SPACE-FILLING CURVES (SFC)

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OVERVIEW

- Introduction
- Type of Space-Filling Curves

    1. The Peano Space-Filling Curves
    2. The Hilbert Space-Filling Curves
    3. The Sierpinski Space-Filling Curves
    4. The Lebesgue Space-Filling Curves

- Applications of Space-Filling Curves

SURJECTIVE MAPPING:

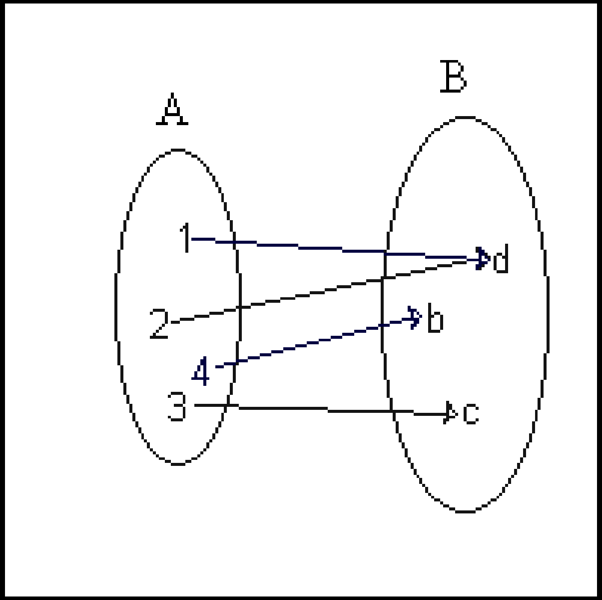
The function is **surjective** (onto) if every element of the codomain is mapped to by at least one element of the domain.

Surjective map from A onto B :

$f(a) = b$

If $f : A \rightarrow B$ then f is said to be surjective if

$\forall\, b \in B\,,\ \exists\, a \in A\,,\ f(a) = b$



INJECTIVE MAPPING:

An **injective** function or injection or one-to-one function is a function that preserves distinctness: it never maps distinct elements of its domain to the same element of its codomain.

Injective map from A onto B :

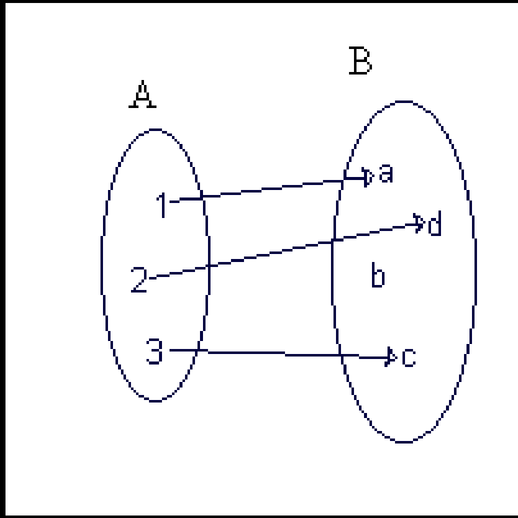
f(a) = b

If f : A→B then f is said to be surjective if

$\forall$ a , b ∈ A , f(a) = f(b) → a = b

or

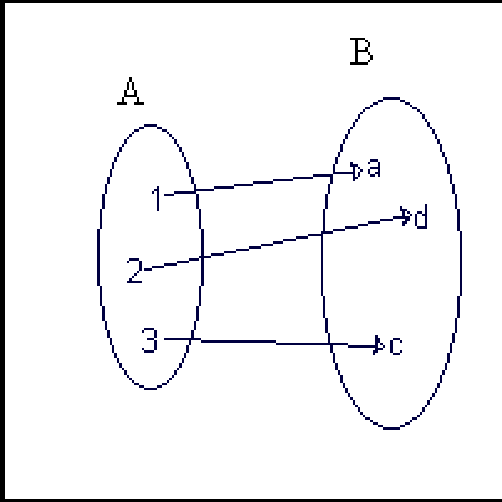
$\forall$ a , b ∈ A , a ≠ b → f(a) ≠ f(b)



BIJECTIVE MAPPING:

A function is **bijective** if it is both injective and surjective. A function is bijective if and only if every possible image is mapped to by exactly one argument.

If f : A→B then f is said to be bijective if
For all b Є B, there's a unique a Є A
Such that f(a) = b



SPACE-FILLING CURVE (SFC) DEFINITION:

Intuitively, a continuous curve in 2 or 3 (or higher) dimensions can be thought of as the path of a continuously moving point.

HISTORY OF THE HILBERT CURVES:

- Hilbert Curves are named after the German mathematician *David Hilbert*. They were first described in 1891.

- A Hilbert curve is a continuous space-filing curve. They are also fractal and are self-similar; If you zoom in and look closely at a section of a higher-order curve, the pattern you see looks just the same as itself.

- An easy way to imagine creation of a Hilbert Curve is to envisage you have a long piece of string and want to lay this over a grid of squares on a table. Your goal is to drape the string over board so that the string passes through each square of the board only once.

THE HILBERT CURVE: GEOMETRIC GENERATION:

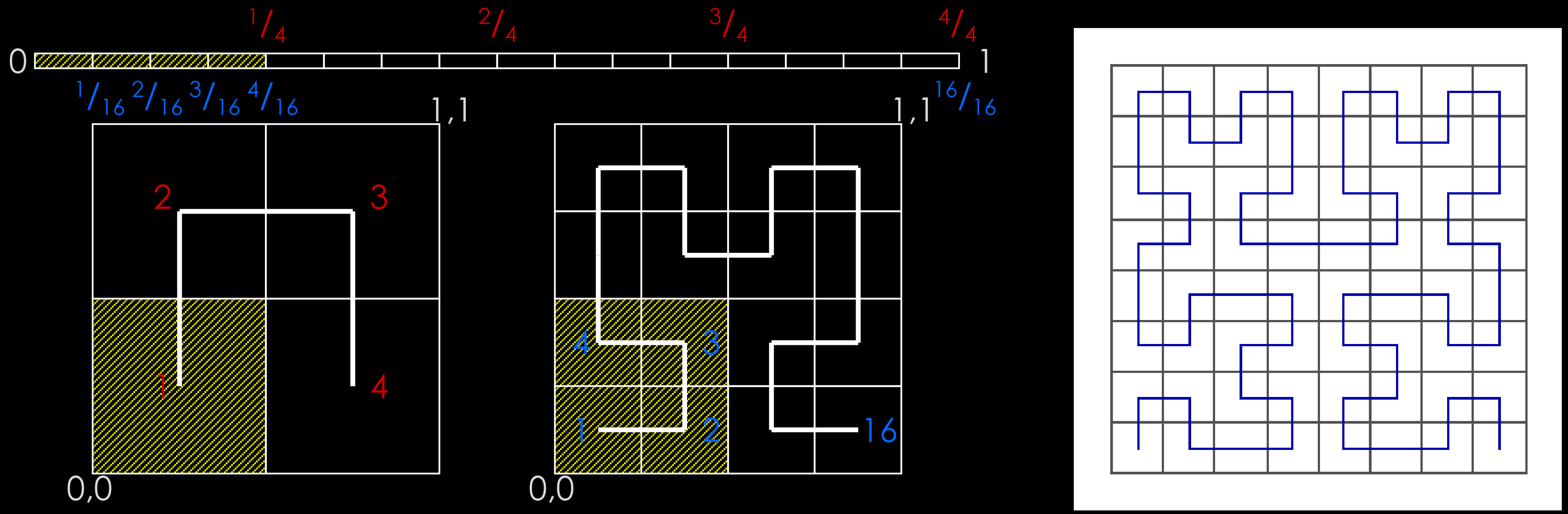
- If *I* can be mapped continuously on Ω, then after partitioning *I* into four congruent subintervals and Ω into four congruent subsquares, each subinterval can be mapped continuously onto one of the subsquares. This partitioning can be carried out ad infinitum.

- The subsquares must be arranged such that adjacent subintervals are mapped onto adjacent subsquares.

- Inclusion relationship: if an interval corresponds to a square, then its subintervals must correspond to the subsquares of that square.

- This process defines a mapping $f_h(I)$, called the Hilbert space-filling curve.

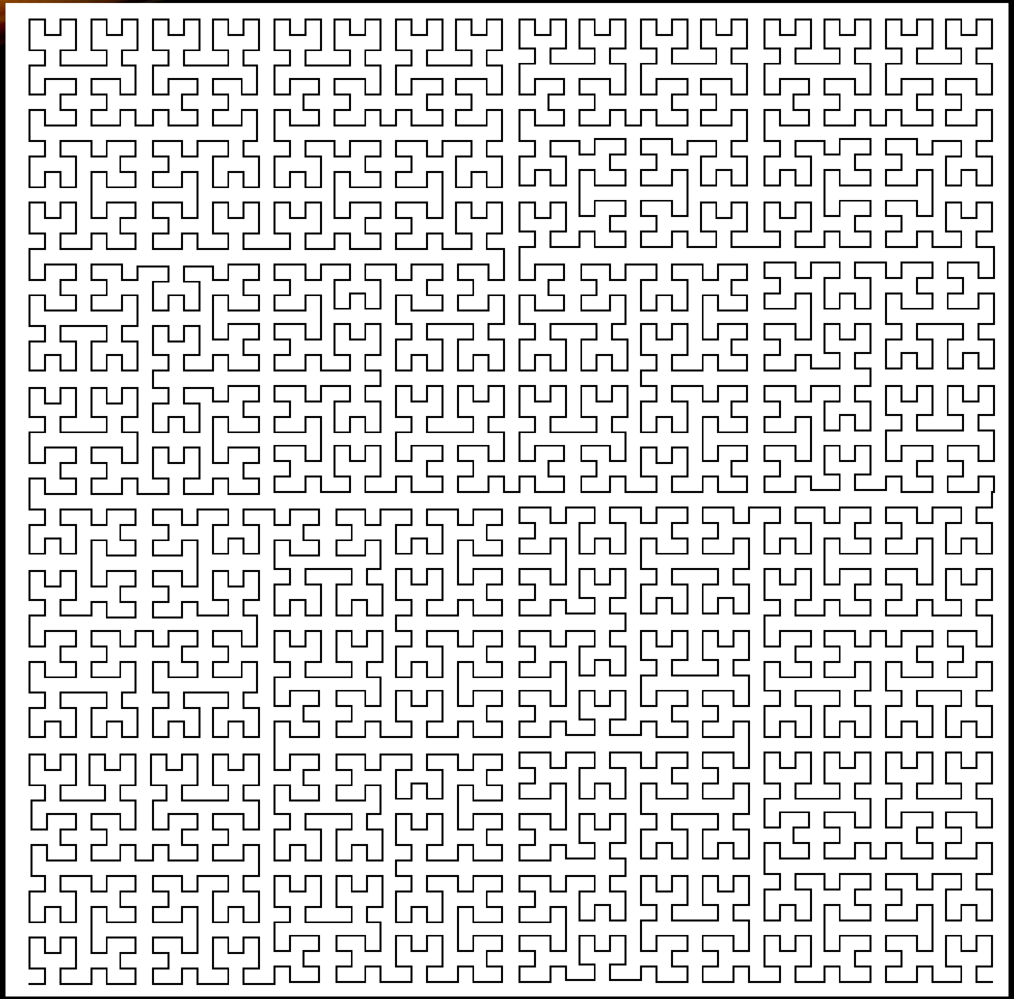
THE HILBERT CURVE: GEOMETRIC GENERATION:



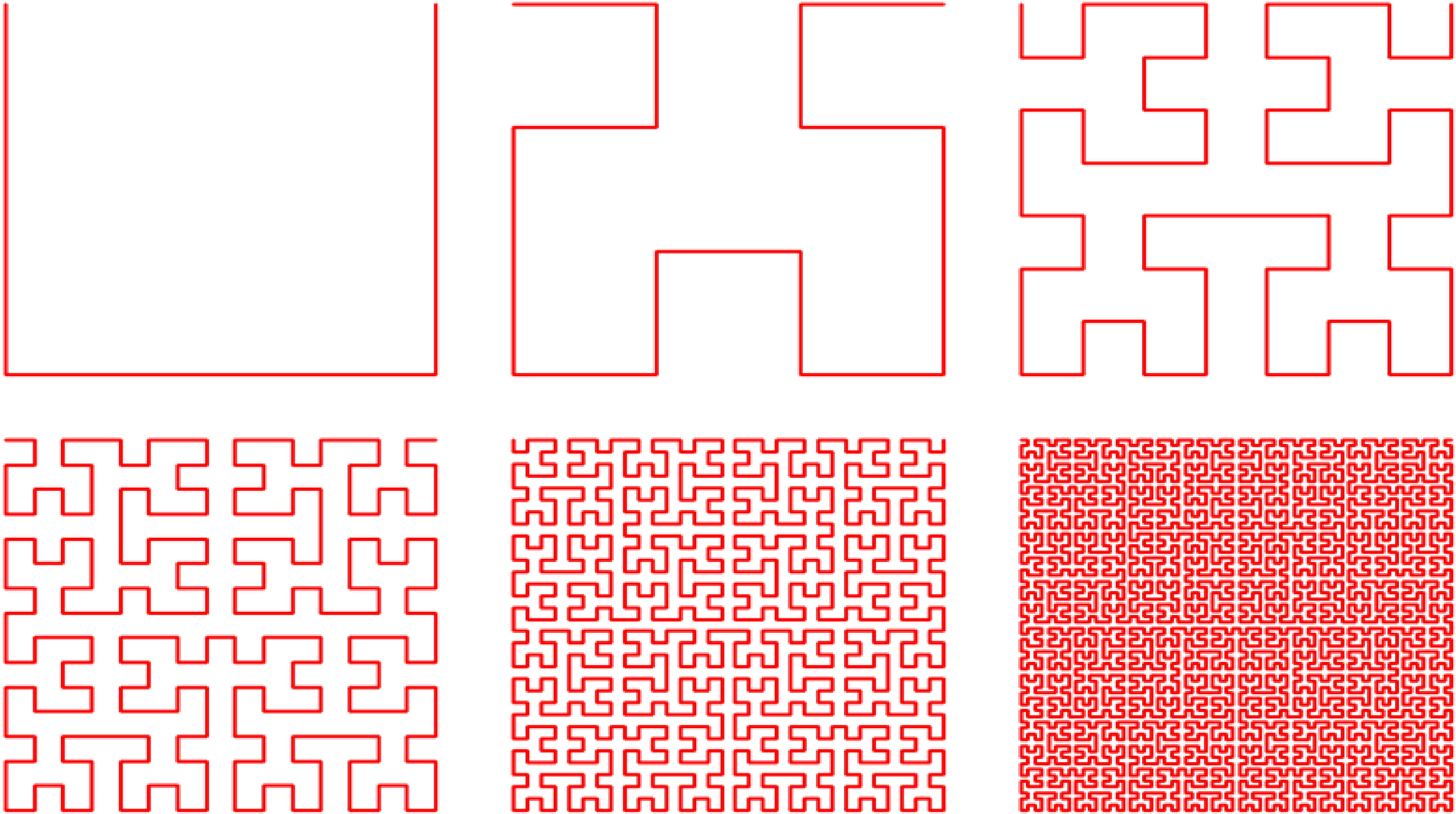
1st iteration

2nd iteration

3rd iteration



6th Iteration

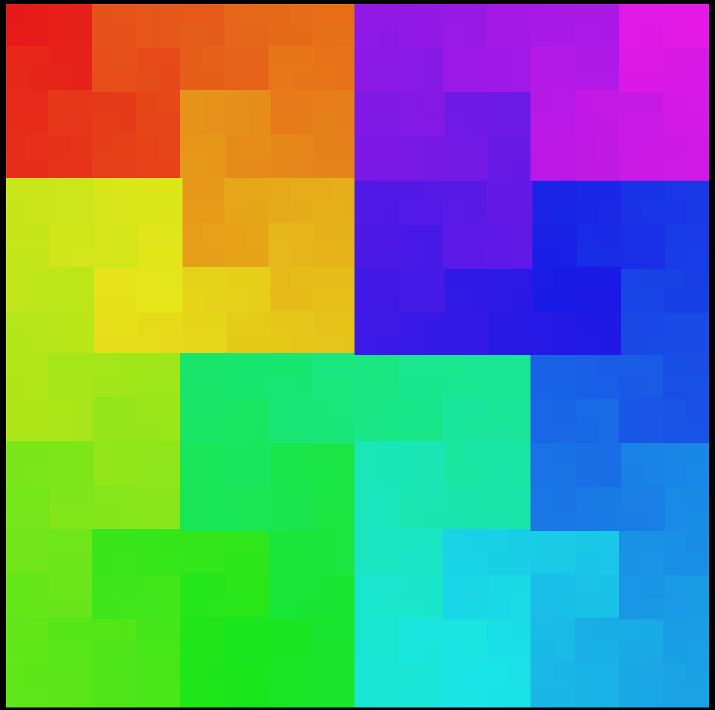


APPLICATION: 1D INTO 2D:

- We find that if we convert the distance (*d*) shown on the tape-measure to a pair of *(x,y)* coordinates on the grid, then we find that other marks on the tape measure (close to *d*) have *(x,y)* coordinates that are very close.

- We have created a useful mapping function between one-dimension and two-dimensions that does a good job of preserving locality.

- To the right you can see the rainbow painted, using a Hilbert Curve. Colors that are close to each other on the spectrum have similar *(x,y)* coordinates.



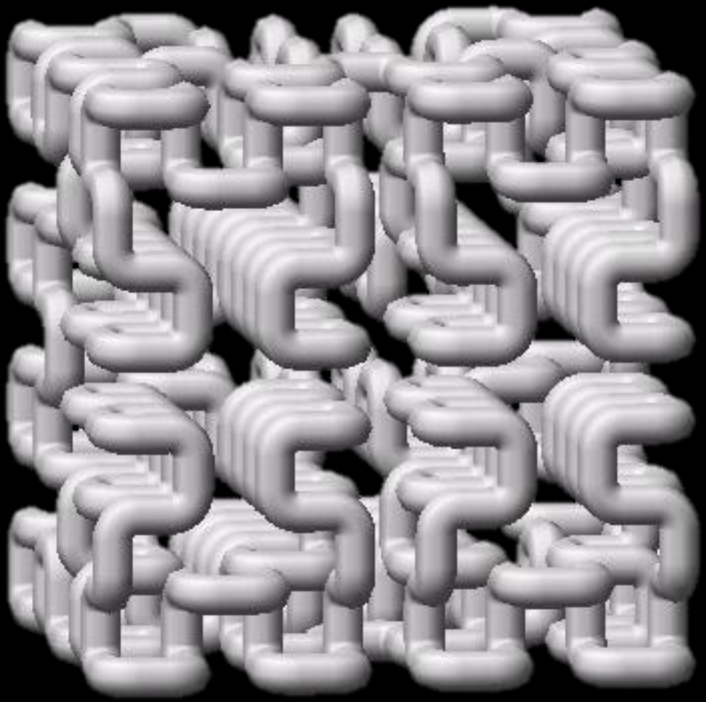
- For more info:

https://www.jasondavies.com/hilbert-curve/

HIGHER DIMENSIONS:

- The characteristics of Hilbert Curves can be extended to more than two dimensions. A 1D line can be wrapped up in as many dimensions as you can imagine.

- A rendering of a curve in 3D can is shown on the left.
  In this shape, points on the line numerically close
  to one another are also close in 3D space.



- It's easy to see how this can be
  extended into higher dimensions.