Computational Geometry Lab:
FEM BASIS FUNCTIONS FOR A TRIANGLE

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1 Introduction

This lab continues the topic of *Computational Geometry*. Having studied triangles and how triangles are used to create triangulations of a region, we will now turn to the use of triangles to define the basis functions used in the finite element method (FEM).

The finite element method is a procedure for approximating and solving partial differential equations. Part of the finite element method involves constructing the triangulation, a topic which is discussed in other labs. Once the triangulation is available, the finite element method uses this mesh to represent functions $f(x, y)$. The representation is *discrete*, that is, it depends on just a finite number of values, but the resulting function is defined over the entire triangulated region; with some restrictions, it can be evaluated, plotted, differentiated or integrated.

If you have ever used a finite difference method to solve differential equations, you will understand an important distinction between these two methods. The finite difference method works with values of a function at given points, but it does not try to "fill in the gaps" between the tabulated points. In contrast, the finite element method may only have exact knowledge of a function at specified points, but it builds a "model" of the function over the entire problem domain.

The key to this model building is the set of **finite element basis functions**. It is the purpose of this lab to understand how these basis functions are defined, evaluated and used to create the finite element functions.

2 Overview

This lab is one of a series on the finite element method. It may help to anticipate where we are going, so that the small results we achieve in this lab are understood to be leading to a much bigger result.

So we suppose that we are given a triangulation **TRI** of some region $\mathcal{R}$. The triangulation is made up, of course, of points and triangles. We will assume there are **NP** points or "nodes", with a typical point being identified as **P** or perhaps $\mathbf{P_i}$ or **p**. If we wish to list the coordinates of **p**, we might write $\mathbf{p} = \{$ p.x, p.y$\}$ or in some mathematical formulas, we may use the notation $\mathbf{p} = \{p_x, p_y\}$.

The triangulation is a set of **NT** triangles, whose vertices are chosen from the set of points, with a typical triangle being **T** or $\mathbf{T_i}$. If we wish to list the points that are the vertices of the triangle, we may write **T**= {a, b, c}.

Let us suppose that we wish to come up with a formula for a function $f(x, y)$, with the requirement that

$$f(P_i) = f_i, \quad i = 1 \ldots NT.$$

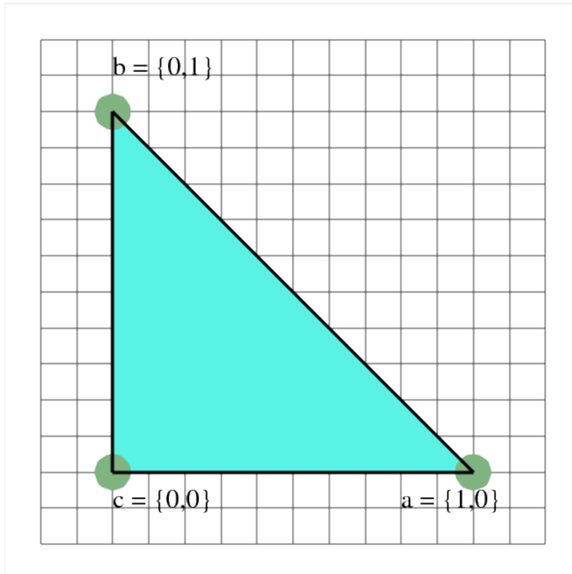

Figure 1: The Reference Triangle

that is, we are going to specify in advance the value of this function at every node in the triangulation.

Our goal is to somehow come up with a formula, or a procedure, which defines $f(x, y)$ for every point (x, y) in $\mathcal{R}$, in such a way that the function is continuous, attains the specified values at the nodes, and is relatively simple to evaluate anywhere in the region. This is an example of what is called **the interpolation problem**.

Our progress in solving the interpolation problem on a triangulation will start very simply. We will look at a "triangulation" that involves a single triangle, called the *reference triangle*. We will investigate how interpolation works in this very simple setting, and we will also "discover" the basis functions that make the answer simple to describe.

We will then transfer this formula to a general triangle. Then we will consider what happens depending on which vertex is chosen to have the value 1 under the formula. When we have understood this problem, we will be able to handle the interpolation problem on the reference triangle, or on any single general triangle.

In later labs, we will consider the effect of setting up these formulas in *every* triangle in the triangulation simultaneously. This might seem to be a recipe for chaos. However, whenever two triangles touch, they share two vertices, and the formula we develop for each triangle will match up continuously along their common boundary (but not differentiably!).

At this point, we will have developed all the machinery that the finite element requires in order to define a function $f(x, y)$ over the finite element mesh with the desired values.

3 The Reference Triangle

Our task is complicated, but we have to start somewhere. Instead of an entire triangulation, we start with a single triangle. Instead of an arbitrary triangle, we start with the "reference triangle", whose definition is simply **Tref**={ a, b, c } = { {1,0}, {0,1}, {0,0} }.

Now suppose we want to define a function $f(x, y)$ over the entire triangle, with the property that its value at each vertex is prescribed in advance:

$$f(a) = fa$$
$$f(b) = fb$$
$$f(c) = fc$$

There are many ways to find such a function; if we make the natural choice that the function be the simplest polynomial possible, we will probably choose a form like:

$$f(x, y) = c_1 + c_2 x + c_3 y$$

where the coefficients c_1, c_2 and c_3 may be chosen to fit our problem.

But the condition $f(c) = fc$ implies that $f(0, 0) = c_1 = fc$. The condition $f(a) = fa$ then implies that $f(1, 0) = fc + c_2 = fa$ which shows that $c_2 = fa - fc$, and we can similarly show that we must have $c_3 = fb - fc$.

Thus, we have solved our interpolation problem for the reference triangle. The function

$$f(x, y) = fc + (fa - fc)x + (fb - fc)y$$

has the correct values at the vertices, is defined and continuous over the entire triangle, and is simple to evaluate.

This certainly doesn't solve our real problem, but it is a helpful guide as to how we want to proceed!

4 Program #1: Interpolation in the Reference Triangle

For the reference triangle **Tref={ a, b, c }** = { {1,0}, {0,1}, {0,0} }, write a program which:

- Reads three vertex function values **fa**, **fb**, and **fc**;
- Reads the coordinates of a point **p**;
- Evaluates and prints the interpolation function **f(p)**.

Test your program with the following data:

- **fa**=17, **fb**=17, and **fc**=17, **p**={0.2, 0.5};
- **fa**=1, **fb**=-2, and **fc**=3, **p**={1.0, 0.0}, {0.0, 1.0}, {0.0, 0.0};
- **fa**=10, **fb**=5, and **fc**=-30, **p**={1/3, 1/3};

Prove or disprove : If a point **p** is contained in the reference triangle, then the value of the linear interpolation function **fp** is bounded below by the minimum, and above by the maximum, of the three data values **fa, fb, fc**. Moreover, if **p** is strictly contained within the reference triangle, and if the three data values are not all equal, then the value **fp** is *strictly* between the given bounds.

5 Basis Functions for the Reference Triangle

When we guessed that our interpolation function would be a linear function of the data values, we wrote out a symbolic formula, which we could regard as describing $f(x, y)$ as a combination of the **basis functions 1**, **x** and **y**. Any linear (actually, affine) function in the plane can be represented as such a combination.

However, there are many equivalent sets of basis functions. Notice how the formula for our solution uses the prescribed value fc several times. What if we rearranged this formula so that each prescribed value showed up exactly once. We'd get something like this:

$$f(x,y) = fa \cdot x + fb \cdot y + fc \cdot (1 - x - y)$$

Now if we look at this formula, we can regard it as using a slightly different set of basis functions than before. Let's actually rename each basis function:

$$\begin{aligned} \phi_a(x,y) =& x \\ \phi_b(x,y) =& y \\ \phi_c(x,y) =& 1 - x - y \end{aligned}$$

This set of basis functions has some useful properties:

- basis function $\phi_a(x,y)$ is 1 at vertex a, and 0 at b and c, with similar statements for the other two basis functions (*the Lagrange basis property*);
- along the triangle's edge {b,c}, basis function $\phi_a(x,y)$ is exactly 0, with similar statements for the other two basis functions;
- at any point in the triangle, the value of each basis function is between 0 and 1; for points strictly inside the triangle, the basis function is strictly between 0 and 1;
- at any point (x,y), the sum of the three basis functions is exactly 1;
- at any point (x,y), the sum of the derivatives of the three basis functions is exactly 0 (which follows from the previous statement).

It should be clear now that if we have any triangle on which we want to solve the interpolation problem, we can write down the solution immediately if we can find a set of basis functions with the Lagrange basis property!

Using our basis function notation, the solution to the interpolation problem has the very nice form:

$$f(x,y) = fa \cdot \phi_a(x,y) + fb \cdot \phi_b(x,y) + fc \cdot \phi_c(x,y)$$

A form like this is useful because it's easy to remember, it suggests what the relationship is between all the terms, and it makes it easy to guess what the corresponding formula might be in more general circumstances (a different triangle, or a problem in 3D involving tetrahedrons!).

Notice, also, that the basis functions $(\phi_a(x,y), \phi_b(x,y), \phi_c(x,y))$ are really the same as considering the barycentric coordinates (ξ_a, ξ_b, ξ_c) as functions of x and y.

6 Basis Function Derivatives for the Reference Triangle

It may seem unnecesary, but for future reference we will note the derivatives with respect to x and y of the basis functions we have determined for the reference triangle.

$$\begin{aligned} &\phi_a(x,y) = x; && \frac{\partial \phi_a}{\partial x} = 1; && \frac{\partial \phi_a}{\partial y} = 0 \\ &\phi_b(x,y) = y; && \frac{\partial \phi_b}{\partial x} = 0; && \frac{\partial \phi_b}{\partial y} = 1 \\ &\phi_c(x,y) = 1 - x - y; && \frac{\partial \phi_c}{\partial x} = -1; && \frac{\partial \phi_c}{\partial y} = -1 \end{aligned}$$

7 Program #2: Basis Functions for the Reference Triangle

For the reference triangle, write a program which

- reads a point **p**;
- evaluates and prints the basis functions ϕ_a, ϕ_b and ϕ_c at **p**;
- prints the sum $\phi_a(p) + \phi_b(p) + \phi_c(p)$.

Test your program with the following data:

- **p**={1.0, 0.0}, {0.0, 1.0}, {0.0, 0.0};
- **p**={0.5, 0.0}, {0.0, 0.2}, {0.25, 0.75};
- **p**={-1/2, 1/3}, {1/3, 1/3}, {1/2, 1/3}, {1.0, 1/3}.

8 Basis Functions for a General Triangle

Now we are ready to consider the interpolation problem on a general triangle, of the form **T={a,b,c}**. It should be clear that our best hope will be to find a set of basis functions ϕ_a, ϕ_b and ϕ_c for this general triangle that work like the ones we found in the reference triangle.

It's actually possible to figure out the formulas for the basis functions based on the simple properties we know about them. Let us find a formula for $\phi_a(x, y)$. We are assuming that $\phi_a(x, y)$ is a linear function. Since ϕ_a is 0 at nodes **b** and **c**, it must also be zero at all points (x, y) on the line between these two nodes.

Now just imagine drawing the line between **b** and **c** and then picking any point $p = (p.x, p.y)$ on that line. The slope of the line is computed by choosing two points on the line, and taking the ratio of the difference between the y and the x coordinates. The result must be the same not matter which two points we choose. Let us compare the slope formulas using first **b** and **c** and then the new point **p** and **c**:

$$\frac{b.y - c.y}{b.x - c.x} = \frac{p.y - c.y}{p.x - c.x}$$

(It's customary to write the slope relationship this way. Should either denominator be zero, we could eliminate the fractions, and have a valid, if less familiar, formula.)

If we subtract one side from the other, and call the resulting function g, we have that:

$$g(p.x, p.y) = (p.x - c.x)(b.y - c.y) - (b.x - c.x)(p.y - c.y)$$

We know that $g(x, y) = 0$ for those points on the line between **b** and **c**. Assuming our triangle is not degenerate, then $g(a.x, a.y)$ must be *nonzero* (because **a** does *not* lie on the line between **b** and **c**!) So $g(x, y)$ is "almost" the basis function $\phi_a(x, y)$ that we're looking for, since it's zero at vertices **b** and **c** and nonzero at vertex **a**. Now if we simply divide this function by its value at vertex **a**, the new function is 1 at vertex **a**, so it satisfies all three conditions. Our basis function formula is:

$$\phi_a(p.x, p.y) = \frac{g(p.x, p.y)}{g(a.x, a.y)} = \frac{(p.x - c.x)(b.y - c.y) - (b.x - c.x)(p.y - c.y)}{(a.x - c.x)(b.y - c.y) - (b.x - c.x)(a.y - c.y)}$$

Now let's make sure this formula does what we want. It's very easy to see that $\phi_a(a.x, a.y) = 1$. Can you also see that $\phi_a(b.x, b.y) = \phi_x(c.x, c.y) = 0$?

Thus we have constructed the formula for the affine function $\phi_a(x, y)$. Similar reasoning will produce the corresponding formulas for $\phi_b(x, y)$ and $\phi_c(x, y)$.

9 Basis Function Derivatives for a General Triangle

For future reference, let us write the basis functions and their x and y derivatives. We have rewritten the numerators to emphasize the factors of x and y, and we have replaced the arguments $(p.x, p.y)$ by the more generic (x, y).

$$
\begin{aligned}
\phi_a(x,y) &= \frac{(b.y - c.y)\ x + (b.x - c.x)\ y + c.x\ b.y - b.x\ c.y)}{(a.x - c.x)(b.y - c.y) - (b.x - c.x)(a.y - c.y)} \\
\frac{\partial \phi_a}{\partial x} &= \frac{b.y - c.y}{(a.x - c.x)(b.y - c.y) - (b.x - c.x)(a.y - c.y)} \\
\frac{\partial \phi_a}{\partial y} &= \frac{c.x - b.x}{(a.x - c.x)(b.y - c.y) - (b.x - c.x)(a.y - c.y)}
\end{aligned}
$$

$$
\begin{aligned}
\phi_b(x,y) &= \frac{(c.y - a.y)\ x + (a.x - c.x)\ y + c.x\ a.y - a.x\ c.y}{(a.x - c.x)(b.y - c.y) - (b.x - c.x)(a.y - c.y)} \\
\frac{\partial \phi_b}{\partial x} &= \frac{c.y - a.y}{(a.x - c.x)(b.y - c.y) - (b.x - c.x)(a.y - c.y)} \\
\frac{\partial \phi_b}{\partial y} &= \frac{a.x - c.x}{(a.x - c.x)(b.y - c.y) - (b.x - c.x)(a.y - c.y)}
\end{aligned}
$$

$$
\begin{aligned}
\phi_c(x,y) &= \frac{(a.y - b.y)\ x + (b.x - a.x)\ y + a.x\ b.y - b.x\ a.y}{(a.x - c.x)(b.y - c.y) - (b.x - c.x)(a.y - c.y)} \\
\frac{\partial \phi_c}{\partial x} &= \frac{a.y - b.y}{(a.x - c.x)(b.y - c.y) - (b.x - c.x)(a.y - c.y)} \\
\frac{\partial \phi_c}{\partial y} &= \frac{b.x - a.x}{(a.x - c.x)(b.y - c.y) - (b.x - c.x)(a.y - c.y)}
\end{aligned}
$$

10 Program #3: Basis Functions for a General Triangle

Write a program which

- reads the definition of a general triangle **T** = **{a,b,c}**;
- reads the definition of a point **p**;
- evaluates and prints the basis functions ϕ_a, ϕ_b and ϕ_c at **p**;
- prints the sum $\phi_a(p) + \phi_b(p) + \phi_c(p)$;
- evaluates and prints the x and y derivatives of the basis functions at **p**;
- prints the sums of the x and y derivatives at **p**.

For **T**, use the triangle **Tex5** defined by {{4,1},{3,5},{0,2}}.
Test your program with the following data:

- **p**={4.0, 1.0}, {3.0, 5.0}, {0.0, 2.0};
- **p**={2.0, 1.5}, {0.6, 1.8}, {3.25, 4.0};
- **p**={-1.0, 3.83}, {7/3, 8/3}, {3.0, 2.5}, {5.0, 2.0}.

11 Basis Functions for a General Triangle by Determinants

It can be shown that a formula for the basis functions can be found as the ratio of two determinants. The determinant in the denominator is essentially the area of the triangle. The determinant in the numerator is formed by replacing the first two elements of column 1, 2 or 3 by the x and y values of the point **p** where basis function ϕ_a, ϕ_b or ϕ_c is to be evaluated. The result is an expression for the area of the triangle formed by the point **p** and two of the vertices. Thus, to evaluate $\phi_a(x, y)$, we write:

$$\phi_a(p) = \phi_a(p.x, p.y) = \frac{\begin{vmatrix} p_x & b_x & c_x \\ p_y & b_y & c_y \\ 1 & 1 & 1 \end{vmatrix}}{\begin{vmatrix} a_x & b_x & c_x \\ a_y & b_y & c_y \\ 1 & 1 & 1 \end{vmatrix}}$$

with corresponding formulas for $\phi_b(p)$ and $\phi_c(p)$.

Exercise: Evaluate the determinant for $\phi_a(p)$, that is, write it out explicitly as an arithmetic formula and compare it to the formula we derived in the previous section.

Exercise: Show that if the triangle **T={a,b,c}** is actually the reference triangle,then the determinant formulas for $\phi_a(p)$, $\phi_b(p)$ and $\phi_c(p)$ give us the basis functions **x**, **y**, and **1-x-y** respectively.

12 Basis Functions for a General Triangle by Mapping

A third way to determine the basis functions for a general triangle relies on the idea of a mapping $\psi_{\text{Tref,T}}(r, s)$ from the reference triangle **Tref** to the general triangle **T={a,b,c}**. To evaluate the basis function $\phi_a(x, y)$ in the general triangle, we map (x, y) back to the corresponding point (r, s) in the reference triangle, and evaluate $\phi_a(r, s)$ there. Now in the reference triangle, the basis functions are **r**, **s** and **1-r-s**. But the inverse map from the general triangle to the reference triangle gives us (r, s), and it's trivial to evaluate $1 - r - s$. This means that if we can find the inverse point (r, s) in the reference triangle corresponding to a given point (x, y) in the general triangle, then we can easily evaluate the basis functions.

The mapping from **Tref** to **T** can be written as

$$\psi_{\text{Tref,T}} \begin{pmatrix} r \\ s \end{pmatrix} = A \cdot \begin{pmatrix} r \\ s \end{pmatrix} + \begin{pmatrix} c_x \\ c_y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

where

$$A = \begin{pmatrix} a_x - c_x & b_x - c_x \\ a_y - c_y & b_y - c_y \end{pmatrix}$$

So the inverse mapping from **T** back to **Tref** can be written as

$$\psi^{-1}_{\text{Tref,T}} \begin{pmatrix} x \\ y \end{pmatrix} = A^{-1} \cdot \begin{pmatrix} x - c_x \\ y - c_y \end{pmatrix} = \begin{pmatrix} r \\ s \end{pmatrix}$$

In the lab on mapping triangles, we showed that:

$$\det A = (a_x - c_x)(b_y - c_y) - (a_y - c_y)(b_x - c_x)$$

Therefore, the inverse of A can then be written as

$$A^{-1} = \frac{1}{\det A} \begin{pmatrix} (b_y - c_y) & -(b_x - c_x) \\ -(a_y - c_y) & (a_x - c_x) \end{pmatrix}$$

so

$$\phi_a(x,y) = r(x,y) = \frac{(b_y - c_y)(x - c_x) - (b_x - c_x)(y - c_y)}{(a_x - c_x)(b_y - c_y) - (a_y - c_y)(b_x - c_x)}$$

and

$$\phi_b(x,y) = s(x,y) = \frac{-(a_y - c_y)(x - c_x) + (a_x - c_x)(y - c_y)}{(a_x - c_x)(b_y - c_y) - (a_y - c_y)(b_x - c_x)}$$

and

$$\phi_c(x,y) = 1 - r(x,y) - s(x,y) = 1 - \phi_a(x,y) - \phi_b(x,y)$$

So, by using the idea of mapping, we have come up with the same formulas that we have already gotten by using linear functions and by using determinants! This is not wasted effort. It's important to be able to see this problem in three different ways, and to understand the relationships between the many ways of describing this problem.

We have considered how to define and evaluate a set of basis functions on a single triangle. This set of basis functions can be used to solve the interpolation problem on a single triangle. In the following lab, we will consider the interpolation problem on a triangulation, and for that lab we will need to use the results we have worked out here!