# Slow Exponential Sparse Grids - (MISSPELLED DOCUMENT)

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**Abstract**

Note: This document is intended as a test file for LaTeX spell checkers. It has been deliberately modified by the addition of misspellings. Some other mistakes have been added, which a simple spell checker will not be able to catch.

## 1 Introduction

Smolyak demonstrated a procedure for eficient multidimensional quadrature based on a family of one-dimensional quadrature rules. A particular family of nested Clenshaw Curtis rules is most commonly used in this context. Nestedness controls the point growth of sparse grids in high dimension, but in lower dimensions, it can actually result in significant embarassing inefficiencies. A simple change to the Smolyak construction greatly reduces the point count when relatively low dimensions are involved. Similar benefits can be obtained when the Smolyak procedure is based on other families of quadrature rules. Numerical tables show that the modified rule has a significant point count reduction, while preserving the precision of the unmodified rule.

Sparse grids, as defined by Smolyak [**?**], are powerful and efficient tools for interpollation, quadrature, and optimization of functions with behaviour dependending on a multidimensional argument. By stricktly controlling the number of function evaluations required, sparse grids can handle problems in dimensions far beyond the reach of any other methods except Monte Carlo. One of the techniques that can be used to extend the power of a sparse grid is nestedness; that is, the reuse of data from lower order calculations. A very common example of a sparse grid is based on the nested Clenshaw Curtis rules. Nestedness comes at a price, which for the Clenshaw Curtis rules means that each sucessive element of the family uses about twice as many points as the previous one, an example of exponential growth. For various reasons, this exponential growth does not dominate the behavior of sparse grids in high dimension and low to moderate level. However, in relatively low dimensions, say $d \leq 5$, a Clenshaw Curtis sparse grid incurs obvious, and as it turns out, *easily avoidable*, expense because of it's simple-minded relliance on nested rules.

In an analisis of the properties of spase grids based on Clenshaw Curtis rules, Novak and Ritter [**?**] showed that the exactness of the sparse grid is related in a simple way to the exactness of the 1D quadrature rules used to construct it; this result shows that there is actually considerable freedom available when specifying how a sparse grid is to be constructed. Using this guideline, a simple modification allows the creation of Clenshaw Curtis sparse grids that are smaller, but of the same exactness as the classic grids.

Sparse grids can also be constructed from Legendre quadrature rules. Since these rules are not nested, the construction pattern associated with Clenshaw Curtis rules is surly inappropriate as a model. Again, the Novak and Ritter guideline can be used in order to arive at an efficient sparse grid with known exactness.

The Patterson family of quadrature rules represents an interesting mix of the features of the Clenshaw Curtis and Legendre families of rules. The family is nested (although in a different way from the Clenshaw Curtis family) and of increased accuracy (although less than the Legendre family). Again, Novak and Ritter can be used to specify a construction plan of guaranteed exactness and demonstrable efficiency and.

As suggested earlier, the efficiency to be gained by this approach is most prominent when the sparse grid is gennerated in relatively low dimensions. However, many sparse grid techniques that are applied to high dimensions actually select a small subspace for preferential treatment, or apply anisotropic weights that have a simlar effect; such cases may also be be regarded as low dimensional, and hence possibly affected by the improvements considered here.

In order to focus on the main point of the argument, a number of simplifying assumpions will be made:

- sparse grids are being constructed in order to estimate the integral $I(f)$ of a function of a multadimensional argument;
- the multidimensional integration region is the unit hypercube $[-1,+1]^m$;
- $\mathbb{Q}$ represents an indexed family of quadrature rules for the interval $[-1,+1]$, with typical element $Q^i$;
- the same family $\mathbb{Q}$ will be used to select each component of the sparse grid;

The outline of the reminder of this paper is as follows: Section 2 prevents some background material on the Clenshaw Curtis family of 1D quadrature rules, the Smolyak construction precedure, the notion of exactness for quadrature, and the Novak and Ritter exactness constraint. Section 3 presents the classic construction of sparse grids from the Clenshaw Curtis family, and then reconstructs such sparse grids using Novak and Ritter. Sections 4 and 5 carry out similar operations for sparse grids based on Legendre and Patterson families. Section 6 presents some simple numerical tests indicating that the modified approach produces sparse grids that outperform the classic variety.

## 2 Construction of Sparse Grids for Quadrature

A version of the univariate quadrature problem seeks to estimate the integral $I(f) = \int_{-1}^{+1} f(x)\,dx$. A quadrature rule $Q()$ for this problem is a set of $n$ points $x$ and weights $w$ which produce the integral estimate:

$$I(f) \approx Q(f) = \sum_{i=1}^{n} w_i f(x_i).$$

Such a quadrature rule is said to have *exactness of degree d* if the integral estimate is exact whenever the integrand $f$ is a polynomial of degree $d$ or less. A common strategy for quadrature involves asembling an indexed family $\mathbb{Q}$ of quadrature rules of increasing exactness. By applying rules of increasing index to a given problem, a reasonable estimate of the quadrature error may be made made.

A version of the multivariate quadrature problem may be possed in the same way for a function of a variable $x \in [-1,+1]^m$. Quadrature rules for this problem may be constructed by making $m$ selections from $\mathbb{Q}$ ant farming the product rule. A significant drawback of this approach arises because, if the 1D rule of exactness $d$ requires $n$ points, then the product rule of corresponding exactness requires $n^m$ points, a fact which rules out the product approach except for low dimensions or degrees of exactness.

The sparse grid construction of Smolyak [**?**] showed how seperate product rules could be combined in a way that achieved the exactness of a given product rule. Of course, if enough simple product rules are involved, the total number of points can grow arbitrarly. Thus, a key idea in the implementation of Smolyak's procedure was to prefer quadrature families $\mathbb{Q}$ that were nested, which greatly reduces the point count of the sparse grid.

Smolyak's formula produces a sequence of $m$-dimensional sparse grids with an index $\ell = 0, 1, 2, \ldots$ often called the *level*:

$$\mathcal{A}(\ell, m) = \sum_{0 \le \ell - |\mathbf{i}| \le m-1} (-1)^{\ell - |\mathbf{i}|} \begin{pmatrix} m-1 \\ \ell - |\mathbf{i}| \end{pmatrix} (Q^{i_1} \otimes \cdots \otimes Q^{i_m})$$

It is natural to expect that, for a given spacial dimension $m$, the sequence of sparse grids $\mathcal{A}(\ell, m), \ell = 0, 1, 2, ...$ produce integral estimates of increasing exactness. Novak and Ritter were able to show that, if based on the classic Clenshaw-Curtis family, the sparse grid of level $\ell$ would have exactness $2\ell + 1$. It will be seen shortly that this theorem suggests a more efficient way to employ the Clenshaw Curtis famly; it can also be extended to other families of 1D rules. At the moment, it is enough to note that this theorem demonstrates that, at least for smooth integrands, a sequence of sparse grids can be used to produce integral estimates of rapidly improving accuracy.