is small enough such that

$$
C_1 - C_2\frac{\alpha\nu\sqrt{\mathbf{d}}}{\sqrt{\mathrm{trace}(\prod)}} > 0, \qquad (5.37)
$$

then $\overrightarrow{e}_u^{k}$ and e_ϕ^k tend to zero in $(H^1(\Omega_S))^d$ and $H^1(\Omega_D)$, respectively. The convergence of e_ϕ^k together with the error equation (5.22) implies the convergence of ε_D^k in $H^{-\frac{1}{2}}(\Gamma)$. Combining the convergence of ε_D^k and e_ϕ^k and the error equation on the interface (5.25), we deduce the convergence of ε_S^k in $H^{-\frac{1}{2}}(\Gamma)$. Combining the convergence of e_ϕ^k and the error equation on the interface (5.27), we deduce the convergence of $\overrightarrow{\varepsilon}_{S\tau}^{k}$ in $H^{-\frac{1}{2}}(\Gamma)$. The convergence of the pressure then follows from the inf-sup condition and (5.23).

Case 2: $\gamma_f < \gamma_p$. Multiplying (5.32) by $\frac{\gamma_f}{\gamma_p}$ and adding it to (5.33), we get

$$\begin{aligned}
&\frac{\gamma_f}{\gamma_p}\|\varepsilon_D^{k+1}\|_\Gamma^2 + \|\varepsilon_S^{k+1}\|_\Gamma^2 \\
&= \left(\frac{\gamma_f}{\gamma_p}\right)^2 \|\varepsilon_D^k\|_\Gamma^2 + \frac{\gamma_f}{\gamma_p}\|\varepsilon_S^k\|_\Gamma^2 + \frac{\gamma_f}{\gamma_p}(\gamma_p^2 - \gamma_f^2)\|\overrightarrow{e}_u^k \cdot \overrightarrow{n}_S\|_\Gamma^2 - 2\frac{\gamma_f}{\gamma_p}(\gamma_f + \gamma_p)a_S(\overrightarrow{e}_u^k, \overrightarrow{e}_u^k) \\
&\quad -2\frac{\gamma_f}{\gamma_p}(\gamma_f+\gamma_p)\ \frac{\alpha\nu\sqrt{\mathbf{d}}}{\sqrt{\text{trace}(\prod)}}\langle P_\tau(\overrightarrow{e}_u^k + \mathbb{K}\nabla e_\phi^{k-1}), P_\tau \overrightarrow{e}_u^k\rangle \\
&\quad + \left(1 - \left(\frac{\gamma_f}{\gamma_p}\right)^2\right)\|ge_\phi^k\|_\Gamma^2 - 2\gamma_f\left(1 + \frac{\gamma_f}{\gamma_p}\right) g\, a_D(e_\phi^k, e_\phi^k) \\
&= \left(\frac{\gamma_f}{\gamma_p}\right)^2 \|\varepsilon_D^k\|_\Gamma^2 + \frac{\gamma_f}{\gamma_p}\|\varepsilon_S^k\|_\Gamma^2 + \frac{\gamma_f}{\gamma_p}(\gamma_p^2 - \gamma_f^2)\|\overrightarrow{e}_u^k \cdot \overrightarrow{n}_S\|_\Gamma^2 + \left(1 - \left(\frac{\gamma_f}{\gamma_p}\right)^2\right)\|ge_\phi^k\|_\Gamma^2 \\
&\quad -2\frac{\gamma_f}{\gamma_p}(\gamma_f+\gamma_p)\ \Big[a_S(\overrightarrow{e}_u^k, \overrightarrow{e}_u^k) + g\, a_D(e_\phi^k, e_\phi^k) \\
&\quad + \frac{\alpha\nu\sqrt{\mathbf{d}}}{\sqrt{\text{trace}(\prod)}}\langle P_\tau(\overrightarrow{e}_u^k + \mathbb{K}\nabla e_\phi^{k-1}), P_\tau \overrightarrow{e}_u^k\rangle\Big].
\end{aligned}$$

Then summing over k from $k = 1$ to N, we deduce

$$\begin{aligned}
0 &\le \frac{\gamma_f}{\gamma_p}\|\varepsilon_D^{N+1}\|_\Gamma^2 + \sum_{k=2}^N \left[\frac{\gamma_f}{\gamma_p} - (\frac{\gamma_f}{\gamma_p})^2\right]\|\varepsilon_D^k\|_\Gamma^2 + \|\varepsilon_S^{N+1}\|_\Gamma^2 + \sum_{k=2}^N\left[1 - \frac{\gamma_f}{\gamma_p}\right]\|\varepsilon_S^k\|_\Gamma^2 \\
&= (\frac{\gamma_f}{\gamma_p})^2\|\varepsilon_D^1\|_\Gamma^2 + \frac{\gamma_f}{\gamma_p}\|\varepsilon_S^1\|_\Gamma^2 + \frac{\gamma_f}{\gamma_p}(\gamma_p^2 - \gamma_f^2)\sum_{k=1}^N \|\overrightarrow{e}_u^k \cdot \overrightarrow{n}_S\|_\Gamma^2 \\
&\quad + \left[1 - \left(\frac{\gamma_f}{\gamma_p}\right)^2\right]\sum_{k=1}^N \|ge_\phi^k\|_\Gamma^2 - 2\frac{\gamma_f}{\gamma_p}(\gamma_f+\gamma_p)\sum_{k=1}^N\Big[a_S(\overrightarrow{e}_u^k, \overrightarrow{e}_u^k) \\
&\quad + \frac{\alpha\nu\sqrt{\mathbf{d}}}{\sqrt{\text{trace}(\prod)}}\langle P_\tau(\overrightarrow{e}_u^k + \mathbb{K}\nabla e_\phi^{k-1}), P_\tau \overrightarrow{e}_u^k\rangle + g a_D(e_\phi^k, e_\phi^k)\Big] \\
&= (\frac{\gamma_f}{\gamma_p})^2\|\varepsilon_D^1\|_\Gamma^2 + \frac{\gamma_f}{\gamma_p}\|\varepsilon_S^1\|_\Gamma^2 + \frac{\gamma_f}{\gamma_p}(\gamma_p^2 - \gamma_f^2)\sum_{k=1}^N \|\overrightarrow{e}_u^k \cdot \overrightarrow{n}_S\|_\Gamma^2 \\
&\quad + \left[1 - \left(\frac{\gamma_f}{\gamma_p}\right)^2\right]\sum_{k=1}^N \|ge_\phi^k\|_\Gamma^2 - 2\frac{\gamma_f}{\gamma_p}(\gamma_f+\gamma_p)\sum_{k=1}^N\Big[a_S(\overrightarrow{e}_u^k, \overrightarrow{e}_u^k) \\
&\quad + \frac{\alpha\nu\sqrt{\mathbf{d}}}{\sqrt{\text{trace}(\prod)}}\langle P_\tau(\overrightarrow{e}_u^k + \mathbb{K}\nabla e_\phi^{k}), P_\tau \overrightarrow{e}_u^k\rangle + g a_D(e_\phi^k, e_\phi^k)] \\
&\quad -2\frac{\gamma_f}{\gamma_p}(\gamma_f+\gamma_p)\sum_{k=1}^N \frac{\alpha\nu\sqrt{\mathbf{d}}}{\sqrt{\text{trace}(\prod)}}\langle P_\tau(\mathbb{K}\nabla e_\phi^{k-1} - \mathbb{K}\nabla e_\phi^k), P_\tau \overrightarrow{e}_u^k\rangle. \qquad (5.38)
\end{aligned}$$

By the trace inequality and the Poincaré inequality, we have

$$\|\overrightarrow{e}_u^k \cdot \overrightarrow{n}_S\|_\Gamma^2 \le C_3\|\overrightarrow{e}_u^k\|_1^2, \qquad (5.39)$$

$$\|ge_\phi^k\|_\Gamma^2 \le g^2 C_4\|e_\phi^k\|_1^2. \qquad (5.40)$$

Suppose α is small enough such that (5.35) is true. Then by (5.35), (5.36), and the above three inequalities, we get

$$\begin{aligned}
0 &\le \frac{\gamma_f}{\gamma_p}\|\varepsilon_D^{N+1}\|_\Gamma^2 + \sum_{k=2}^{N}\left[\frac{\gamma_f}{\gamma_p} - (\frac{\gamma_f}{\gamma_p})^2\right]\|\varepsilon_D^k\|_\Gamma^2 + \|\varepsilon_S^{N+1}\|_\Gamma^2 + \sum_{k=2}^{N}\left[1 - \frac{\gamma_f}{\gamma_p}\right]\|\varepsilon_S^k\|_\Gamma^2 \\
&\le (\frac{\gamma_f}{\gamma_p})^2\|\varepsilon_D^1\|_\Gamma^2 + \frac{\gamma_f}{\gamma_p}\|\varepsilon_S^1\|_\Gamma^2 + \frac{\gamma_f}{\gamma_p}(\gamma_p^2 - \gamma_f^2)C_3\sum_{k=1}^{N}\|\overrightarrow{e}_u^k\|_1^2 \\
&\quad + \left[1 - \left(\frac{\gamma_f}{\gamma_p}\right)^2\right] g^2 C_4 \sum_{k=1}^{N}\|e_\phi^k\|_1^2 + 2\frac{\gamma_f}{\gamma_p}(\gamma_f + \gamma_p)C_1\left(\|\overrightarrow{e}_u^0\|_1^2 + \|e_\phi^0\|_1^2\right) \\
&\quad -2\frac{\gamma_f}{\gamma_p}(\gamma_f + \gamma_p)(C_1 - C_2\frac{\alpha\nu\sqrt{\mathbf{d}}}{\sqrt{\mathrm{trace}(\prod)}})\sum_{k=0}^{N}\left(\|\overrightarrow{e}_u^k\|_1^2 + \|e_\phi^k\|_1^2\right). \qquad (5.41)
\end{aligned}$$

Fix any number $s \in (0,2)$. Suppose γ_f and γ_p are chosen such that

$$\begin{aligned}
&\frac{\gamma_f}{\gamma_p}(\gamma_p^2 - \gamma_f^2)C_3 - s\frac{\gamma_f}{\gamma_p}(\gamma_f + \gamma_p)(C_1 - C_2\frac{\alpha\nu\sqrt{\mathbf{d}}}{\sqrt{\mathrm{trace}(\prod)}}) \le 0, \\
&\left[1 - \left(\frac{\gamma_f}{\gamma_p}\right)^2\right] g^2 C_4 - s\frac{\gamma_f}{\gamma_p}(\gamma_f + \gamma_p)(C_1 - C_2\frac{\alpha\nu\sqrt{d}}{\prod}) \le 0,
\end{aligned}$$

which are equivalent to

$$\gamma_p - \gamma_f \le \frac{s(C_1 - C_2\frac{\alpha\nu\sqrt{\mathbf{d}}}{\sqrt{\mathrm{trace}(\prod)}})}{C_3}, \qquad (5.42)$$

$$\frac{1}{\gamma_f} - \frac{1}{\gamma_p} \le \frac{s(C_1 - C_2\frac{\alpha\nu\sqrt{\mathbf{d}}}{\sqrt{\mathrm{trace}(\prod)}})}{g^2 C_4}. \qquad (5.43)$$

Then, we get

$$\begin{aligned}
0 &\le \frac{\gamma_f}{\gamma_p}\|\varepsilon_D^{N+1}\|_\Gamma^2 + \sum_{k=2}^{N}\left[\frac{\gamma_f}{\gamma_p} - (\frac{\gamma_f}{\gamma_p})^2\right]\|\varepsilon_D^k\|_\Gamma^2 + \|\varepsilon_S^{N+1}\|_\Gamma^2 + \sum_{k=2}^{N}\left[1 - \frac{\gamma_f}{\gamma_p}\right]\|\varepsilon_S^k\|_\Gamma^2 \\
&\le (\frac{\gamma_f}{\gamma_p})^2\|\varepsilon_D^1\|_\Gamma^2 + \frac{\gamma_f}{\gamma_p}\|\varepsilon_S^1\|_\Gamma^2 + 2\frac{\gamma_f}{\gamma_p}(\gamma_f + \gamma_p)C_1\left(\|\overrightarrow{e}_u^0\|_1^2 + \|e_\phi^0\|_1^2\right) \\
&\quad -(2-s)\frac{\gamma_f}{\gamma_p}(\gamma_f + \gamma_p)(C_1 - C_2\frac{\alpha\nu\sqrt{\mathbf{d}}}{\sqrt{\mathrm{trace}(\prod)}})\sum_{k=0}^{N}\left[\|\overrightarrow{e}_u^k\|_1^2 + \|e_\phi^k\|_1^2\right].
\end{aligned}$$

With the same argument as the end of the Case 1, we obtain the convergence for Case 2.

Now we derive a geometric convergence rate for Case 2. Similar to (5.35), we still need to assume that α is small enough such that

$$\begin{aligned}
&a_S(\overrightarrow{e}_u^k, \overrightarrow{e}_u^k) + g\, a_D(e_\phi^{k-1}, e_\phi^{k-1}) + \frac{\alpha\nu\sqrt{\mathbf{d}}}{\sqrt{\mathrm{trace}(\prod)}}\langle P_\tau(\overrightarrow{e}_u^k + \mathbb{K}\nabla e_\phi^{k-1}), P_\tau\overrightarrow{e}_u^k\rangle \\
&\ge C_1\left(\|\overrightarrow{e}_u^k\|_1^2 + \|e_\phi^{k-1}\|_1^2\right). \qquad (5.44)
\end{aligned}$$

Also, it is easy to see

$$a_D(e_\phi^{k-1}, e_\phi^{k-1}) \le C_5 \|e_\phi^{k-1}\|_1^2. \tag{5.45}$$

where C_5 depends on K. Plugging (5.33) into (5.32) and using (5.39), (5.40), (5.44), and (5.45), we have

$$\begin{aligned}
&\|\varepsilon_D^{k+1}\|_\Gamma^2 \\
&= \left(\frac{\gamma_f}{\gamma_p}\right)^2 \|\varepsilon_D^{k-1}\|_\Gamma^2 + \left(1 - \left(\frac{\gamma_f}{\gamma_p}\right)^2\right) \|g e_\phi^{k-1}\|_\Gamma^2 - 2\gamma_f \left(1 + \frac{\gamma_f}{\gamma_p}\right) g\, a_D(e_\phi^{k-1}, e_\phi^{k-1}) \\
&\quad + (\gamma_p^2 - \gamma_f^2) \|\vec{e}_u^{\,k} \cdot \vec{n}_S\|_\Gamma^2 - 2(\gamma_f + \gamma_p) a_S(\vec{e}_u^{\,k}, \vec{e}_u^{\,k}) \\
&\quad - 2(\gamma_f + \gamma_p)\, \frac{\alpha\nu\sqrt{\mathbf{d}}}{\sqrt{\mathrm{trace}(\prod)}} \langle P_\tau(\vec{e}_u^{\,k} + \mathbb{K}\nabla e_\phi^{k-1}), P_\tau \vec{e}_u^{\,k}\rangle \\
&= \left(\frac{\gamma_f}{\gamma_p}\right)^2 \|\varepsilon_D^{k-1}\|_\Gamma^2 + \left(1 - \left(\frac{\gamma_f}{\gamma_p}\right)^2\right) \|g e_\phi^{k-1}\|_\Gamma^2 \\
&\quad + 2(\gamma_f + \gamma_p) \left(1 - \frac{\gamma_f}{\gamma_p}\right) g\, a_D(e_\phi^{k-1}, e_\phi^{k-1}) + (\gamma_p^2 - \gamma_f^2) \|\vec{e}_u^{\,k} \cdot \vec{n}_S\|_\Gamma^2 \\
&\quad - 2(\gamma_f + \gamma_p) \Big[a_S(\vec{e}_u^{\,k}, \vec{e}_u^{\,k}) + g\, a_D(e_\phi^{k-1}, e_\phi^{k-1}) \\
&\quad + \frac{\alpha\nu\sqrt{\mathbf{d}}}{\sqrt{\mathrm{trace}(\prod)}} \langle P_\tau(\vec{e}_u^{\,k} + \mathbb{K}\nabla e_\phi^{k-1}), P_\tau \vec{e}_u^{\,k}\rangle \Big] \\
&\le \left(\frac{\gamma_f}{\gamma_p}\right)^2 \|\varepsilon_D^{k-1}\|_\Gamma^2 + \left(1 - \left(\frac{\gamma_f}{\gamma_p}\right)^2\right) g^2 C_4 \|e_\phi^{k-1}\|_1^2 \\
&\quad + 2(\gamma_f + \gamma_p) \left(1 - \frac{\gamma_f}{\gamma_p}\right) g C_5 \|e_\phi^{k-1}\|_1^2 + (\gamma_p^2 - \gamma_f^2) C_3 \|\vec{e}_u^{\,k}\|_1^2 \\
&\quad - 2(\gamma_f + \gamma_p) C_1 \left(\|\vec{e}_u^{\,k}\|_1^2 + \|e_\phi^{k-1}\|_1^2 \right).
\end{aligned} \tag{5.46}$$

Fix any number $s \in (0, 2)$. Suppose γ_f and γ_p are chosen such that

$$\begin{aligned}
&(\gamma_p^2 - \gamma_f^2) C_3 - s(\gamma_f + \gamma_p) C_1 \le 0, \\
&\left[1 - \left(\frac{\gamma_f}{\gamma_p}\right)^2\right] g^2 C_4 + 2(\gamma_f + \gamma_p) \left(1 - \frac{\gamma_f}{\gamma_p}\right) g C_5 - s(\gamma_f + \gamma_p) C_1 \le 0,
\end{aligned}$$

which are equivalent to

$$\gamma_p - \gamma_f \le \frac{sC_1}{C_3}, \tag{5.47}$$

$$\gamma_p - \gamma_f \le \frac{s\gamma_p^2 C_1}{g^2 C_4 + 2\gamma_p g C_5}. \tag{5.48}$$

Then, we have

$$\|\varepsilon_D^{k+1}\|_\Gamma^2 + (2 - s)(\gamma_f + \gamma_p) C_1 \left(\|\vec{e}_u^{\,k}\|_1^2 + \|e_\phi^{k-1}\|_1^2 \right) \le \left(\frac{\gamma_f}{\gamma_p}\right)^2 \|\varepsilon_D^{k-1}\|_\Gamma^2.$$

Hence, we get the geometric convergence rate $\frac{\gamma_f}{\gamma_p}$ for ε_D^k, $\vec{e}_u^{\,k}$, and e_ϕ^k. Using (5.22)-(5.27), we obtain the geometric convergence rate $\frac{\gamma_f}{\gamma_p}$ for ε_S^k, e_p^k and $\vec{\varepsilon}_{S\tau}^{\,k}$.

Combining the results of Case 1 and Case 2, we have proved the following theorem.

Theorem 1 *Assume $\gamma_f \leq \gamma_p$, $\mathbb{K}$ is isotropic, and α is small enough such that* (5.35) *and* (5.37) *are true. If $\gamma_f < \gamma_p$, assume that γ_f and γ_p are close to each other such that* (5.42) *and* (5.43) *are true. Then,*

$$\phi_D^k \xrightarrow{X_D} \phi_D \qquad \overrightarrow{u}_S^k \xrightarrow{X_S} \overrightarrow{u}_S \qquad p_f^k \xrightarrow{Q_S} p_S,$$
$$\eta_p^k \overset{H^{-\frac{1}{2}}(\Gamma)}{\longrightarrow} \gamma\overrightarrow{u}_S\cdot\overrightarrow{n}_S + g\phi_D = -\gamma\mathbb{K}\nabla\phi_D\cdot\overrightarrow{n}_D + g\phi_D,$$
$$\eta_f^k \overset{H^{-\frac{1}{2}}(\Gamma)}{\longrightarrow} \gamma\overrightarrow{u}_S\cdot\overrightarrow{n}_S - g\phi_D = \overrightarrow{n}_S\cdot(\mathbb{T}(\overrightarrow{u}_S,p_S)\cdot\overrightarrow{n}_S) + \gamma\overrightarrow{u}_S\cdot\overrightarrow{n}_S,$$
$$\overrightarrow{\eta}_{f\tau}^k \overset{H^{-\frac{1}{2}}(\Gamma)}{\longrightarrow} \frac{\alpha\nu\sqrt{\boldsymbol{d}}}{\sqrt{trace(\prod)}}P_\tau\mathbb{K}\nabla\phi_D.$$

Specifically, for the case of $\gamma_f < \gamma_p$, if γ_f and γ_p are close to each other such that (5.47) *and* (5.48) *are true, then we have geometric convergence rate $\frac{\gamma_f}{\gamma_p}$ for ε_D^k, ε_S^k, $\overrightarrow{e}_u^k$, e_ϕ^k, e_p^k and $\overrightarrow{\varepsilon}_{S\tau}^k$.*

6 Convergence of the serial Robin-Robin DDM

In this section, we similarly demonstrate the convergence of the serial Robin-Robin domain decomposition method for appropriate choice of parameters γ_p and γ_f. Most of the notations are the same as those of the previous section. First, the error functions still satisfy (5.22)-(5.26), but (5.27) is changed to be

$$\overrightarrow{\varepsilon}_{S\tau}^k = \frac{\alpha\nu\sqrt{\mathbf{d}}}{\sqrt{\mathrm{trace}(\prod)}}P_\tau(\mathbb{K}\nabla e_\phi^k). \tag{6.49}$$

With similar arguments for Lemma 2, we get the following lemma.

Lemma 3 *The error functions satisfy*

$$\|\varepsilon_D^{k+1}\|_\Gamma^2 = \|\varepsilon_S^k\|_\Gamma^2 + (\gamma_p^2-\gamma_f^2)\|\overrightarrow{e}_u^k\cdot\overrightarrow{n}_S\|_\Gamma^2 - 2(\gamma_f+\gamma_p)a_S(\overrightarrow{e}_u^k, \overrightarrow{e}_u^k)$$
$$-2(\gamma_f+\gamma_p)\,\frac{\alpha\nu\sqrt{\boldsymbol{d}}}{\sqrt{trace(\prod)}}\langle P_\tau(\overrightarrow{e}_u^k + \mathbb{K}\nabla e_\phi^k), P_\tau\overrightarrow{e}_u^k\rangle, \tag{6.50}$$

$$\|\varepsilon_S^{k+1}\|_\Gamma^2 = \left(\frac{\gamma_f}{\gamma_p}\right)^2\|\varepsilon_D^k\|_\Gamma^2 + \left(1-\left(\frac{\gamma_f}{\gamma_p}\right)^2\right)\|ge_\phi^k\|_\Gamma^2$$
$$-2\gamma_f\left(1+\frac{\gamma_f}{\gamma_p}\right)g\,a_D(e_\phi^k, e_\phi^k). \tag{6.51}$$

Following the same idea as in the previous section, we are now ready to demonstrate the convergence of our serial Robin-Robin domain decomposition method. Again the convergence analysis for $\gamma_f = \gamma_p$ and $\gamma_f < \gamma_p$ will be treated separately.

Case 1: $\gamma_f = \gamma_p = \gamma$. In this case, we have

$$\|\varepsilon_D^{k+1}\|_\Gamma^2 = \|\varepsilon_S^k\|_\Gamma^2 - 4\gamma a_S(\overrightarrow{e}_u^k, \overrightarrow{e}_u^k) - 4\gamma\frac{\alpha\nu\sqrt{\mathbf{d}}}{\sqrt{\mathrm{trace}(\prod)}}\langle P_\tau(\overrightarrow{e}_u^k + \mathbb{K}\nabla e_\phi^k), P_\tau\overrightarrow{e}_u^k\rangle$$
$$\|\varepsilon_S^{k+1}\|_\Gamma^2 = \|\varepsilon_D^k\|_\Gamma^2 - 4\gamma g a_D(e_\phi^k, e_\phi^k).$$

Adding the two equations and summing over k from $k=0$ to N, we deduce

$$\|\varepsilon_D^{N+1}\|_\Gamma^2+\|\varepsilon_S^{N+1}\|_\Gamma^2=\|\varepsilon_D^0\|_\Gamma^2+\|\varepsilon_S^0\|_\Gamma^2-4\gamma\sum_{k=0}^{N}(a_S(\overrightarrow{e}_u^k,\overrightarrow{e}_u^k)+g\,a_D(e_\phi^k,e_\phi^k)$$
$$+\frac{\alpha\nu\sqrt{\mathbf{d}}}{\sqrt{\mathrm{trace}(\prod)}}\langle P_\tau(\overrightarrow{e}_u^k+\mathbb{K}\nabla e_\phi^k),P_\tau\overrightarrow{e}_u^k\rangle).$$

Suppose α is small enough such that (5.35) is true. Then,

$$0\le\|\varepsilon_D^{N+1}\|_\Gamma^2+\|\varepsilon_S^{N+1}\|_\Gamma^2\le\|\varepsilon_D^0\|_\Gamma^2+\|\varepsilon_S^0\|_\Gamma^2-4\gamma C_1\sum_{k=0}^{N}(\left\|\overrightarrow{e}_u^k\right\|_1^2+\left\|e_\phi^k\right\|_1^2)$$

which leads to

$$4\gamma C_1\sum_{k=0}^{N}(\left\|\overrightarrow{e}_u^k\right\|_1^2+\left\|e_\phi^k\right\|_1^2)\le\|\varepsilon_D^0\|_\Gamma^2+\|\varepsilon_S^0\|_\Gamma^2\ \text{ for arbitrary positive integer } N.$$

This implies that $\overrightarrow{e}_u^k$ and e_ϕ^k tend to zero in $(H^1(\Omega_S))^d$ and $H^1(\Omega_D)$, respectively. With the same arguments in the previous section, we obtain the convergence for Case 1.

Case 2: $\gamma_f<\gamma_p$. Multiplying (6.50) by $\frac{\gamma_f}{\gamma_p}$ and adding it to (6.51), we get

$$\frac{\gamma_f}{\gamma_p}\|\varepsilon_D^{k+1}\|_\Gamma^2+\|\varepsilon_S^{k+1}\|_\Gamma^2$$
$$=\left(\frac{\gamma_f}{\gamma_p}\right)^2\|\varepsilon_D^k\|_\Gamma^2+\frac{\gamma_f}{\gamma_p}\|\varepsilon_S^k\|_\Gamma^2+\frac{\gamma_f}{\gamma_p}(\gamma_p^2-\gamma_f^2)\|\overrightarrow{e}_u^k\cdot\overrightarrow{n}_S\|_\Gamma^2-$$
$$2\frac{\gamma_f}{\gamma_p}(\gamma_f+\gamma_p)a_S(\overrightarrow{e}_u^k,\overrightarrow{e}_u^k)-2\frac{\gamma_f}{\gamma_p}(\gamma_f+\gamma_p)\ \frac{\alpha\nu\sqrt{\mathbf{d}}}{\sqrt{\mathrm{trace}(\prod)}}\langle P_\tau(\overrightarrow{e}_u^k+\mathbb{K}\nabla e_\phi^k),P_\tau\overrightarrow{e}_u^k\rangle$$
$$+\left[1-\left(\frac{\gamma_f}{\gamma_p}\right)^2\right]\|ge_\phi^k\|_\Gamma^2-2\gamma_f\left(1+\frac{\gamma_f}{\gamma_p}\right)g\,a_D(e_\phi^k,e_\phi^k)$$
$$=\left(\frac{\gamma_f}{\gamma_p}\right)^2\|\varepsilon_D^k\|_\Gamma^2+\frac{\gamma_f}{\gamma_p}\|\varepsilon_S^k\|_\Gamma^2+\frac{\gamma_f}{\gamma_p}(\gamma_p^2-\gamma_f^2)\|\overrightarrow{e}_u^k\cdot\overrightarrow{n}_S\|_\Gamma^2+\left[1-\left(\frac{\gamma_f}{\gamma_p}\right)^2\right]\|ge_\phi^k\|_\Gamma^2$$
$$-2\frac{\gamma_f}{\gamma_p}(\gamma_f+\gamma_p)\ \Big[a_S(\overrightarrow{e}_u^k,\overrightarrow{e}_u^k)+ga_D(e_\phi^k,e_\phi^k)$$
$$+\frac{\alpha\nu\sqrt{\mathbf{d}}}{\sqrt{\mathrm{trace}(\prod)}}\langle P_\tau(\overrightarrow{e}_u^k+\mathbb{K}\nabla e_\phi^k),P_\tau\overrightarrow{e}_u^k\rangle\Big]. \tag{6.52}$$

Then summing over k from $k=0$ to N, we deduce

$$0\le\frac{\gamma_f}{\gamma_p}\|\varepsilon_D^{N+1}\|_\Gamma^2+\sum_{k=1}^{N}\left[\frac{\gamma_f}{\gamma_p}-\left(\frac{\gamma_f}{\gamma_p}\right)^2\right]\|\varepsilon_D^k\|_\Gamma^2+\|\varepsilon_S^{N+1}\|_\Gamma^2+\sum_{k=1}^{N}\left[1-\frac{\gamma_f}{\gamma_p}\right]\|\varepsilon_S^k\|_\Gamma^2$$
$$=\left(\frac{\gamma_f}{\gamma_p}\right)^2\|\varepsilon_D^0\|_\Gamma^2+\frac{\gamma_f}{\gamma_p}\|\varepsilon_S^0\|_\Gamma^2+\frac{\gamma_f}{\gamma_p}(\gamma_p^2-\gamma_f^2)\sum_{k=0}^{N}\|\overrightarrow{e}_u^k\cdot\overrightarrow{n}_S\|_\Gamma^2$$
$$+\left[1-\left(\frac{\gamma_f}{\gamma_p}\right)^2\right]\sum_{k=0}^{N}\|ge_\phi^k\|_\Gamma^2-2\frac{\gamma_f}{\gamma_p}(\gamma_f+\gamma_p)\sum_{k=0}^{N}\Big[a_S(\overrightarrow{e}_u^k,\overrightarrow{e}_u^k)+ga_D(e_\phi^k,e_\phi^k)$$
$$+\frac{\alpha\nu\sqrt{\mathbf{d}}}{\sqrt{\mathrm{trace}(\prod)}}\langle P_\tau(\overrightarrow{e}_u^k+\mathbb{K}\nabla e_\phi^k),P_\tau\overrightarrow{e}_u^k\rangle\Big]. \tag{6.53}$$

Suppose α is small enough such that (5.35) is true. Then by (5.35), (5.39), (5.40), and the above inequality we get

$$\begin{aligned}
0 &\le \frac{\gamma_f}{\gamma_p}\|\varepsilon_D^{N+1}\|_\Gamma^2 + \sum_{k=1}^{N}\left[\frac{\gamma_f}{\gamma_p} - (\frac{\gamma_f}{\gamma_p})^2\right]\|\varepsilon_D^k\|_\Gamma^2 + \|\varepsilon_S^{N+1}\|_\Gamma^2 + \sum_{k=1}^{N}\left[1 - \frac{\gamma_f}{\gamma_p}\right]\|\varepsilon_S^k\|_\Gamma^2 \\
&\le (\frac{\gamma_f}{\gamma_p})^2\|\varepsilon_D^0\|_\Gamma^2 + \frac{\gamma_f}{\gamma_p}\|\varepsilon_S^0\|_\Gamma^2 + \frac{\gamma_f}{\gamma_p}(\gamma_p^2 - \gamma_f^2)C_3\sum_{k=0}^{N}\|\overrightarrow{e}_u^k\|_1^2 \\
&\quad + \left[1 - \left(\frac{\gamma_f}{\gamma_p}\right)^2\right]g^2C_4\sum_{k=0}^{N}\|e_\phi^k\|_1^2 - 2\frac{\gamma_f}{\gamma_p}(\gamma_f + \gamma_p)C_1\sum_{k=0}^{N}\left[\|\overrightarrow{e}_u^k\|_1^2 + \|e_\phi^k\|_1^2\right].
\end{aligned}$$

Fix any number $s \in (0, 2)$. Suppose γ_f and γ_p are chosen such that

$$\begin{aligned}
&\frac{\gamma_f}{\gamma_p}(\gamma_p^2 - \gamma_f^2)C_3 - s\frac{\gamma_f}{\gamma_p}(\gamma_f + \gamma_p)C_1 \le 0, \\
&\left[1 - \left(\frac{\gamma_f}{\gamma_p}\right)^2\right]g^2C_4 - s\frac{\gamma_f}{\gamma_p}(\gamma_f + \gamma_p)C_1 \le 0
\end{aligned}$$

which are equivalent to

$$\gamma_p - \gamma_f \le \frac{sC_1}{C_3}, \tag{6.54}$$

$$\frac{1}{\gamma_f} - \frac{1}{\gamma_p} \le \frac{sC_1}{g^2C_4}. \tag{6.55}$$

Then we get

$$\begin{aligned}
0 &\le \frac{\gamma_f}{\gamma_p}\|\varepsilon_D^{N+1}\|_\Gamma^2 + \sum_{k=1}^{N}\left[\frac{\gamma_f}{\gamma_p} - (\frac{\gamma_f}{\gamma_p})^2\right]\|\varepsilon_D^k\|_\Gamma^2 + \|\varepsilon_S^{N+1}\|_\Gamma^2 + \sum_{k=1}^{N}\left[1 - \frac{\gamma_f}{\gamma_p}\right]\|\varepsilon_S^k\|_\Gamma^2 \\
&= (\frac{\gamma_f}{\gamma_p})^2\|\varepsilon_D^0\|_\Gamma^2 + \frac{\gamma_f}{\gamma_p}\|\varepsilon_S^0\|_\Gamma^2 - (2 - s)(\gamma_f + \gamma_p)C_1\sum_{k=0}^{N}\left[\|\overrightarrow{e}_u^k\|_1^2 + \|e_\phi^k\|_1^2\right].
\end{aligned}$$

With similar argument in the previous section, we finish the convergence for Case 2.

Now we derive a geometric convergence rate for Case 2. We still need to assume that α is small enough such that (5.35) is true; then, substituting (5.35), (5.39), and (5.40) into (6.52) we get

$$\begin{aligned}
&\frac{\gamma_f}{\gamma_p}\|\varepsilon_D^{k+1}\|_\Gamma^2 + \|\varepsilon_S^{k+1}\|_\Gamma^2 \\
&\le \left(\frac{\gamma_f}{\gamma_p}\right)^2\|\varepsilon_D^k\|_\Gamma^2 + \frac{\gamma_f}{\gamma_p}\|\varepsilon_S^k\|_\Gamma^2 + \left(1 - \left(\frac{\gamma_f}{\gamma_p}\right)^2\right)g^2C_4\|e_\phi^k\|_1^2 + \frac{\gamma_f}{\gamma_p}(\gamma_p^2 - \gamma_f^2)C_3\|\overrightarrow{e}_u^k\|_1^2 \\
&\quad -2\frac{\gamma_f}{\gamma_p}(\gamma_f + \gamma_p)C_1\left(\|\overrightarrow{e}_u^k\|_1^2 + \|e_\phi^k\|_1^2\right).
\end{aligned}$$

Suppose γ_f and γ_p are chosen such that

$$\begin{aligned}
&\frac{\gamma_f}{\gamma_p}(\gamma_p^2 - \gamma_f^2)C_3 - s\frac{\gamma_f}{\gamma_p}(\gamma_f + \gamma_p)C_1 \le 0, \\
&\left[1 - \left(\frac{\gamma_f}{\gamma_p}\right)^2\right]g^2C_4 - s\frac{\gamma_f}{\gamma_p}(\gamma_f + \gamma_p)C_1 \le 0
\end{aligned}$$

which are equivalent to

$$\gamma_p - \gamma_f \leq \frac{sC_1}{C_3}, \tag{6.56}$$

$$\frac{1}{\gamma_f} - \frac{1}{\gamma_p} \leq \frac{sC_1}{g^2 C_4}. \tag{6.57}$$

Then, we have

$$\begin{aligned} &\left(\frac{\gamma_f}{\gamma_p}\|\varepsilon_D^{k+1}\|_\Gamma^2 + \|\varepsilon_S^{k+1}\|_\Gamma^2\right) + (2-s)\frac{\gamma_f}{\gamma_p}(\gamma_f+\gamma_p)C_1\left(\|\overrightarrow{e}_u^k\|_1^2 + \|e_\phi^k\|_1^2\right) \\ \leq\ &\frac{\gamma_f}{\gamma_p}\left(\frac{\gamma_f}{\gamma_p}\|\varepsilon_D^k\|_\Gamma^2 + \|\varepsilon_S^k\|_\Gamma^2\right). \end{aligned}$$

Hence, we get the geometric convergence rate $\frac{\gamma_f}{\gamma_p}$ for ε_D^k, ε_S^k, $\overrightarrow{e}_u^k$, and e_ϕ^k. Using (5.22)-(5.26) and (6.49), we obtain the geometric convergence rate $\frac{\gamma_f}{\gamma_p}$ for e_p^k and $\overrightarrow{\varepsilon}_{S\tau}^k$. Combining the results of Case 1 and Case 2, we have proved the following theorem.

Theorem 2 *Assume $\gamma_f \leq \gamma_p$, $\mathbb{K}$ is isotropic, and α is small enough such that* (5.35) *is true. If $\gamma_f < \gamma_p$, assume that γ_f and γ_p are close to each other such that* (6.54) *and* (6.55) *are true. Then,*

$$\begin{aligned} &\phi_D^k \xrightarrow{X_D} \phi_D \qquad \overrightarrow{u}_S^k \xrightarrow{X_S} \overrightarrow{u}_S \qquad p_f^k \xrightarrow{Q_S} p_S, \\ &\eta_p^k \xrightarrow{H^{-\frac{1}{2}}(\Gamma)} \gamma\overrightarrow{u}_S\cdot\overrightarrow{n}_S + g\phi_D = -\gamma\mathbb{K}\nabla\phi_D\cdot\overrightarrow{n}_D + g\phi_D, \\ &\eta_f^k \xrightarrow{H^{-\frac{1}{2}}(\Gamma)} \gamma\overrightarrow{u}_S\cdot\overrightarrow{n}_S - g\phi_D = \overrightarrow{n}_S\cdot(\mathbb{T}(\overrightarrow{u}_S,p_S)\cdot\overrightarrow{n}_S) + \gamma\overrightarrow{u}_S\cdot\overrightarrow{n}_S, \\ &\overrightarrow{\eta}_{f\tau}^k \xrightarrow{H^{-\frac{1}{2}}(\Gamma)} \frac{\alpha\nu\sqrt{\boldsymbol{d}}}{\sqrt{trace(\prod)}}P_\tau\mathbb{K}\nabla\phi_D. \end{aligned}$$

Specifically, for the case of $\gamma_f < \gamma_p$, if γ_f and γ_p are close to each other such that (6.56) *and* (6.57)*are true, we then have the geometric convergence rate $\frac{\gamma_f}{\gamma_p}$ for ε_D^k, ε_S^k, $\overrightarrow{e}_u^k$, e_ϕ^k, e_p^k and $\overrightarrow{\varepsilon}_{S\tau}^k$.*

7 Computational examples

Consider the model problem (2.1)-(2.6) on $\Omega = [0,1]\times[-0.25, 0.75]$ where $\Omega_D = [0,1]\times[0,0.75]$ and $\Omega_S = [0,1]\times[-0.25,0]$. Choose $\frac{\alpha\nu\sqrt{\mathbf{d}}}{\sqrt{\mathrm{trace}(\prod)}} = 1$, $\nu = 1$, $g = 1$, $z = 0$, and $\mathbb{K} = kI$ where I the identity matrix and $k = 1$. The boundary condition data functions and the source terms are chosen such that the exact solution of the Stokes-Darcy system with the BJ interface condition is given by

$$\begin{cases} \phi_D = [2-\pi\sin(\pi x)][-y + cos(\pi(1-y))], \\ \overrightarrow{u}_S = [x^2y^2 + e^{-y},\ -\frac{2}{3}xy^3 + 2 - \pi\sin(\pi x)]^T, \\ p_S = -[2-\pi\sin(\pi x)]\cos(2\pi y). \end{cases}$$

We use a uniform grid with $h = \frac{1}{32}$. The Taylor-Hood element pair is used for the Stokes equation and quadratic finite elements are used for the Darcy equation.

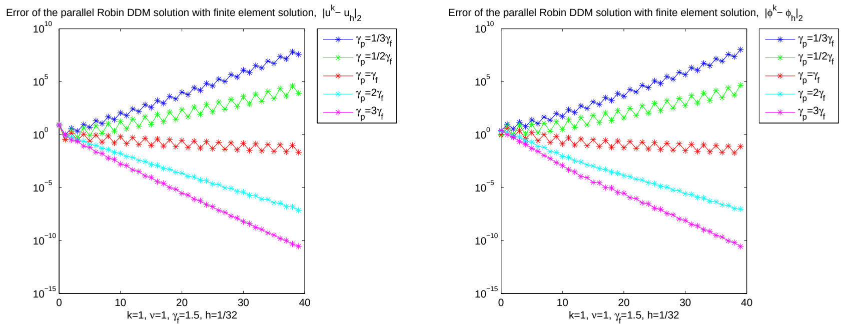

Fig. 2 L^2 velocity (left) and hydraulic head (right) errors of the iterates versus the iteration counter k for the parallel Robin-Robin domain decomposition method for $\gamma_f = 1.5$.

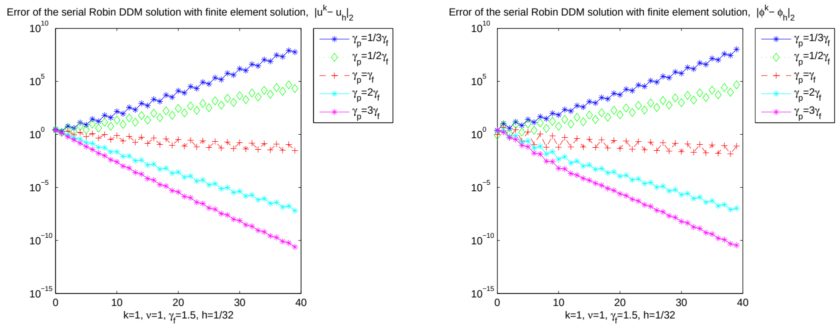

Fig. 3 L^2 velocity (left) and hydraulic head (right) errors of the iterates versus the iteration counter k for the serial Robin-Robin domain decomposition method for $\gamma_f = 1.5$.

Figure 2 shows the errors of the velocity and hydraulic head for the parallel DDM for $\gamma_f = 1.5$. We can see that the parallel domain decomposition method is convergent for $\gamma_f \le \gamma_p$, which computationally verifies the conclusions given in Section 5.

Figures 3 shows the errors of the velocity and hydraulic head in the serial DDM for $\gamma_f = 1.5$. We can see that the serial domain decomposition method is convergent for $\gamma_f \le \gamma_p$, which computationally verifies the conclusions given in Section 6.

Table 1 shows the number of iterations K for different grid sizes for both domain decomposition methods. Here, we set $\gamma_f = 1.5$ and $\gamma_p = 3\gamma_f$. The criteria used to stop the iteration, i.e., to determine the value K, is $\|\overrightarrow{u}_S^k - \overrightarrow{u}_S^{k-1}\|_{l^2} + \|\phi_D^k - \phi_D^{k-1}\|_{l^2} + \|p_S^k - p_S^{k-1}\|_{l^2} < \varepsilon$, where the tolerance $\varepsilon = 10^{-5}$ and $\|v\|_{l^2}$ denotes the l^2 norm of a vector formed by the values of a function v at the grid nodes. We can see that the number of iteration steps k is almost independent of the grid size h.

8 Conclusions

This article discusses two iterative domain decomposition methods for solving the steady Stokes-Darcy system with the Beavers-Joseph interface condition. Both the analyses and numerical experiments show that the domain decomposition solutions converge to the solution of the coupled system. Of course, more extensive computa-

Table 1 The iteration counter K versus the grid size h for both the parallel and the serial Robin-Robin domain decomposition methods.

h	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{32}$	$\frac{1}{64}$	$\frac{1}{128}$
K for the parallel DDM	22	24	24	26	28
K for the serial DDM	24	24	26	26	28

tional testing on more complicated geometries and with nonuniform meshes is needed and is underway.

So far, all studies concerning domain decomposition methods for the Stokes-Darcy system address the steady-state case, but most real-world applications are time-dependent. Thus, domain decomposition methods for the time-dependent Stokes-Darcy system are a subject of future research.

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