

Analysis and finite element approximation of a coupled, continuum pipe-flow/Darcy model for flow in porous media with embedded conduits*

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Abstract

We consider the continuum Darcy/pipe flow model for flows in a porous matrix containing embedded conduits; such coupled flows are present in, e.g., karst aquifers. The mathematical well-posedness of the coupled problem as well as convergence rates of finite element approximation are established in the two-dimensional case. Computational results are also provided.

1 Introduction

Conventionally, the purely hydraulic interaction between flow in a porous media (also called a fissured system in the geological community) and the free flow in a conduit network embedded in the porous media is simulated by employing three types of modeling approaches [27]. First, multiple sets of fractures may be coupled to each other in order to represent the different hydraulic properties of the porous media and the conduit system [14]. Alternatively, double-continuum or multi-continuum models have been applied wherein the cross flow between the two systems depends on the corresponding pressure differences via linear exchange terms [2, 11, 25, 26]. As a third approach, discrete networks of flow

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paths can be coupled to a continuum model in order to model the dualistic flow patterns [12, 19, 22].

The last approach, the most used for geological studies, is referred to as the *coupled continuum pipe flow* (CCPF) model. As its name indicates, it is a coupled system consisting of a two or three-dimensional continuum, the porous matrix, with an embedded one-dimensional conduit flow that is governed by the pipe flow model.¹ The flow in the matrix is governed by a Darcy-type system. It is used in [4, 6, 20] to study the genesis of karst aquifers and has achieved certain degree of success and acceptance.

1.1 The coupled pipe flow/Darcy model

We first review the formulation of the coupled pipe flow/Darcy model. The flow in the porous matrix is modeled by a continuum approach using the Boussinesq equation [5]

$$-\nabla \cdot (\mathbb{K} \nabla h_m) = S \frac{\partial h_m}{\partial t} - \gamma, \quad (1.1)$$

where h_m denotes the hydraulic head in the porous matrix, \mathbb{K} the hydraulic conductivity tensor, S the storativity, and γ represents the volumetric rate of fluid transfer to the porous matrix system from the conduit system per unit length. We assume that $\mathbb{K} \in \mathbb{R}^{d \times d}$, $d = 2$ or 3 , is positive definite, i.e., there exist positive constants $c_1, c_2 > 0$ such that, for any $\mathbf{v} \in \mathbb{R}^d$,

$$c_1 \mathbf{v} \mathbf{v}^T \leq \mathbf{v} \mathbb{K} \mathbf{v}^T \leq c_2 \mathbf{v} \mathbf{v}^T.$$

For the conduit flow, the discharge can be related to the head difference in the tube by applying the Darcy-Weisbach equation [7]

$$\frac{\partial h_c}{\partial \tau} = -\lambda \frac{u|u|}{2dg}, \quad (1.2)$$

where τ denotes the tangential direction along the one-dimensional pipe/channel conduit, d the pipe diameter (channel width for 2D),

$$u = \begin{cases} 4Q/(\pi d^2) & \text{for 3D} \\ Q/d & \text{for 2D} \end{cases}$$

the average velocity, g the gravitational acceleration, and Q the total discharge in the pipe. The friction factor, λ , depends on the velocity in the pipe via the Reynolds number $Re = |u|d/\nu$, with ν the kinematic viscosity of water. For low flow velocities, laminar flow is assumed and the Hagen-Poiseuille equation can be applied. The friction factor for laminar flow is calculated as

$$\lambda = \begin{cases} \frac{64}{Re} = \frac{64\nu}{d|u|} & \text{for 3D} \\ \frac{24}{Re} = \frac{24\nu}{d|u|} & \text{for 2D.} \end{cases}$$

¹Strictly speaking, for a two-dimensional problem, the fluid in the conduit is actually a channel flow. We refer to the conduit flow, no matter in two- or three-dimensional, as pipe flow for conciseness.

Substituting the above relation into (1.2), we have

$$Q = -D \frac{\partial h_c}{\partial \tau}, \quad \text{where } D = \begin{cases} \frac{\pi d^4 g}{128\nu} & \text{for 3D} \\ \frac{d^3 g}{12\nu} & \text{for 2D.} \end{cases}$$

Conservation of mass in the pipe implies that $\frac{\partial Q}{\partial \tau} = -\gamma$. Hence, we have

$$-\frac{\partial}{\partial \tau} \left(D \frac{\partial h_c}{\partial \tau} \right) = -\gamma \quad (1.3)$$

The matrix and conduit flows are coupled at their intersection by the quasi-steady-state exchange term [2, 24]

$$\gamma = \alpha_{ex}(h_c - h_m), \quad (1.4)$$

where $\alpha_{ex} > 0$ is the exchange coefficient. This is equivalent to saying that the process in the conduit is enslaved by that in the porous matrix.

As a result, in the steady-state case, we have the system

$$\begin{cases} -\nabla(\mathbb{K}\nabla h_m) & = -\alpha_{ex}(h_m - h_c)\delta_{\Omega_c} + f_m & \text{in } \Omega_m \\ -\frac{\partial}{\partial \tau} \left(D \frac{\partial h_c}{\partial \tau} \right) & = \alpha_{ex}(h_m - h_c) + f_c & \text{in } \Omega_c, \end{cases} \quad (1.5)$$

where f_m and f_c denote the external source or sink, Ω_m and Ω_c denote the regions for the porous matrix and pipe conduit, respectively, and δ_{Ω_c} represents the Dirac δ function concentrated on Ω_c . (1.5) is essentially two coupled Poisson equations in different dimensions.

The time-dependent version of (1.5) first appeared in [3, 4, 6], where karst aquifer genesis is studied. The model was originally borrowed from dual-porosity models [2, 11, 25, 26]. The finite element discretization for the coupled nonlinear Richard's equation and pipe (or plane) flow is discussed in [22, 28] for variably saturated porous media. However, no mathematical theory is available to guarantee existence of solutions and the coupling condition of the form (1.4) is not incorporated. Combining the idea of dual porosity and mimicking the numerical schemes in [22, 28], the model (1.5) in [3, 4, 6] is solved by the Carbonate Aquifer Void Evolution (CAVE) code [13]. CAVE solves the flow in the porous matrix by a finite difference scheme using MODFLOW [18] and the flow in conduit by a nonlinear finite difference discretization.² Then, the coupling exchange condition is reached by iteratively solving in the matrix and conduit regions. The exchange of fluid is only allowed at the discrete finite difference node points. No rigorous mathematical theory is presented therein to show well-posedness or guarantee convergence.

In this paper, a systematic presentation of the analysis and numerical analysis of a simplified, *two-dimensional* problem is presented. The stationary case

²The nonlinearity for the conduit only occurs for the turbulent case. Since we only consider laminar flow in the conduit, a nonlinear solver is not necessary.

with fully saturated flow in the matrix and linear flow in the conduit is considered. The following geometrical setup is used. The matrix continuum is assumed to occupy the square $\Omega_m := (0, 1) \times (-1/2, 1/2) \subset \mathbb{R}^2$ and the one-dimensional conduit pipe lies in the middle $\Omega_c := I := (0, 1) \times \{0\}$. Then, (1.5) becomes

$$\begin{cases} -\nabla(\mathbb{K}\nabla h_m) &= -\alpha_{ex}(h_m(x, 0) - h_c(x))\delta(y) + f_m & \text{in } \Omega_m \\ -\frac{d}{dx}(D\frac{dh_c}{dx}) &= \alpha_{ex}(h_m|_{y=0} - h_c) + f_c & \text{in } \Omega_c, \end{cases} \quad (1.6)$$

where $\delta(y)$ denotes the Dirac delta function in y . In addition, we impose Dirichlet (fixed heads) boundary conditions

$$\begin{cases} h_m &= g_m & \text{on } \partial\Omega_m \\ h_c &= g_c & \text{on } \partial\Omega_c, \end{cases} \quad (1.7)$$

where g_m and g_c are given functions. We remark that we may also consider a three-dimensional porous matrix coupled with two-dimensional free-flow plane. The proof of well-posedness remains the same, whereas the regularity and numerical analysis can be treated in a very similar fashion. For the coupling of a three-dimensional porous matrix with a one-dimensional conduit, the rigorous mathematical treatment is not straightforward since the coupling introduces a forcing term to the Darcy equation governing the matrix having a strong singularity. Despite the lack of mathematical rigor, such three-dimensional/one-dimensional couplings are used, at the discrete level, in engineering practice.

2 Well-posedness and regularity

In what follows, we only consider the homogeneous boundary case, i.e., we assume that $g_m \equiv 0$ and $g_c \equiv 0$. The non-homogeneous case can be converted to the homogeneous case through the standard procedure of extending the boundary data to the whole domain and then considering the difference between the original unknown and the extension.

The variational formulation for the problem (1.6) can be derived in the usual manner by multiplying the equation by test functions and performing integration by parts. This leads to the bilinear form $a(\cdot, \cdot)$ on $\mathbf{H} \times \mathbf{H}$, where $\mathbf{H} := H_0^1(\Omega_m) \times H_0^1(\Omega_c)$, defined by

$$\begin{aligned} a(\mathbf{h}, \mathbf{v}) &:= \int_{\Omega} \mathbb{K}\nabla h_m(x, y) \cdot \nabla v_m(x, y) dx dy + \int_0^1 Dh'_c(x)v'_c(x) dx \\ &\quad + \alpha_{ex} \int_0^1 (h_m(x, 0) - h_c(x))v_m(x, 0) dx \\ &\quad - \alpha_{ex} \int_0^1 (h_m(x, 0) - h_c(x))v_c(x) dx, \end{aligned} \quad (2.1)$$

where $\mathbf{h} = (h_m, h_c)$, $\mathbf{v} = (v_m, v_c) \in \mathbf{H}$.

We say that $\mathbf{h} \in \mathbf{H}$ is a weak solution of (1.6) if it satisfies

$$a(\mathbf{h}, \mathbf{v}) = (f_m, v_m)_{L^2(\Omega)} + (f_c, v_c)_{L^2(I)} \quad \forall \mathbf{v} \in \mathbf{H}. \quad (2.2)$$

Proposition 2.1. *Assume that $f_m \in H^{-1}(\Omega)$ and $f_c \in H^{-1}(I)$. Then, a weak solution \mathbf{h} of (1.6) exists, is unique, and satisfies the estimate*

$$\|\mathbf{h}\|_{\mathbf{H}} \leq C(\|f_m\|_{H^{-1}(\Omega)} + \|f_c\|_{H^{-1}(I)}), \quad (2.3)$$

where C is a constant independent of f_m and f_c .

Proof. The proof is a straightforward application of the Lax-Milgram theorem. We only need to verify the continuity and coercivity of the bilinear form $a(\cdot, \cdot)$ on $\mathbf{H} \times \mathbf{H}$.

The verification of the continuity is trivial and follows directly from the trace theorem since $\|h(\cdot, 0)\|_{L^2(I)} \leq C\|h\|_{H^1(\Omega)}$. The coercivity is also straightforward because, for $\mathbf{h} \in \mathbf{H}$, we have that

$$\begin{aligned} a(\mathbf{h}, \mathbf{h}) &= \int_{\Omega} \mathbb{K} \nabla h_m(x, y) \cdot \nabla h_m(x, y) dx dy + \int_0^1 D(h'_c(x))^2 dx \\ &\quad + \alpha_{ex} \int_0^1 (h_m(x, 0) - h_c(x)) h_m(x, 0) dx \\ &\quad - \alpha_{ex} \int_0^1 (h_m(x, 0) - h_c(x)) h_c(x) dx \\ &\geq \int_{\Omega} \mathbb{K} \nabla h_m(x, y) \cdot \nabla h_m(x, y) dx dy + \int_0^1 D(h'_c(x))^2 dx. \end{aligned}$$

□

Next, we consider the regularity of the weak solution.

Proposition 2.2. *Let $\mathbf{h} = (h_m, h_c)$ denote the weak solution of (1.6). Assume that $f_m \in H^{-\frac{1}{2}}(\Omega)$ and $f_c \in L^2(I)$. Then, $\mathbf{h} \in H^{\frac{3}{2}-\epsilon}(\Omega) \times H^2(I)$ for all $\epsilon \in (0, \frac{1}{2})$ and*

$$\|\mathbf{h}\|_{H^{\frac{3}{2}-\epsilon}(\Omega) \times H^2(I)} \leq C(\epsilon)(\|f_m\|_{H^{-\frac{1}{2}}(\Omega)} + \|f_c\|_{L^2(I)}). \quad (2.4)$$

This regularity result is nearly optimal in the sense that there is no $\beta > 0$ such that $h_m \in H^{\frac{3}{2}+\beta}(\Omega)$, even if f_m and f_c are smooth.

Proof. The fact that $h_c \in H^2(I)$ is obvious. Indeed, by setting the test function v_m for the porous media identically to zero in the weak formulation (2.2), we see that h_c solves the one dimensional elliptic problem

$$\int_0^1 (Dh'_c(x)v'_c(x) + \alpha_{ex}h_c(x)v_c(x))dx = \int_0^1 (\alpha_{ex}h_m(x, 0) + f_c(x))v_c(x)dx.$$

The regularity of h_c follows from classical elliptic regularity, the assumption that $f_c \in L^2(I)$, and the fact that $h_m(x, 0) \in H^{\frac{1}{2}}(I) \subset L^2(I)$.

As for the regularity of h_m , we notice that h_m solves the equation

$$\begin{aligned} & \int_{\Omega} \mathbb{K} \nabla h_m(x, y) \cdot \nabla v_m(x, y) dx dy \\ &= - \int_0^1 \alpha_{ex}(h_m(x, 0) - h_c(x)) v_m(x, 0) dx + \int_{\Omega} f_m v_m dx dy, \forall v_m \in H_0^1(\Omega). \end{aligned} \quad (2.5)$$

We further notice that, for $\epsilon \in (0, \frac{1}{2})$,

$$\begin{aligned} \int_0^1 (h_m(x, 0) - h_c(x)) v(x, 0) dx &\leq (\|h_m(\cdot, 0)\|_{L^2(I)} + \|h_c\|_{L^2(I)}) \|v(\cdot, 0)\|_{L^2(I)} \\ &\leq C(\epsilon) (\|h_m\|_{H^1(\Omega)} + \|h_c\|_{L^2(I)}) \|v\|_{H^{\frac{1}{2}+\epsilon}(\Omega)}. \end{aligned}$$

Therefore, $-\int_0^1 \alpha_{ex}(h_m(x, 0) - h_c(x)) v_m(x, 0) dx$ defines a bounded linear functional $\tilde{f}_m \in H^{-\frac{1}{2}-\epsilon}(\Omega)$ on $H_0^{\frac{1}{2}+\epsilon}(\Omega)$, and

$$\|\tilde{f}_m\|_{H^{-\frac{1}{2}-\epsilon}(\Omega)} \leq C(\epsilon) (\|f_m\|_{H^{-1}(\Omega)} + \|f_c\|_{H^{-1}(I)}).$$

Henceforth, thanks to classical elliptic regularity results for domains with corners [17] and with the help of interpolation inequalities if necessary [1], we have that $h_m \in H^{\frac{3}{2}-\epsilon}(\Omega)$ because the right-hand side of (2.5) $\tilde{f}_m + f_m \in H^{-\frac{1}{2}-\epsilon}(\Omega)$. Higher regularity of f_m or f_c does not improve the results.

As for the near optimal regularity, we notice that a simple calculation indicates that

$$\delta(y) \in H^{-\frac{1}{2}-\epsilon}(\Omega) \quad \forall \epsilon > 0$$

but not any better (unless one resorts to anisotropic Sobolev spaces or non-Hilbert spaces). Now, if h_m were more regular and belonged to $H^{\frac{3}{2}+\beta}(\Omega)$ for some $\beta > 0$, it would imply that³

$$\delta(y)(h_m(x, y) - h_c(x)) \in H^{-\frac{1}{2}-\epsilon}(\Omega), \quad \forall \epsilon > 0.$$

This contradicts the first equation in (1.6) (which is valid in the distributional sense) since the regularity of the left-hand side would be $H^{-\frac{1}{2}+\beta}(\Omega)$ under the assumption that $h_m \in H^{\frac{3}{2}+\beta}(\Omega)$, and this is not consistent with the optimal regularity of the right-hand side ($H^{-\frac{1}{2}-\epsilon}(\Omega)$). \square

Remark 2.3. *In the case of homogeneous isotropic matrix, i.e. $\mathbb{K} = k\mathbb{I}$, one can easily check via separation of variable that we have higher order piecewise regularity in the sense that $h_m|_{\Omega \cap \{y>0\}} \in H^{2+\epsilon}(\Omega \cap \{y > 0\})$, $h_m|_{\Omega \cap \{y<0\}} \in H^{2+\epsilon}(\Omega \cap \{y < 0\})$ for any $\epsilon \in (0, \frac{1}{2})$.*

³This is true because, for any $f_1 \in H_0^{\frac{1}{2}+\epsilon}(\Omega)$, $\epsilon > 0$ and $f_2 \in H^{1+\beta}(\Omega)$, $\beta > 0$, their product, $f_1 f_2$, belongs to $H_0^{\frac{1}{2}+\epsilon}(\Omega)$ [17].

3 Finite element approximations

In this section, we construct and analyze finite element approximations for the weak solution of (1.6). Let Ω_h denote a regular triangulation [8] of Ω and I_h be a regular partition of I . We construct the finite element subspaces $H_m^h \subset H_0^1(\Omega)$ and $H_0^1(I) \subset H_c^h$ consisting of continuous piecewise polynomials of order $k \geq 1$ with respect to Ω_h and I^h , respectively, where h denotes a measure of the mesh size. The finite element approximation $\mathbf{h}^h = (h_m^h, h_c^h) \in \mathbf{H}^h := H_m^h \times H_c^h$ of the weak solution \mathbf{h} of (1.6) is determined from

$$a(\mathbf{h}^h, \mathbf{v}^h) = (f_m, v_m^h)_{L^2(\Omega)} + (f_c, v_c^h)_{L^2(I)}, \quad \forall \mathbf{v}^h = (v_m^h, v_c^h) \in \mathbf{H}^h. \quad (3.1)$$

By the regularity result of Proposition 2.2 and a standard argument for finite element approximations, we have the following error estimate.

Theorem 3.1. (3.1) admits a unique solution $\mathbf{h}^h = (h_m^h, h_c^h) \in \mathbf{H}^h$. Moreover, for $0 < \epsilon < \frac{1}{2}$, there exists a constant $C(\epsilon)$ independent of h such that

$$\begin{aligned} \|\mathbf{h} - \mathbf{h}^h\|_{\mathbf{H}} &\leq C(\epsilon)h^{\frac{1}{2}-\epsilon} \\ \|\mathbf{h} - \mathbf{h}^h\|_{L^2(\Omega) \times L^2(I)} &\leq C(\epsilon)h^{1-2\epsilon}. \end{aligned} \quad (3.2)$$

Furthermore, if a given exact solution h_m is piecewise smooth in the sense that $h_m|_{y>0} \in H^{k+1}(\Omega \cap \{y > 0\})$, $h_m|_{y<0} \in H^{k+1}(\Omega \cap \{y < 0\})$, and $h_c \in H^{k+1}(I)$, and if the finite elements are chosen so that I does not intersect with the interior of any element from Ω_h , we have the following estimates with improved convergence rates:

$$\begin{aligned} \|\mathbf{h} - \mathbf{h}^h\|_{\mathbf{H}} &\leq Ch^k, \\ \|\mathbf{h} - \mathbf{h}^h\|_{L^2(\Omega) \times L^2(I)} &\leq C(\epsilon)h^{k+\frac{1}{2}-\epsilon}. \end{aligned} \quad (3.3)$$

Proof. The proof of the well-posedness of the finite element problem (3.1) is the same as that for the well-posedness of the PDE given in Proposition 2.1.

Recall that we have the well-known approximation inequalities

$$\begin{aligned} \|\mathbf{h} - \mathcal{I}_k^h(\mathbf{h})\|_{\mathbf{H}} &\leq Ch^l \|\mathbf{h}\|_{H^{l+1}(\Omega) \times H^{l+1}(I)} \\ \|\mathbf{h} - \mathcal{I}_k^h(\mathbf{h})\|_{\mathbf{H}} &\leq C \|\mathbf{h}\|_{\mathbf{H}} \end{aligned}$$

for all $0 \leq l \leq k$, where C is a constant with value independent of \mathbf{h} and h , and \mathcal{I}_k^h denotes the classical interpolation operator by continuous piecewise polynomials of order k ([8, Chapter 4]). Utilizing the interpolation theorem [21], we obtain

$$\|\mathbf{h} - \mathcal{I}_k^h(\mathbf{h})\|_{\mathbf{H}} \leq C(\epsilon)h^{\frac{1}{2}-\epsilon} \|\mathbf{h}\|_{H^{\frac{3}{2}-\epsilon}(\Omega) \times H^{\frac{3}{2}-\epsilon}(I)}, \forall k.$$

A standard finite element error estimates together with the continuity and coercivity of the bilinear form $a(\cdot, \cdot)$ on $\mathbf{H} \times \mathbf{H}$ leads to

$$\begin{aligned} \|\mathbf{h} - \mathbf{h}^h\|_{\mathbf{H}} &\leq C \inf_{\tilde{\mathbf{h}}^h \in \mathbf{H}^h} \|\mathbf{h} - \tilde{\mathbf{h}}^h\|_{\mathbf{H}} \\ &\leq C \|\mathbf{h} - \mathcal{I}_k^h(\mathbf{h})\|_{\mathbf{H}} \\ &\leq C(\epsilon)h^{\frac{1}{2}-\epsilon} \|\mathbf{h}\|_{H^{\frac{3}{2}-\epsilon}(\Omega) \times H^{\frac{3}{2}-\epsilon}(I)}. \end{aligned}$$

As for the L^2 error estimate in (3.2), we apply the classical Aubin-Nitsche technique to deduce that

$$\begin{aligned}
\|\mathbf{h} - \mathbf{h}^h\|_{L^2(\Omega) \times L^2(I)}^2 &= a(\mathbf{h} - \mathbf{h}^h, \phi) \\
&= a(\mathbf{h} - \mathbf{h}^h, \phi - \psi^h) \quad \forall \psi^h \in \mathbf{H}^h \\
&\leq C \|\mathbf{h} - \mathbf{h}^h\|_{\mathbf{H}} \inf_{\psi^h \in \mathbf{H}^h} \|\phi - \psi^h\|_{\mathbf{H}} \\
&\leq C(\epsilon) h^{1-2\epsilon} \|\mathbf{h}\|_{H^{\frac{3}{2}-\epsilon}(\Omega) \times H^{\frac{3}{2}-\epsilon}(I)} \|\phi\|_{H^{\frac{3}{2}-\epsilon}(\Omega) \times H^{\frac{3}{2}-\epsilon}(I)} \\
&\leq C(\epsilon) h^{1-2\epsilon} \|\mathbf{h} - \mathbf{h}^h\|_{L^2(\Omega) \times L^2(I)},
\end{aligned}$$

where $\phi \in \mathbf{H}$ solves the adjoint problem

$$a(\mathbf{v}, \phi) = (\mathbf{h} - \mathbf{h}^h, \mathbf{v})_{L^2(\Omega) \times L^2(I)}, \quad \forall \mathbf{v} \in \mathbf{H}$$

and we have applied the regularity estimate (2.4).

If the exact solution \mathbf{h} is piecewise smooth as assumed, then we have the better estimates stated in (3.3). Indeed, in this case we have

$$\begin{aligned}
\|\mathbf{h} - \mathbf{h}^h\|_{\mathbf{H}} &\leq C \inf_{\tilde{\mathbf{h}}^h \in \mathbf{H}^h} \|\mathbf{h} - \tilde{\mathbf{h}}^h\|_{\mathbf{H}} \leq C \|\mathbf{h} - \mathcal{I}_k^h(\mathbf{h})\|_{\mathbf{H}} \\
&\leq Ch^k \left(\|\mathbf{h}\|_{H^{k+1}(\Omega \cap \{y>0\}) \times H^{k+1}(I)} + \|\mathbf{h}\|_{H^{k+1}(\Omega \cap \{y<0\}) \times H^{k+1}(I)} \right),
\end{aligned}$$

where in the last step we have used the assumption that I does not intersect with the interior of any element from Ω_h and hence h_m is smooth on each element so that leads the classical interpolation error can be applied.

We also have, by the Nitsche-Aubin technique,

$$\begin{aligned}
\|\mathbf{h} - \mathbf{h}^h\|_{L^2(\Omega) \times L^2(I)}^2 &\leq C \|\mathbf{h} - \mathbf{h}^h\|_{\mathbf{H}} \inf_{\psi^h \in \mathbf{H}^h} \|\phi - \psi^h\|_{\mathbf{H}} \\
&\leq C(\epsilon) h^{k+\frac{1}{2}-\epsilon} \|\phi\|_{H^{\frac{3}{2}-\epsilon}(\Omega) \times H^{\frac{3}{2}-\epsilon}(I)} \\
&\leq C(\epsilon) h^{k+\frac{1}{2}-\epsilon} \|\mathbf{h} - \mathbf{h}^h\|_{L^2(\Omega) \times L^2(I)}.
\end{aligned}$$

□

Remark 3.2. *If we assume further regularity of solutions to the PDE (2.2) in the sense that, for all $H^{k-1}(\Omega_m)$ data, we have piecewise $H^{k+1}(\Omega \cap \{y > 0\})$ and $H^{k+1}(\Omega \cap \{y < 0\})$ solutions for h_m and*

$$\begin{aligned}
\|\mathbf{h}\|_{H^{k+1}(\Omega \cap \{y>0\}) \times H^{k+1}(I)} + \|\mathbf{h}\|_{H^{k+1}(\Omega \cap \{y<0\}) \times H^{k+1}(I)} \\
\leq C(\|f_m\|_{H^{k-1}(\Omega)} + \|f_c\|_{H^{k-1}(I)}),
\end{aligned}$$

then we would obtain the optimal convergence rate $k + 1$ in L^2 for approximations with piecewise polynomials of order k . Such piecewise regularity holds for homogeneous isotropic matrix with $k = 1$ according to remark (2.3).

4 Computational results

We consider the two-dimensional setting. For convenience, we set all parameters in (1.5), including the hydraulic conductivity, to unity. We adjust the forcing functions f_m and f_c in Ω_m and Ω_c , respectively, so that the exact solution is given by

$$\begin{cases} h_c &= 2 \sin(2\pi x) & \text{in } \Omega_c \\ h_m &= \sin(2\pi x) & \text{in } (0, 1) \times (-1/2, 0] \subset \Omega_m \\ h_m &= (-\frac{\alpha_{ex}}{\mathbb{K}}y + 1) \sin(2\pi x) & \text{in } (0, 1) \times [0, 1/2) \subset \Omega_m. \end{cases} \quad (4.1)$$

We use the piecewise linear or piecewise quadratic finite element spaces, i.e., $k = 1$ or 2 respectively, based on uniform grids. The computational errors obtained and the estimated convergence rates are summarized in Table 4.1 and 4.2.

Table 4.1: Convergence rate for steady state problem with piecewise linear elements.

h	$\ h_c - h_c^h\ _0$	$\ h_m - h_m^h\ _0$	$ h_c - h_c^h _1$	$ h_m - h_m^h _1$
2^{-2}	$3.045E-1$	$1.648E-1$	3.867	2.250
2^{-3}	$7.959E-2$	$4.244E-2$	1.994	1.164
2^{-4}	$2.012E-2$	$1.069E-2$	1.005	$5.869E-1$
2^{-5}	$5.045E-3$	$2.677E-3$	$5.033E-1$	$2.941E-1$
2^{-6}	$1.262E-3$	$6.696E-4$	$2.518E-1$	$1.471E-1$
rate of conv.	1.981	1.987	0.987	0.985

The second-order convergence rate for the L^2 norm of the error and the first-order convergence rate for the H^1 norm for piecewise linear elements completely agrees with our analysis since we are dealing with homogeneous isotropic matrix in this numerical example and hence we have piecewise regularity; see remarks (2.3 and 3.2).

Table 4.2: Convergence rate for steady state problem with piecewise quadratic elements

h	$\ h_c - h_c^h\ _0$	$\ h_m - h_m^h\ _0$	$ h_c - h_c^h _1$	$ h_m - h_m^h _1$
2^{-2}	$3.047E-2$	$1.644E-2$	$7.888E-1$	$6.906E-2$
2^{-3}	$3.907E-3$	$2.092E-3$	$2.025E-1$	$1.517E-2$
2^{-4}	$4.915E-4$	$2.628E-4$	$5.096E-2$	$3.582E-3$
2^{-5}	$6.153E-5$	$3.290E-5$	$1.276E-2$	$8.818E-4$
2^{-6}	$7.694E-6$	$4.114E-6$	$3.191E-3$	$2.198E-4$
rate of conv.	2.989	2.992	1.987	2.070

The second order convergence rate in the H^1 norm agrees with our theoretical prediction with $k = 2$ since the chosen exact solution is piecewise smooth

in the upper and lower regions and I does not intersect with the interior of any element from Ω_h . The better-than-predicted convergence rate in the L^2 norm may be explained via Remark 3.2, although it is not known if the piecewise regularity assumed in that remark for the solution of (2.2) is true in general.

5 Concluding remarks

In this paper, we addressed several mathematical issues regarding the CCFP model that is of increasing interest in the geological science community. A weak formulation of the linear stationary CCFP model is presented and its well-posedness is demonstrated. Finite element discretizations are then constructed and analyzed. Optimal convergence rates in the L^2 and H^1 norms are derived for the piecewise linear element case, and an optimal H^1 convergence rate for piecewise quadratic elements is derived under the assumption that the exact solution is piecewise smooth. However, we obtain better-than-expected numerical convergence rate in the L^2 norm. This is a likely consequence of the proper alignment of the grids and possible piecewise higher regularity of the solution for generic forcing.

Comparisons of results obtained using the coupled continuum/pipe flow model to the coupled Stokes-Darcy system with Beavers-Joseph or the simplified Beavers-Joseph-Saffman interface boundary condition (see [9, 10, 15, 16, 23]) are underway. The investigation of the three-dimensional problem is also underway. As they stand, the analyses presented in this paper do not hold for the three-dimensional case due to the stronger (compared to the two-dimensional case) singular nature of the line Dirac delta function in three dimensions.

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