Crank Nicolson Scheme for the Heat Equation

The goal of this section is to derive a 2-level scheme for the heat equation which has no stability requirement and is second order in both space and time. From our previous work we expect the scheme to be implicit. This scheme is called the Crank-Nicolson method and is one of the most popular methods in practice.

1 CN Scheme

We write the equation at the point $(x_i, t^{n+\frac{1}{2}})$. Then

$$u_t(x_i, t^{n+\frac{1}{2}}) \approx \frac{u(x_i, t^{n+1}) - u(x_i, t^n)}{\Delta t}$$

is a centered difference approximation for u_t at $(x_i, t^{n+\frac{1}{2}})$ and therefore should be $\mathcal{O}(\Delta t^2)$.

To approximate the term $u_{xx}(x_i, t^{n+\frac{1}{2}})$ we use the average of the second centered differences for $u_{xx}(x_i, t^{n+1})$ and $u_{xx}(x_i, t^n)$; i.e.,

$$u_{xx}(x_i, t^{n+\frac{1}{2}}) \approx \frac{1}{2}\Big[\frac{u(x_{i+1}, t^{n+1}) - 2u(x_i, t^{n+1}) + u(x_{i-1}, t^{n+1})}{(\Delta x)^2} + \frac{u(x_{i+1}, t^n) - 2u(x_i, t^n) + u(x_{i-1}, t^n)}{(\Delta x)^2}\Big]$$

We now define the CN scheme for the IBVP

$$\begin{aligned} u_t &= \nu u_{xx} \quad (x,t) \in (0,1) \times (0,T] \\ u(x,0) &= u_0 \\ u(0,t) &= u(1,t) = 0 \end{aligned}$$

as

Set

$$U_i^0 = u_0(x_i) \quad i = 0, 1, \ldots, M$$

For $n = 0, 1, \dots,$

$$\begin{aligned} \frac{U_i^{n+1} - U_i^n}{\Delta t} &= \frac{\nu}{2}\Big[\frac{U_{i+1}^{n+1} - 2U_i^{n+1} + U_{i-1}^{n+1}}{(\Delta x)^2} + \frac{U_{i+1}^n - 2U_i^n + U_{i-1}^n}{(\Delta x)^2}\Big] \quad \text{for } i = 1, \dots, M-1 \\ U_0^{n+1} &= U_M^{n+1} = 0 \end{aligned}$$

Remark: If the original PDE has a source term $f(x,t)$ then we usually handle this by using

$$\frac{1}{2}(f(x_i, t^n) + f(x_i, t^{n+1}))$$

Our scheme can be rewritten as

Set

$$U_i^0 = u_0(x_i) \quad i = 0, 1, \dots, M$$

For $n = 0, 1, \dots,$

$$\begin{aligned} -\lambda U_{i+1}^{n+1} + (2 + 2\lambda)U_i^{n+1} - \lambda U_{i-1}^{n+1} &= \lambda U_{i+1}^n + (2 - 2\lambda)U_i^n + \lambda U_{i-1}^n \quad \text{for } i = 1, \dots, M-1 &\quad (1) \\ U_0^{n+1} &= U_M^{n+1} = 0 &\quad (2) \end{aligned}$$

For simplicity we will often write the difference equation as

$$L^h U_i^{n+1} = R^h U_i^n$$

where

$$L^h U_i^{n+1} = -\lambda U_{i+1}^{n+1} + (2 + 2\lambda)U_i^{n+1} - \lambda U_{i-1}^{n+1}$$

and

$$R^h U_i^n = \lambda U_{i+1}^n + (2 - 2\lambda)U_i^n + \lambda U_{i-1}^n$$

Remark: Note that we can no longer solve for U_1^{n+1}, then U_2^{n+1} even if we know the solution at the previous time step. Instead, we must solve for all values at a specific timestep at once, i.e., we must solve a system of linear equations. Such a scheme is called an implicit scheme.

From (1)–(2) we set $U_i^0 = u_0(x_i)$ for $i = 0, 1, \ldots, M$ and for each value of $n = 0, 1, \ldots$ solve the system

$$
\begin{pmatrix}
2+2\lambda & -\lambda & 0 & \cdots & 0 \\
-\lambda & 2+2\lambda & -\lambda & 0 & 0 \\
0 & \ddots & \ddots & \ddots & \\
0 & \ddots & -\lambda & 2+2\lambda & -\lambda \\
0 & \cdots & 0 & -\lambda & 2+2\lambda
\end{pmatrix}
\begin{pmatrix}
U_1^{n+1} \\ U_2^{n+1} \\ \vdots \\ \vdots \\ U_{M-1}^{n+1}
\end{pmatrix}
=
\begin{pmatrix}
\lambda U_0^n + (2-2\lambda)U_1^n + \lambda U_2^n + \lambda U_0^{n+1} \\
\lambda U_1^n + (2-2\lambda)U_2^n + \lambda U_3^n \\
\vdots \\
\vdots \\
\lambda U_{M-2}^n + (2-2\lambda)U_{M-1}^n + \lambda U_M^n + \lambda U_M^{n+1}
\end{pmatrix}
\tag{3}
$$

Of course, on the right hand side of the first and last equations, the terms involving

$$U_0^n, \quad U_0^{n+1}, \quad U_M^n, \quad U_M^{n+1}$$

are zero due to the homogeneous boundary conditions for our IBVP.

The system (3) can be written symbolically as

$$A\vec{U}^{n+1} = \vec{b}^n$$

Remark: The matrix A is tridiagonal, and symmetric positive definite and thus can be solve by the same method as the standard implicit scheme which we discussed in the previous section.

Remark: The matrix A does not change at each timestep (as long as the timestep remains constant).

Exercise How do we know that this matrix is symmetric positive definite?

Exercise Why are we guaranteed that this system has a unique solution?

We now want to investigate the stability, consistency (and thus accuracy), and convergence of the CN scheme. Recall that we wanted a scheme which was second order accurate in space and time and which was unconditionally stable.

2 Stability of Crank-Nicolson Scheme

We show stability in the norm $\|\cdot\|_{2,\Delta x}$ where

$$\|x\|_{2,\Delta x} = \Big[\sum_{i=1}^{M-1} x_i^2 \Delta x\Big]^{1/2}$$

Note here that the sum begins at $i = 1$ and ends at $i = M - 1$ because we are imposing homogeneous Dirichlet boundary data.

Lemma. Let $\vec{U}^n$ be the solution of (3). Let $\vec{u_0}$ be defined by

$$\vec{u_0} = \begin{pmatrix} u_0(x_1) \\ u_0(x_2) \\ \vdots \\ u_0(x_{M-1}) \end{pmatrix}$$

Then

$$\|\vec{U}^n\|_{2,\Delta x} \leq \|\vec{u_0}\|_{2,\Delta x} \tag{4}$$

Remark: This results says that the CN scheme is unconditionally stable i.e., there is no condition on λ required for stability.

proof

From the scheme we have

$$-\lambda U_{i+1}^{n+1} + (2+2\lambda)U_i^{n+1} - \lambda U_{i-1}^{n+1} = \lambda U_{i+1}^n + (2-2\lambda)U_i^n + \lambda U_{i-1}^n \quad \text{for } i = 1, \ldots, M-1$$

so that

$$U_i^{n+1} - U_i^n = \frac{\lambda}{2}\Big[\big(U_{i+1}^{n+1} + U_{i+1}^n\big) + \big(U_{i-1}^{n+1} + U_{i-1}^n\big) - \frac{2\lambda}{2}\big(U_i^{n+1} + U_i^n\big)\Big]$$

for $1 \leq i \leq M-1$ and

$$U_0^{n+1} = U_0^n = 0 \qquad U_M^{n+1} = U_M^n = 0$$

For each i multiply each equation by $(U_i^{n+1} + U_i^n)\Delta x$ to obtain

$$\Delta x\Big[(U_i^{n+1})^2 - U_i^n)^2\Big] = \frac{\lambda}{2}\Big[\big(U_{i+1}^{n+1} + U_{i+1}^n\big) + \big(U_{i-1}^{n+1} + U_{i-1}^n\big) - \frac{2\lambda}{2}\,\big(U_i^{n+1} + U_i^n\big)\Big](U_i^{n+1} + U_i^n)\Delta x$$

Now sum over $i = 1, \ldots, M-1$

$$\sum_{i=1}^{M-1}(U_i^{n+1})^2\Delta x - \sum_{i=1}^{M-1}(U_i^n)^2\Delta x = \frac{\lambda\Delta x}{2}\sum_{i=1}^{M-1}\Big[\big(U_{i+1}^{n+1} + U_{i+1}^n\big) + \big(U_{i-1}^{n+1} + U_{i-1}^n\big) - \frac{2\lambda}{2}\,\big(U_i^{n+1} + U_i^n\big)\Big](U_i^{n+1} + U_i^n)$$

Our goal here is to show that the rhs of this expression is ≤ 0. If we do this, then

$$\sum_{i=1}^{M-1}(U_i^{n+1})^2\Delta x \le \sum_{i=1}^{M-1}(U_i^n)^2\Delta x$$

and thus

$$\|\vec{U}^{n+1}\|_{2,\Delta x}^2 \le \|\vec{U}^n\|_{2,\Delta x}^2 \implies \|\vec{U}^{n+1}\|_{2,\Delta x} \le \|\vec{U}^n\|_{2,\Delta x}$$

If we show this then we can apply it repeatedly to get the desired result.

To show this we simplify our equation by writing

$$B_i = U_i^{n+1} + U_i^n$$

where we know $B_0 = B_M = 0$ from the homogenous boundary data.

We have

$$
\begin{aligned}
\|\vec{U}^{n+1}\|_{2,\Delta x}^2 - \|\vec{U}^{n}\|_{2,\Delta x}^2 &= \frac{\lambda}{2}\Delta x \sum_{i=1}^{M-1} \left[B_{i+1} + B_{i-1} - 2B_i\right] B_i \\
&= \frac{\lambda}{2}\Delta x \sum_{i=1}^{M-1} \left[B_{i+1}B_i + B_{i-1}B_i - 2B_i^2\right] \\
&= \frac{\lambda}{2}\Delta x \Big[(B_1B_2 + B_2B_3 + \cdots + B_{M-1}B_M) \\
&\qquad + (B_0B_1 + B_1B_2 + \cdots + B_{M-2}B_{M-1}) - 2\sum_{i=1}^{M-1} B_i^2\Big] \\
&= \frac{\lambda}{2}\Delta x \Big[2\sum_{i=1}^{M-2} B_iB_{i+1} - 2\sum_{i=1}^{M-1} B_i^2\Big] \\
&= \frac{\lambda}{2}\Delta x \left[2\sum_{i=1}^{M-1} B_iB_{i+1} - \sum_{i=1}^{M-1} B_i^2 - \sum_{i=1}^{M-1} B_{i+1}^2 - B_1^2\right] \\
&= \frac{\lambda}{2}\Delta x \left[-\sum_{i=1}^{M-1} (B_i - B_{i+1})^2 - B_1^2\right] \\
&= -\frac{\lambda}{2}\Delta x \left[\sum_{i=1}^{M-1} (B_i - B_{i+1})^2 + B_1^2\right] \\
&\leq 0
\end{aligned}
$$

3 Consistency of Crank-Nicolson Scheme

In this section we show that the CN scheme is consistent and its order of accuracy is $(2, 2)$.

Let u satisfy the homogeneous Dirichlet IBVP for the 1-D heat equation in $Q = (0, 1) \times (0, T]$ where $u \in C^{6,3}(\bar{Q})$ and let U_i^n be the solution to the CN equation $L^h U_i^{n+1} = R^h U_i^n$. Then there exists constants, independent of h, k, u such that

$$\max_{\substack{1\le i\le M-1\\ 0\le n\le N-1}} \left|L^h u(x_i,t^{n+1}) - R^h u(x_i,t^n)\right| \le C\Delta t\big[(\Delta x)^2 + (\Delta t)^2\big] \max_{(x,t)\in\bar{Q}} \left(\frac{\partial^4 u}{\partial x^4} + \frac{\partial^6 u}{\partial x^6}\right) \tag{5}$$

Proof

As before we plug the exact solution into the difference equation and expand using Taylor series about the point (x_i, t^n). We have

$$\begin{aligned} L^h u(x_i,t^{n+1}) - R^h u(x_i,t^n) &= -\lambda u(x_{i+1},t^{n+1}) + (2+2\lambda)u(x_i,t^{n+1}) - \lambda u(x_{i-1},t^{n+1}) \\ &\quad -\big[\lambda u(x_i,t^n) + (2-2\lambda)u(x_i,t^n) + \lambda u(x_{i-1},t^n)\big] \end{aligned}$$

Expanding in terms of Taylor series we obtain

$$\begin{aligned} L^h u(x_i,t^{n+1}) &= -\lambda\Big[u(x_i,t^{n+1}) + \Delta x u_x(x_i,t^{n+1}) + \frac{(\Delta x)^2}{2!}u_{xxx}(x_i,t^{n+1}) + \frac{(\Delta x)^3}{3!}u_{xxx}(x_i,t^{n+1}) + \frac{(\Delta x)^4}{4!}u_{xxxx}(\Theta_1,t^{n+1})\Big] \\ &\quad +2u(x_i,t^{n+1}) + 2\lambda u(x_i,t^{n+1}) \\ &\quad -\lambda\Big[u(x_i,t^{n+1}) - \Delta x u_x(x_i,t^{n+1}) + \frac{(\Delta x)^2}{2!}u_{xxx}(x_i,t^{n+1}) - \frac{(\Delta x)^3}{3!}u_{xxx}(x_i,t^{n+1}) + \frac{(\Delta x)^4}{4!}u_{xxxx}(\Theta_2 t^{n+1})\Big] \\ &= 2u(x_i,t^{n+1}) - \lambda(\Delta x)^2 u_{xx}(x_i,t^{n+1}) - \lambda\frac{(\Delta x)^4}{4!}\big(u_{xxxx}(\Theta_1,t^{n+1}) + u_{xxxx}(\Theta_2,t^{n+1})\big) \end{aligned}$$

and similarly

$$\begin{aligned} R^h u(x_i,t^n) &= \lambda u(x_i,t^n) + (2-2\lambda)u(x_i,t^n) + \lambda u(x_{i-1},t^n) \\ &= 2u(x_i,t^n) + \lambda(\Delta x)^2 u_{xx}(x_i,t^n) + \lambda\frac{(\Delta x)^4}{4!}\left(u_{xxxx}(\Theta_3,t^n) + u_{xxxx}(\Theta_4,t^n)\right) \end{aligned}$$

Expanding $L^h u(x_i, t^{n+1})$ in time about t^n gives us

$$\begin{aligned}
L^h u(x_i, t^{n+1}) &= 2u(x_i, t^{n+1}) - \frac{\lambda}{\nu}(\Delta x)^2 u_t(x_i, t^{n+1}) - \lambda\frac{(\Delta x)^4}{4!}\left(u_{xxxx}(\Theta_1, t^{n+1}) + u_{xxxx}(\Theta_2, t^{n+1})\right) \\
&= 2\Big[u(x_i, t^n) + \Delta t u_t(x_i, t^n) + \frac{(\Delta t)^2}{2!}u_{tt}(x_i, t^n) + \frac{(\Delta t)^3}{3!}u_{ttt}(x_i, \tau_1)\Big] \\
&\quad -\frac{\lambda}{\nu}(\Delta x)^2\Big[u_t(x_i, t^n) + \Delta t u_{tt}(x_i, t^n) + \frac{(\Delta t)^2}{2!}u_{ttt}(x_i, \tau_2)\Big] \\
&\quad -\lambda\frac{(\Delta x)^4}{4!}\left(u_{xxxx}(\Theta_1, \tau_3) + u_{xxxx}(\Theta_2, \tau_4)\right) \\
&= 2u(x_i, t^n) + 2\Delta t u_t + 2\frac{(\Delta t)^2}{2!}u_{tt} + 2\frac{(\Delta t)^3}{3!}u_{ttt}(x_i, \tau_1) \\
&\quad -\Delta t u_t - (\Delta t)^2 u_{tt} - \frac{(\Delta t)^3}{2}u_{ttt}(x_i, \tau_2) \\
&\quad -\lambda\frac{(\Delta x)^4}{4!}\left(u_{xxxx}(\Theta_1, t^n) + u_{xxxx}(\Theta_2, t^n)\right) \\
&= 2u(x_i, t^n) + \Delta t u_t + 2\frac{(\Delta t)^3}{3!}u_{ttt}(x_i, \tau_1) - \frac{(\Delta t)^3}{2}u_{ttt}(x_i, \tau_2) \\
&\quad -\lambda\frac{(\Delta x)^4}{4!}\left(u_{xxxx}(\Theta_1, \tau_3) + u_{xxxx}(\Theta_2, \tau_4)\right)
\end{aligned}$$

∞

where we have used the fact that $u_t = \nu u_{xx}$ and $\lambda = \nu\Delta t/(\Delta x)^2$

Combining we arrive at

$$
\begin{aligned}
L^h u(x_i, t^{n+1}) - R^h u(x_i, t^n) &= 2u(x_i, t^n) + \Delta t u_t + 2\frac{(\Delta t)^3}{3!} u_{ttt}(x_i, \tau_1) - \frac{(\Delta t)^3}{2} u_{ttt}(x_i, \tau_2) \\
&\quad -\lambda \frac{(\Delta x)^4}{4!} \left(u_{xxxx}(\Theta_1, \tau_3) + u_{xxxx}(\Theta_2, \tau_4)\right) \\
&\quad -2u(x_i, t^n) - \lambda (\Delta x)^2 u_{xx} - \lambda \frac{(\Delta x)^4}{4!} \left(u_{xxxx}(\Theta_3, t^n) + u_{xxxx}(\Theta_4, t^n)\right) \\
&= \Delta t u_t - \frac{(\Delta t)^2}{2!} \nu^2 u_{xxxx}(x_i, t^n) + 2\frac{(\Delta t)^3}{3!} \left[\frac{\partial^6 u}{\partial x^6}(x_i, \tau_1) + \frac{\partial^6 u}{\partial x^6}(x_i, \tau_2)\right] \\
&\quad -\lambda \frac{(\Delta x)^4}{4!} \left(u_{xxxx}(\Theta_1, t^n) + u_{xxxx}(\Theta_2, t^n)\right) \\
&\quad -\frac{\nu \Delta t}{(\Delta x)^2} (\Delta x)^2 \frac{1}{\nu} u_t - \lambda \frac{(\Delta x)^4}{4!} \left(u_{xxxx}(\Theta_3, t^n) + u_{xxxx}(\Theta_4, t^n)\right)
\end{aligned}
$$

where again we have used the equations $u_t = \nu u_{xx}$, $u_{tt} = \nu u_{xxt} = \nu (u_t)_{xx} = \nu^2 u_{xxxx}$ and $u_{ttt} = \nu^2 \partial^6 u / \partial x^6$. Thus we have that

$$
L^h u(x_i, t^{n+1}) - R^h u(x_i, t^n) \leq \Delta t \Big((\Delta t)^2 (\Delta x)^2 \Big) C \max \left\{ \left| \frac{\partial^4 u}{\partial x^4} \right|, \left| \frac{\partial^6 u}{\partial x^6} \right| \right\}
$$

Remark: We have illustrated that the CN scheme is a 2 time level scheme which is accurate of order $(2,2)$ so we say it is second order in space and time.

4 Convergence of CN