

A BRIEF HISTORY OF THE BEGINNING OF THE FINITE ELEMENT METHOD

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SUMMARY

This paper presents summaries of the works of several authors associated with the invention of the analysis technique now referred to as the finite element method. It stresses the notion of first development from which subsequent ideas evolved and gives what is believed to be an accurate record of the historical sequence of published papers in the international literature.

KEY WORDS: history; finite elements; structural mechanics

INTRODUCTION

The purpose of this paper is to trace the development of the ideas which led to the method of analysis in which the field equations of mathematical physics are approximated over simple regions (triangles, quadrilaterals, tetrahedrons, etc.), and then assembled together so that equilibrium or continuity is satisfied at the interconnecting nodal points of the domains. The technique thus realized is now referred to as the finite element method (FEM). It is natural to assume that if the size of the approximating domains becomes infinitely small, the solutions so obtained tending to this limit by successive mesh refinement converge towards the analytic solution. It is also noted that the subdivision should in addition reproduce the constant function space (e.g. strain, curvature, potential), since the representation of this state must be independent of mesh size (i.e. a larger mesh is obtained from a smaller one by simple magnification, but the function field remains constant). There are five groups of papers which may be considered in the development of the FEM and in one of these the name originated. They are the papers by Courant,¹ Argyris,² Turner *et al.*,³ Clough^{4–6} and Zienkiewicz.⁷ The present paper examines the contribution of each of these five groups of papers and attempts to put them in their place in the history of the FEM.

In this survey of the papers the motivation is to examine the contributions by the following criteria:

- (1) Is the discretization technique clearly explained so that the reader will be capable of implementing it to model any practical problem with complex geometry and loading?

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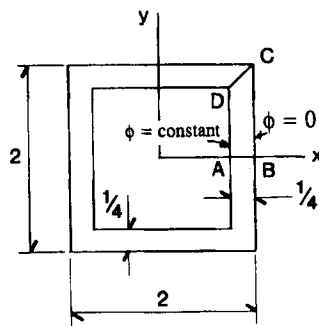
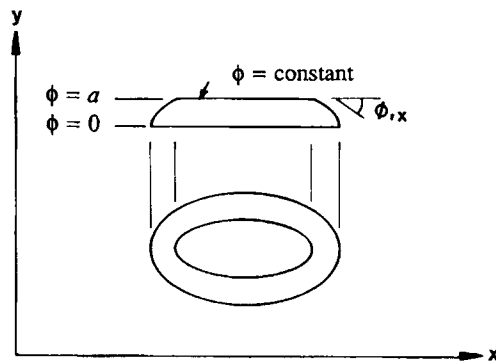


Figure 1. Square hollow shaft

Figure 2. Stress function, ϕ , with slopes $\phi_{,x}$, $\phi_{,y}$

- (2) Has the paper explained how the solution convergence may be achieved?
- (3) Can the presentation in the paper be adapted routinely to automatic computation?

DISCUSSION OF THE ORIGINAL CONTRIBUTIONS

This section contains detailed discussions and reviews of the various individual contributions of the above five groups of papers.

Courant

Courant¹ developed the idea of the minimization of a functional using linear approximation over subregions, with the values being specified at discrete points which in essence become the node points of a mesh of elements. In his paper, Courant shows the mesh subdivision with 1, 2, 3, 5 and 9 approximating points used to solve the St Venant's torsion of a square hollow box of (2×2) with wall thickness of $\frac{1}{4}$, Figure 1. Courant notes that the St Venant's torsion problem can be solved using a stress function ϕ which has a zero value on the external boundary, and constant value on any closed internal boundary. The shear stresses (τ_{zx} , τ_{zy}) in the shaft are given as the first derivatives of the stress function ϕ , such that, see Figure 2,

$$\tau_{zx} = \frac{\partial \phi}{\partial y} = \phi_{,y}; \quad \tau_{zy} = -\frac{\partial \phi}{\partial x} = -\phi_{,x} \quad (1)$$

with $\phi = 0$ on an external boundary and $\phi = \text{constant}$ on the internal boundary.

The strain energy V for the shaft Figure 2 is given by (see Reference 8)

$$V = \frac{1}{2G} \iint (\tau_{zx}^2 + \tau_{zy}^2) dx dy = \frac{1}{2G} \iint (\phi_{,x}^2 + \phi_{,y}^2) dx dy \quad (2)$$

The torque T is found from the integral

$$T = \iint (\tau_{zy}x - \tau_{zx}y) dy dx = - \iint (\phi_{,x}x + \phi_{,y}y) dx dy \quad (3)$$

Integrating over the region gives the torque

$$T = 2 \iint \phi dx dy = 2 \times \text{volume under } \phi \text{ surface} \quad (4)$$

The potential energy U of the system, for a twist per unit length θ , is given by

$$U = -T\theta = -2 \iint \phi dx dy \theta \quad (5)$$

The condition of stationary potential energy is expressed as

$$\delta(V + U) = 0 \quad (6)$$

That is,

$$\delta \left\{ \frac{1}{2G} \iint (\phi_{,x}^2 + \phi_{,y}^2) dx dy - 2 \iint \phi dx dy \theta \right\} = 0 \quad (7)$$

If ϕ is described in terms of a number of discrete parameters a_i , the stationary condition (7) leads to the set of linear equations (8);

$$\frac{\partial(V + U)}{\partial a_i} = 0 \quad (8)$$

In the Appendix of his paper, Courant applies the above theory to the solution for the stress function ϕ for a (2×2) hollow square shaft, with an opening of $(1\frac{1}{2} \times 1\frac{1}{2})$ as shown in Figure 1. In Figure 1, the quadrilateral ABCD represents $\frac{1}{8}$ of the tube, and if this surface is assumed to be planar, only $\phi = a$, the value along AD, is unknown. Courant thus starts to explore the Rayleigh–Ritz method, first using the approximation

$$\phi = a(1 - x) \quad (9)$$

for which, with $a = 0.44045$,

$$T = 2 \iint \phi dx dy = 2 \times 0.339 \times 2G\theta \quad (10)$$

From the theory of elasticity, it is known that the shear stresses are zero at the external corner C , and have a singularity at the internal corner D . It is evident that the ϕ function should have zero slopes at C , and infinite slopes at D , and that the function in equation (9) does not satisfy these conditions. Courant then goes on with the Rayleigh–Ritz method with a second functional with two unknowns,

$$\phi = a(1 - x)[1 + \alpha(x - 3/4)y] \quad (11)$$

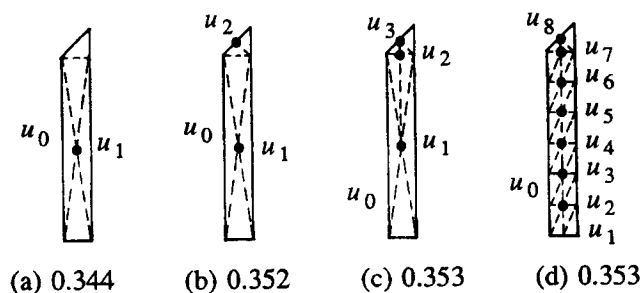


Figure 3. Cross-section subdivisions

A careful examination of this functional shows that it is little better than that in equation (9). Not surprisingly, the value for the torsion stiffness obtained by Courant is now 0.340, a slight improvement. It is then that Courant changes his approach and introduces the idea of linear approximation to ϕ over triangular areas as shown in Figure 3 of the paper. These triangular areas are shown in Figures 3(a)–3(d). The values for the torsion stiffness are given in the Figure 3, and we see that there is a modest improvement. Courant gives the following explanation of the procedure:

‘These results were checked with those obtained by our generalized method of finite differences where arbitrary triangular nets are permitted. The diagrams are self-explanatory. Unknown are the net point values, $u_i (i = 0)$. In the net triangles our functions were chosen as linear, so that the variational problem results in linear equations in the u_i . These results show in themselves, and by comparison, that the generalized methods of nets seems to have advantages. It was applied with similar success to the case of a square with four holes, and is obviously adaptable to any type of domain, much more so than the Rayleigh–Ritz procedure in which the construction of admissible functions would usually offer decisive obstacles.’

Courant goes on:

‘In a separate publication it will be shown how the method can be extended also to problems of plates and to other problems involving higher derivatives.’

Since this work appears never to have been published, we must presume that it was never completed. One might speculate that the solution of the Poisson equation, $\nabla^2 \phi = \text{constant}$, for the stress function ϕ in the torsion problem, is perhaps the most simple two-dimensional continuum problem, whereas the solution of the biharmonic equation in plate bending is a more formidable task. May be Courant found either its formulation or the solution of the large number of linear equations required beyond the computational power at that time.

It is also worthwhile to examine the Courant meshes shown in Figures 3(a)–(d). As pointed out previously, the ϕ surface has derivative singularities at the reentrant corner D, and also zero values at C. Thus along the diagonal DC there must be a marked variation from linearity. Also, along AB, the shear stress τ_{xy} varies nearly linearly, so that $\phi_{,x}$ will also be linear, implying that there is a nearly parabolic variation on ϕ along AB. Thus the set of triangles shown in Figure 3(a) is a poor choice as it really does not reflect any of these features. The meshes (b) and (c) are slightly better, but (d) gives no further advantages. We conclude that the meshes chosen show little understanding of the physics of the problem and certainly do not represent a study of convergence of the solution.

Since Courant has not given any of the mathematical details of his piecewise linear approximation to the ϕ surface all that we can say is that he indicated a procedure which could apparently be used in the minimization of the total potential energy of the torsion problem.

In Reference 9, Felippa gives an interesting overview of the Courant paper with some insights into Courant's life and with his observations that in the Courant paper, 'Flares of brilliance are interwoven with expanses of flat material'.

Argyris

The series of papers under the title, '*Energy Theorems and Structural Analysis*' by Argyris² is a significant landmark in the history of structural mechanics. This paper develops the matrix theory of structures for the discrete elements and then goes on to show that this is only a particular case of the general continuum in which stresses and strains have been specified. This innovation leads to the concept of flexibility and stiffness:

$$\text{Flexibility: } [F] = \int_{\text{vol}} [b]^T [f] [b] dV \quad (12)$$

$$\text{Stiffness: } [K] = \int_{\text{vol}} [a]^T [k] [a] dV \quad (13)$$

in which $[f]$ and $[k]$ are material constitutive equations and $[b]$ and $[a]$ are stress-force and strain-displacement relationships, respectively. These equations have become standard formulations in structural mechanics. Argyris recognizes the concept of duality between equilibrium and compatibility. That is, if equilibrium between nodal forces $\{R\}$ and member forces $\{S\}$ is expressed by the linear transformation,

$$\{S\} = [b]\{R\} \quad (14)$$

it follows automatically that the corresponding displacements are related by the transposed expression,

$$\{r\} = [b]^T \{v\} \quad (15)$$

Alternatively if compatibility of deformations is expressed as

$$\{v\} = [a]\{r\} \quad (16)$$

it follows automatically that equilibrium is expressed by

$$\{R\} = [a]^T \{S\} \quad (17)$$

The four equations (14)–(17) are given completely by Argyris² in their infinitesimal forms, relating stresses and strains to loads and displacements. Argyris then applies the theory to the displacements of the rectangular panel (d, l) shown in Figure 4, for the case of nodal displacements which vary linearly along the edges of the element thus calculating its stiffness matrix. He thus develops the first element using serendipity interpolation functions.

The paper determines the stiffness k_{jk} of the panel shown in Figure 4, for unit displacements in the z - and s -directions. The stiffness is an 8×8 matrix. Argyris notes: 'Assume that the displacements vary linearly between nodal points. This assumption offends against the equilibrium conditions but its effect upon stiffness is not pronounced as long as we keep the unit panels reasonably small.' From this quote we see that Argyris anticipates convergence with mesh refinement, although his examples do not explicitly prove the point.

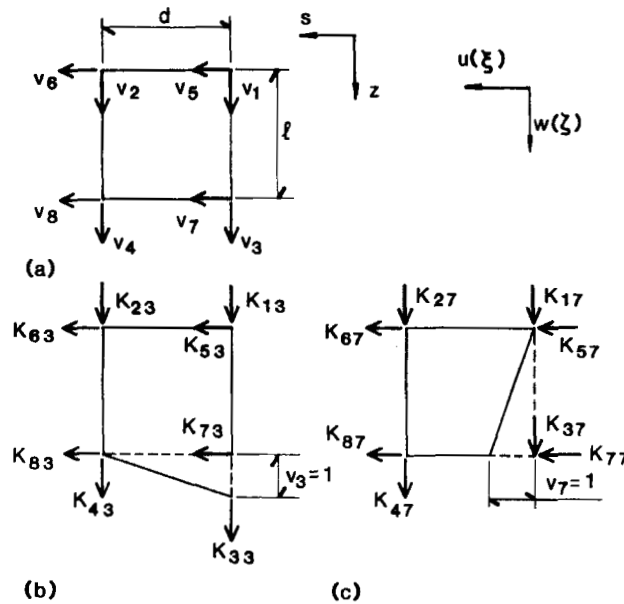


Figure 4. Rectangular plane stress panel: (a) nodal displacements; (b) unit displacements V_3 ; (c) unit displacement U_7

Consider the state of strain and stress arising from a unit displacement,

$$v_3 = 1 \quad (18)$$

Following our assumption, the internal displacements are given, using ξ , ζ for the s , z -coordinates, by

$$u_3 = 0; \quad w_3 = \frac{\zeta}{l} \left(1 - \frac{\xi}{d} \right) \quad (19)$$

Note: From this, when $\xi = d$, $w_3 = 0$, and when $\xi = 0$, $\zeta = l$; then $w_3 = v_3$, so that Argyris has developed the concept of linear displacement interpolation along the element boundaries (i.e. it is a displacement compatible element).

The strains ϵ_h and stresses σ_j at any point, using the notation of the Argyris paper, are given by

$$\epsilon_{zs3} = \frac{\partial w}{\partial \xi} = -\frac{\zeta}{ld}; \quad \sigma_{zs3} = -\frac{G\zeta}{ld} \quad (20a)$$

$$\epsilon_{zz3} = \frac{\partial w}{\partial \zeta} = \frac{1}{l} \left(1 - \frac{\xi}{d} \right); \quad \sigma_{zz3} = \frac{E'}{l} \left(1 - \frac{\xi}{d} \right) \quad (20b)$$

$$\epsilon_{ss3} = 0; \quad \sigma_{ss3} = \frac{\nu E'}{l} \left(1 - \frac{\xi}{d} \right) \quad (20c)$$

where $E' = E/(1 - \nu^2)$; E , ν and G are Young's modulus, Poisson's ratio and shear modulus, respectively.

Argyris then makes use of the developed matrix theory using the following expression for the stiffness coefficients:

$$1 \times k_{jh} = t \int_0^d \int_0^l \sigma_j \epsilon_h d\zeta d\zeta \quad (21)$$

in which t is the thickness of the panel. Hence for example

$$k_{33} = \frac{tE'}{d^2l} \int_0^l \int_0^d (d - \zeta)^2 d\zeta d\zeta + \frac{tG}{l^2d^2} \int_0^l \int_0^d \zeta^2 d\zeta d\zeta = \frac{E'dt}{3l} + \frac{Gdt}{3l} \quad (22)$$

The total stiffness matrix for shear and direct strain is written as

$$[K] = [K_s] + [K_d] \quad (23)$$

where

$$[K_s] = Gt \begin{bmatrix} K_{s11} & K_{s12} \\ K_{s21} & K_{s22} \end{bmatrix} \quad (24)$$

with

$$[K_{s11}] = \frac{l}{6d} \begin{bmatrix} 2 & -1 & -1 & -1 \\ -1 & 2 & -1 & -1 \\ -1 & -1 & 2 & -1 \\ -1 & -1 & -1 & 2 \end{bmatrix}; \quad [K_{s12}] = [K_{s21}]^T = \frac{1}{4} \begin{bmatrix} 1 & 1 & -1 & -1 \\ -1 & -1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ -1 & -1 & 1 & 1 \end{bmatrix} \quad (25)$$

$$[K_{s22}] = \frac{d}{6l} \begin{bmatrix} 2 & 1 & -2 & -1 \\ 1 & 2 & -1 & -2 \\ -2 & -1 & 2 & 1 \\ -1 & -2 & 1 & 2 \end{bmatrix} \quad (26)$$

and

$$[K_d] = E't \begin{bmatrix} K_{d11} & K_{d12} \\ K_{d21} & K_{d22} \end{bmatrix} \quad (27)$$

$$[K_{d11}] = \frac{d}{6l} \begin{bmatrix} 2 & 1 & -2 & -1 \\ 1 & 2 & -1 & -2 \\ -2 & -1 & 2 & 1 \\ -1 & -2 & 1 & 2 \end{bmatrix}; \quad [K_{d12}] = [K_{d21}]^T = \frac{v}{4} \begin{bmatrix} 1 & -1 & 1 & -1 \\ 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \\ -1 & 1 & -1 & 1 \end{bmatrix} \quad (28)$$

$$[K_{d22}] = \frac{l}{6d} \begin{bmatrix} 2 & -2 & 1 & -1 \\ -2 & 2 & -1 & 1 \\ 1 & -1 & 2 & -2 \\ -1 & 1 & -2 & 2 \end{bmatrix} \quad (29)$$

Argyris writes: 'Naturally the grid does not have to be restricted to this definition and we can always choose a finer one if the stiffeners are widely spaced so that the assumption of linear variation between adjacent nodal points can represent adequately the displacement pattern.'

Thus Argyris has developed the rectangular panel stiffness matrix in the state of plane stress from the point of view of element interpolation functions in terms of nodal displacements.

Turner

This pioneering paper by Turner *et al.*³ starts with a discussion of the truss member and derives the stiffness matrix in global co-ordinates:

$$[K] = \frac{AE}{L} \begin{bmatrix} \lambda^2 & -\lambda^2 & \lambda\mu & -\lambda\mu \\ -\lambda^2 & \lambda^2 & -\lambda\mu & \lambda\mu \\ \lambda\mu & -\lambda\mu & \mu^2 & -\mu^2 \\ -\lambda\mu & \lambda\mu & -\mu^2 & \mu^2 \end{bmatrix} \quad (30)$$

where λ, μ are member direction cosines, L is the member length and A its area of cross-section.

After some discussion on rectangular plate elements it then turns to triangular elements and writes: 'The triangle is not only simpler to handle than the rectangle but later it will be used as the basic 'building block' for calculating stiffness matrices for plates of arbitrary shape'.

The triangular element is shown in Figures 5(a) and 5(b). It is interesting to note that, although the transformation in equation (30) is used for the bar element from its local axis to global axes, no attempt is made to connect 5(a) and 5(b) in this way. The paper starts by assuming constant strains

$$\varepsilon_x = a = \frac{1}{E} (\sigma_x - \nu\sigma_y) = \frac{\partial u}{\partial x} \quad (31a)$$

$$\varepsilon_y = b = \frac{1}{E} (\sigma_y - \nu\sigma_x) = \frac{\partial v}{\partial y} \quad (31b)$$

$$\gamma_{xy} = c = \frac{1}{G} (\tau_{xy}) = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \quad (31c)$$

Integrating we find the displacements to be

$$u = ax + Ay + B \quad (32a)$$

$$v = by + (c - A)x + C \quad (32b)$$

where (A, B, C) are constants of integration which define rigid body translation and rotation of the triangle. Hence, the triangle can displace as a rigid body in its own plane and undergo uniform

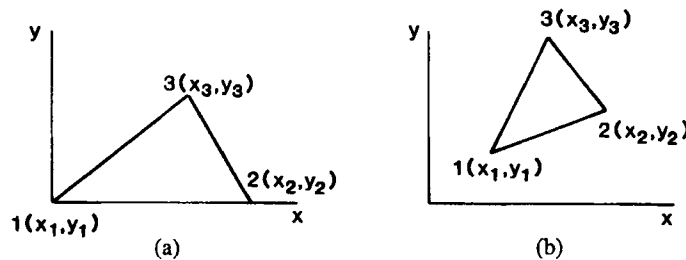


Figure 5. Triangular element, co-ordinate reference frames

straining according to equations (31). Displacements at the nodes can be determined by inserting applicable node co-ordinates into equations (32). In this way six equations occur which are just sufficient for uniquely determining the six constants in equations (32).

Thus let u_1, u_2, u_3 be the x -displacements of nodes 1, 2, 3 respectively; then from equation (32a),

$$\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{bmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{bmatrix} \begin{Bmatrix} a \\ A \\ B \end{Bmatrix} \quad (33)$$

Solving these equations, we get

$$\begin{Bmatrix} a \\ A \\ B \end{Bmatrix} = \frac{1}{2A} \begin{bmatrix} y_2 - y_3 & y_3 - y_1 & y_1 - y_2 \\ x_3 - x_2 & x_1 - x_3 & x_2 - x_1 \\ x_2 y_3 - x_3 y_2 & x_3 y_1 - x_1 y_3 & x_1 y_2 - x_2 y_1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} \quad (34)$$

and for v

$$\begin{Bmatrix} v_1 \\ v_2 \\ v_3 \end{Bmatrix} = \begin{bmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{bmatrix} \begin{Bmatrix} c \\ b \\ C \end{Bmatrix} - \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} A \quad (35)$$

so that

$$\begin{Bmatrix} c \\ b \\ C \end{Bmatrix} = \frac{1}{2A} \begin{bmatrix} y_2 - y_3 & y_3 - y_1 & y_1 - y_2 \\ x_3 - x_2 & x_1 - x_3 & x_2 - x_1 \\ x_2 y_3 - x_3 y_2 & x_3 y_1 - x_1 y_3 & x_1 y_2 - x_2 y_1 \end{bmatrix} \left[\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} + \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} A \right] \quad (36)$$

Hence the three straining constants are calculated to be

$$\begin{Bmatrix} a \\ b \\ c \end{Bmatrix} = \frac{1}{2A} \begin{bmatrix} y_2 - y_3 & 0 & y_3 - y_1 & 0 & y_1 - y_2 & 0 \\ 0 & x_3 - x_2 & 0 & x_1 - x_3 & 0 & x_2 - x_1 \\ x_3 - x_2 & y_2 - y_3 & x_1 - x_3 & y_3 - y_1 & x_2 - x_1 & y_1 - y_2 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix} \quad (37)$$

Now, using the equations (31), the stresses are expressed as

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \frac{E}{1 - \nu^2} \frac{1}{2A} \begin{bmatrix} b_1 & \nu a_1 & b_2 & \nu a_2 & b_3 & \nu a_3 \\ \nu b_1 & a_1 & \nu b_2 & a_2 & \nu b_3 & a_3 \\ \frac{1 - \nu}{2} a_1 & \frac{1 - \nu}{2} a_2 & \frac{1 - \nu}{2} a_3 & \frac{1 - \nu}{2} b_1 & \frac{1 - \nu}{2} b_2 & \frac{1 - \nu}{2} b_3 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix} \quad (38)$$

where $b_1 = y_2 - y_3$, $a_1 = x_3 - x_2$, etc., in cyclic permutation. Thus for example comparing our equation (38) with equation (18a) of Turner *et al.*

$$\frac{b_1}{2A} = \frac{-y_3}{y_3 x_2} = -\frac{1}{x_2} \quad (39)$$

for the triangle shown in Figure 5(a).

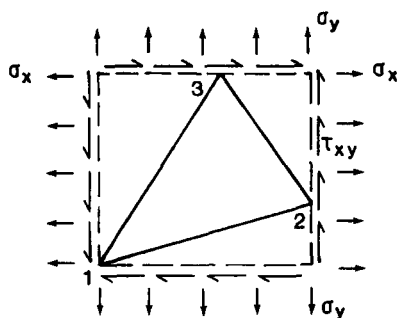


Figure 6. Stress components on sides of triangle

Writing equation (38) in matrix symbolic form

$$\{\sigma\} = [S]\{\delta\} \quad (40)$$

To obtain the nodal force-displacement relationship reference is made to the statics argument used by Turner *et al.*

From Figure 6 the statically equivalent nodal forces can be calculated by first principles:

$$F_{x1} = \frac{y_1 - y_3}{2} \sigma_x t + \frac{y_2 - y_1}{2} \sigma_x t - \frac{x_2 - x_1}{2} \tau_{xy} t + \frac{x_3 - x_1}{2} \tau_{xy} t$$

That is,

$$F_{x1} = \frac{y_2 - y_3}{2} \sigma_x t + \frac{x_3 - x_2}{2} \tau_{xy} t = \frac{b_1}{2} \sigma_x t + \frac{a_1}{2} \tau_{xy} t \quad (41)$$

Similarly

$$F_{y1} = -\frac{x_2 - x_1}{2} \sigma_y t + \frac{x_3 - x_1}{2} \sigma_y t - \frac{y_3 - y_1}{2} \tau_{xy} t + \frac{y_2 - y_1}{2} \tau_{xy} t \quad (42)$$

$$F_{y1} = \frac{x_3 - x_2}{2} \sigma_y t + \frac{y_2 - y_3}{2} \tau_{xy} t = \frac{a_1}{2} \sigma_y t + \frac{b_1}{2} \tau_{xy} t \quad (43)$$

If we use equation (17) of Argyris, noting that the integral of a constant over the area is simply the area of the triangle, the expressions in (41), (43) come directly from the transpose of the first column in equation (37). Thus we see that either from the statics argument presented in Figure 6, or from the integral of the transposed relationship in equation (37), the nodal forces are obtained as

$$\begin{Bmatrix} F_{x1} \\ F_{y1} \\ F_{x2} \\ F_{y2} \\ F_{x3} \\ F_{y3} \end{Bmatrix} = \frac{t}{2} \begin{bmatrix} y_2 - y_3 & 0 & x_3 - x_2 \\ 0 & x_3 - x_2 & y_2 - y_3 \\ y_3 - y_1 & 0 & x_1 - x_3 \\ 0 & x_1 - x_3 & y_3 - y_1 \\ y_1 - y_2 & 0 & x_2 - x_1 \\ 0 & x_2 - x_1 & y_1 - y_2 \end{bmatrix} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} \quad (44)$$

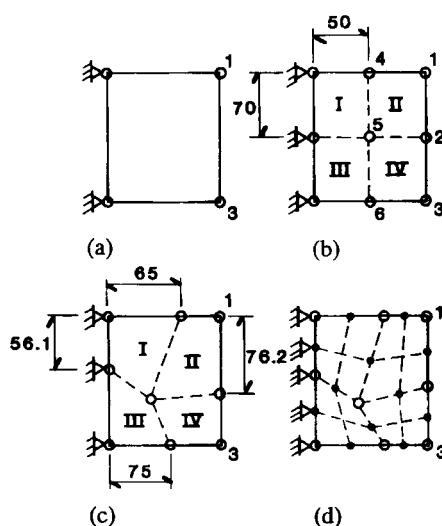


Figure 7. Nodes and supports for clamped rectangular plate: (a) solution (3); (b) solution (4); (c) solution (5); (d) solution (6)

Table I

Solution No.	Method	Figure	u_1	u_2	u_3	u_4	u_5	v_1	v_3	v_4
			(multiply all values by 10^{-4})							
1	Relaxation	13	2.703	2.607	2.703	1.391	1.248	0.686	-0.685	0.562
2	Simple theory	13	2.721	2.721	2.721	1.460	1.360	0.635	-0.635	
3	Plate K -matrix	13(a)	2.595		2.595			0.740	-0.740	
4	Plate K -matrix	13(b)	2.692	2.578	2.692	1.355	1.199	0.680	-0.680	0.568
5	Plate K -matrix	13(c)	2.718		2.697			0.686	-0.717	
6	Plate K -matrix	13(d)	2.714		2.712			0.688	-0.691	

Noting that in Figure 5(a), $x_1 = y_1 = y_2 = 0$, equation 20(a) of the Turner paper is obtained from equation (44). Turner writes equation (44) as

$$\{F\} = [T]\{\sigma\} = [T][S]\{\delta\} \quad (45)$$

The element stiffness matrix is then given as

$$[K] = [T][S] \quad (46)$$

which is equivalent to equation (13), developed independently by Argyris. The Turner paper uses the above element in the study of the deflections of a plate showing that for an irregular mesh, composed of quadrilaterals, each made of four triangles, the errors tend to disappear as the mesh is refined. Thus the Turner paper addresses the question of convergence. The Turner meshes are shown in Figure 7 and the results in Table I.

Clough

In his most interesting paper⁴ Professor Clough outlines the research programme undertaken at the Boeing Company in 1952–1953 for the calculation of flexibility coefficients for low aspect

ratio wing structures for dynamic analysis. Clough gives full credit to Jon Turner for the invention of the triangular plane stress element in a state of constant stress which has been outlined in this paper. Clough's own contribution at Berkeley is significant because of his extension of Turner's work from 1957 onwards to the calculation of stresses and the verification that for known geometries and loading these stresses converged to the corresponding analytic solution. Clough outlines how he first invented the name finite element method in References 5 and 6 because he wished to show the distinction between the continuum analysis and the matrix methods of structural analysis. Thus the choice of name is reserved for Professor Clough.

Zienkiewicz and Cheung

The development of the non-structural applications by means of minimization of the total potential energy of a system is developed systematically, for the first time in the paper by Zienkiewicz and Cheung,⁷ in which heat transfer and St Venant's torsion of prismatic shafts are analysed. In this paper, the approximation to the functional in terms of the nodal values of the triangular domain, into which the region is subdivided, is set up. Following the paper by Clough,⁴ the approximation is now referred to as the finite element method. Using the notation developed by Zienkiewicz, the approximation to the function ϕ (see equation (2)) is given for one triangle by the expression

$$\phi = [N]^T \{\bar{\phi}\} \quad (47)$$

in which $\{\bar{\phi}\}^T = [\phi_1, \phi_2, \phi_3]$, are the nodal values and

$$[N]^T = \frac{1}{2A} \begin{bmatrix} 1 & x & y \end{bmatrix} \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} \quad (48)$$

In this equation, (a_1, a_2, a_3) represent the rigid body motion of the Turner paper (A, B, C) and disappear when first partial derivatives with respect to x and y are taken. From equation (47), derivatives with respect to x are written as

$$\phi_{,x} = \frac{\partial N^T}{\partial x} \bar{\phi} \quad (49)$$

and hence

$$\phi_{,x}^2 = \{\bar{\phi}\}^T \left\{ \frac{\partial N}{\partial x} \right\} \left\{ \frac{\partial N}{\partial x} \right\}^T \{\bar{\phi}\} \quad (50)$$

Similarly,

$$\{\phi_{,y}\}^2 = \{\bar{\phi}\}^T \left\{ \frac{\partial N}{\partial y} \right\} \left\{ \frac{\partial N}{\partial y} \right\}^T \{\bar{\phi}\} \quad (51)$$

Taking derivatives with respect to the $\bar{\phi}$ values which are now to be chosen to make the functional a minimum,

$$\frac{\partial \phi_{,x}^2}{\partial \bar{\phi}} = 2 \left\{ \frac{\partial N}{\partial x} \right\} \left\{ \frac{\partial N}{\partial x} \right\}^T \bar{\phi} \quad (52)$$

$$\frac{\partial \phi_{,y}^2}{\partial \bar{\phi}} = 2 \left\{ \frac{\partial N}{\partial y} \right\} \left\{ \frac{\partial N}{\partial y} \right\}^T \bar{\phi} \quad (53)$$

From equation (51),

$$\frac{\partial N^T}{\partial x} = \frac{1}{2A} [b_1 \ b_2 \ b_3] \quad (54)$$

$$\frac{\partial N^T}{\partial y} = \frac{1}{2A} [c_1 \ c_2 \ c_3] \quad (55)$$

It follows that for the whole domain, integration of each triangle and minimization leads to

$$\sum \frac{1}{4A} \left\{ \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} [b_1 \ b_2 \ b_3] + \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} [c_1 \ c_2 \ c_3] \right\} \{\phi_e\} \quad (56)$$

The summation takes place over all triangles. Because each triangle contributes to only three nodes the summation in equation (56) evidently leads to sparse equations, which, as noted by Zienkiewicz, can be highly banded. Thus the notion of the topology of the system has also now been introduced. The concept of continuity of the function along common element boundaries is also discussed.

CONCLUSION

A careful examination of the paper by Courant¹ shows that he has apparently used a finite element type of procedure in a potential energy minimization of a functional for the torsion stress function using grid point values as the unknown parameters. We use the word 'apparently' with caution because no details of the calculations were given. There is no indication that the calculation of the mesh integrals could be made in a repetitive fashion using the mesh topology. From an examination of the meshes used in Figure 3, there appears to have been little thought given to the choice of nodal points so as to achieve a refinement of the approximation to the ϕ surface. Because of this lack of any guidance as to how the procedures were carried out we find it difficult to attribute the origin of the FEM to Courant.

The Argyris paper² is in a different category. Here the numerical techniques necessary for the application of the principles of virtual displacements and forces are set out in consistent matrix form. These matrix methods can become the basis for nearly all stress analysis applications of the FEM. Argyris successfully demonstrates how the method may be applied to a rectangular planar elasticity element for which inter-element displacement compatibility is assured by the choice of functions linear along element edges. The publication, 'Energy Theorems and Structural Analysis' is a significant work in the development of structural mechanics. However Argyris does not undertake convergence studies on his rectangular element, although the notion is anticipated in his paper when he writes:

'Naturally the grid does not have to be restricted to this definition and we can always choose a finer one if the stiffeners are widely spaced so that the assumption of linear variation between adjacent nodal points can represent adequately the displacement pattern.'

Argyris also gave lecture courses on his methods in the early 1950s.

The triangular element stiffness matrix was developed independently by Turner *et al.*³ An important feature of this paper is the study of the displacement convergence characteristics of planar elements, comparing the new approach with simple theory and the relaxation method, for the deflections of a cantilever beam. This pioneering paper was published in 1956. Clough in Reference 4 acknowledges the Turner contribution when he writes:

'Also it should be recognized that the principal credit for conceiving the procedure should go to M. J. Turner, who not only led the developmental effort for the two critical years of 1952–1953, but who also provided the inspiration to use assumed strains to define the stiffness of triangular plane stress elements.'

Clough's contribution was to continue convergence studies on stress components and to popularize the ideas by giving the name finite element method. Clough also gave lectures on the method in the spring of 1958.

The function minimization techniques, referred to so obliquely by Courant, were finally clarified by Zienkiewicz in 1965 and opened the way to the analysis of field problems by the FEM. The first text book popularizing FEM¹⁰ was published by Zienkiewicz in 1957 and contained some 270 pages. The current edition of this book is in two volumes and is 1400 pages long.¹¹

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