Monte Carlo Simulation

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Math 1103: BIG Problems
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312 ANNUAL REGISTER, 1830.

BILLS OF MORTALITY, *from December* 12, 1829, *to December* 15, 1830.

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Decreased in the Burials reported this year, 1879.

The Monty Hall Paradox

Should you switch your choice or stick to it?

Not a Proof from the Book

Even Paul Erdös thought you should stick to your original choice.

Logical arguments didn't persuade him,

Finally, he watched a computer simulation that compared both strategies, verifying that switching was twice as good.

"OK", he said, *"but that is not a proof from the Book!"*

A Monte Carlo analysis can help us to visualize a problem, and estimate its behavior. It is not a proof, although it can suggest where to look!

Monte Carlo Simulations for 3 through 10 doors

Number of trials = 100000

Doors	No-switch		Switch	
	Observed	Predicted	Observed	Predicted
n		1/n		n-1/n-2 * 1/n
3	0.3364	0.3333	0.6679	0.6667
4	0.2507	0.2500	0.3740	0.3750
5	0.2011	0.2000	0.2669	0.2667
6	0.1652	0.1667	0.2043	0.2083
7	0.1424	0.1429	0.1714	0.1714
8	0.1267	0.1250	0.1457	0.1458
9	0.1126	0.1111	0.1287	0.1270
10	0.1021	0.1000	0.1129	0.1125

Explanation of Predicted Results

For a no-switch strategy, you simply have $\frac{1}{n}$ chance of picking the prize door.

For switch, you have $\frac{1}{n}$ chance of switching from the prize door, so a $\frac{n-1}{n}$ chance of having a second chance at the prize. In this second chance, there are $n-2$ doors, and so you have a $\frac{1}{n-2}$ chance of success. Thus, overall, you have a $\frac{n-1}{n} * \frac{1}{n-2}$ chance of rejecting a nonprize initial selection and successfully choosing the prize on your second guess.

Thus, the advantage of switching is most obvious when $n = 3$, and diminishes with an increasing number of doors.

The Monte Carlo method suggested switching helped, and after we worked out the probabilities, it verified our results.

Fooling with Dueling

Take turns or blast away now? Stop or keep going? Who has the best chance?

One Simultaneous Shot Each

Our duelers are Alphonse (A) and Beauregard (B).

On a single shot, A has a 40% chance of hitting B; B has a 30% chance of hitting A.

If they both fire simultaneously, once, then the odds are:

```
          0.40                  0.60
       +-----------------------------
0.30   | 0.12                  0.18
       | A dead, B dead        A dead, B OK
       |-----------------------------
0.70   | 0.28                  0.42
       | A OK,   B dead        A OK,   B OK
```

Harder Questions

1. Suppose A fires first, and then B (if still alive!).
2. Since B is the weaker player, what happens if B goes first?
3. What happens, with either ordering, if the players continue to fire until someone is killed?
4. What happens to the survival odds if A fires once, then B twice, then A three times, and so on.

A Dueling Program

```
A_alive = true, A_accuracy = 0.40
B_alive = true, B_accuracy = 0.30
shots = 0

repeat

  shots = shots + 1
  if ( random() < A_accuracy )
    B_alive = false
    break

  shots = shots + 1
  if ( random ( ) < B_accuracy )
    A_alive = false
    break

print ( A_alive, B_alive, shots )
```

Results of AB and BA orderings

The two orderings were compared, with 1000 simulations.

If A goes first, A wins with probablity 0.667, average 2.8 shots.

If B goes first, B wins wtth probability 0.513, average 2.9 shots.

A Three-Way Duel

Surviving a three way duel requires a strategy!

The Rule for the Duel

1. A, B, and C are going to fight a three way duel.
2. In each first round, the first player may fire once at any opponent (or none!), then the second has a turn, then the third.
3. More rounds will take place until there is only one survivor.
4. The players know the accuracy ratings of their opponents. We will say A $= \frac{5}{6}$, B $= \frac{4}{6}$, C $= \frac{2}{6}$.
5. What is a good strategy for each player?
6. What are the chances that C will survive, for order (A,B,C)?
7. What are the chances that C will survive, for order (C,B,A)?

Sample Strategies

1. C should shoot at the "next" shooter.
2. C should shoot at the "previous" shooter.
3. C should always shoot at the best shooter.
4. C should not fire if there are two other players; otherwise, shoot at the remaining opponent;
5. C should choose a target at random;

Although the two-person duel can easily be analyzed exactly, the addition of strategies in the three person duel makes tha analysis much harder. However, it is trivial to vary the strategies in a Monte Carlo simulation, so to experimentally search for the best approach.

MATLAB code for one turn for player # I

Here, each player I aims at best opponent, with accuracy P(I).

```
if ( 0 < p(i) )                          <-- I is still alive?
  turn_num = turn_num + 1;
  p_save = p(i);
  p(i) = 0.0;                            <-- "Hide" our P
  [ pmax, target ] = max ( p );          <-- TARGET is max P
  r = rand ( );
  if ( r <= p_save )
    p(target) = 0.0;
    if ( sum ( p ) == 0.0 )              <-- If all dead...
      survivor = i;
      break;
    end
  end
  p(i) = p_save;                         <-- Put back P
end
```

Some results

Using the order C, B, A, we compare various strategies for C.

C Strategy 1: Shoot at highest
C Strategy 2: Shoot next opponent
C Strategy 3: Shoot at random opponent
C Strategy 4: Don't shoot until only 1 opponent

Player	Accuracy	Strat 1	Strat 2	Strat 3	Strat 4
C	0.33	0.29	0.21	0.17	0.41
B	0.67	0.58	0.40	0.35	0.41
A	0.83	0.13	0.39	0.48	0.18

Gambler's Ruin

How long until one of you is ruined?

Gambler's Ruin

Two gamblers, **A** and **B** compete.

A has \$3 and **B** has \$7.

They decide to flip a coin. If it comes up heads, **A** wins \$1 from **B**, while tails works the other way.

They continue to bet \$1 at a time until bankruptcy.

Gambler's Ruin

Here are some questions we can ask:

1. What is the probability that **A** will win?
2. What is the expected value of the game to **A**, that is, how much will A win or lose, on average?
3. How long will a typical game last?
4. What are the chances that the winner will be ahead the entire game?
5. What if one player has an infinite supply of cash?
6. What happens if we change the amount of money that **A** and **B** have?
7. What happens if the coin is slightly unfair, turning up heads 55% of the time?
8. What happens if three people want to play the game?

In this simple case, we could answer these questions exactly. But we will try to estimate the results using a Monte Carlo approach. We will play many games and average the results.

Gambler's Ruin

Here's a typical game, in which **A** starts with \$3 and **B** with \$7:

H/T:		H	H	T	T	T	H	T	T	H	T	H	H	T	T	T
A	3	4	5	4	3	2	3	2	1	2	1	2	3	2	1	0
B	7	6	5	6	7	8	7	8	9	8	9	8	7	8	9	10
#	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15

A loses after 15 tosses, having tossed 6 heads and 9 tails.

Gambler's Ruin - Program (Play One Game)

```
a = 3;
b = 7;
step = 0;
while ( 0 < a && 0 < b )
  step = step + 1;
  if ( rand ( ) <= 0.5 )      <-- Coin is fair
    a = a + 1;
    b = b - 1;
  else
    a = a - 1;
    b = b + 1;
  end
end
```

Gambler's Ruin - Results of 1000 Games

$A	$B	Av Length	Max Length	A Prob	Flips	Max Flips
$3	$7	21	135	0.29	1.7	14
$30	$70	2,010	12,992	0.28	19.7	189
$10	$10	101	722	0.49	4.6	59
$1	$100	104	10,472	0.01	0.5	1

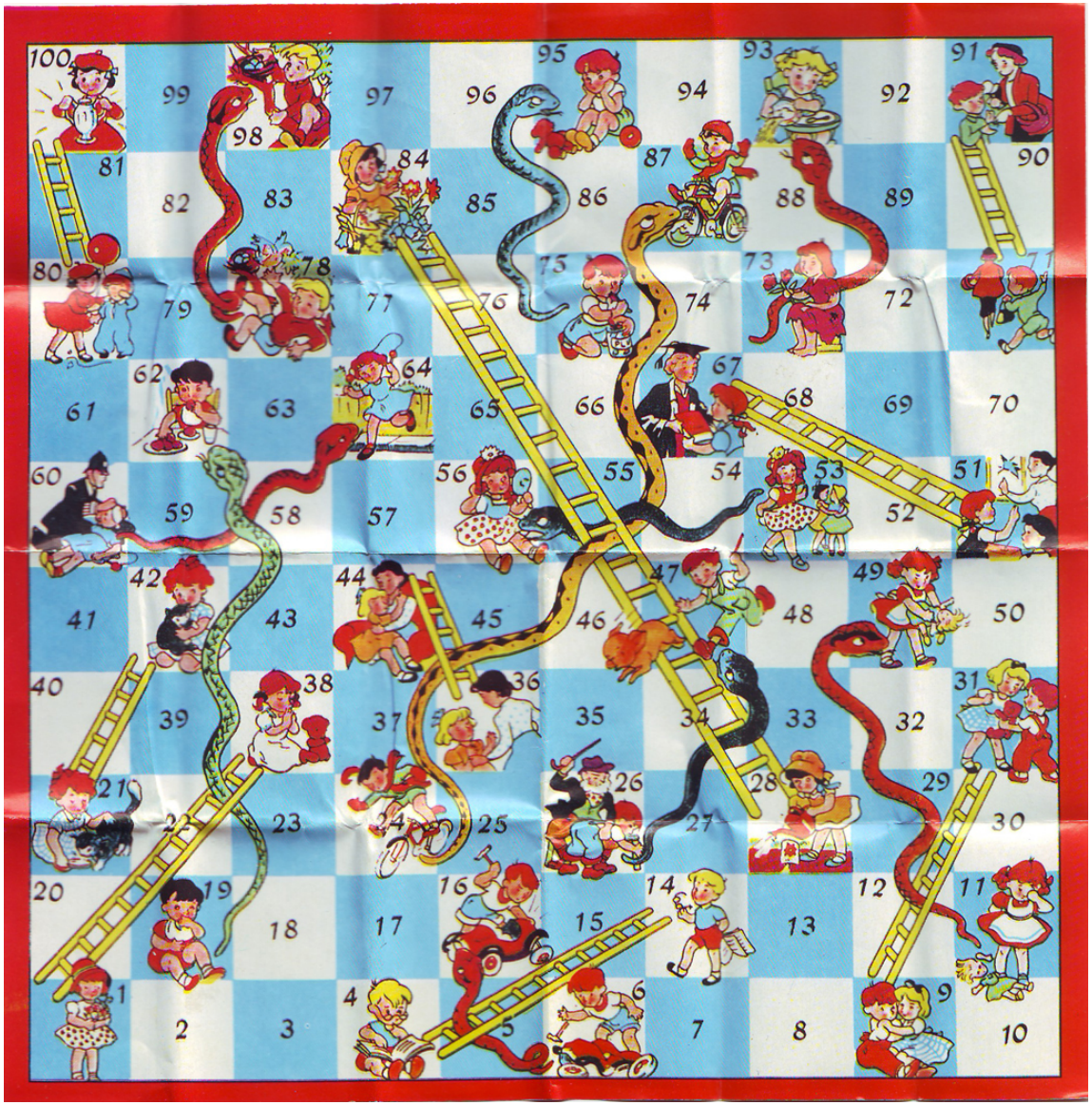
Monte Carlo results allow us to

1. Guess a formula for the average length of a game.
2. Guess a formula for the probability that A wins.

The maximum length is theoretically ∞. Predicting the average number of flips in who is ahead is a hard question.

Snakes and Ladders

The rules change depending on your current status.



Rules Depend on Current State

Let's imagine a single player version of the game. You start at square 1, roll one die, and move ahead. Landing on a ladder square moves you further forward, while a snake moves you back. You win by reaching the final square.

We know you'll win, but the unknown is how long it will take, that is, how many rolls of the die are required, on average. We can estimate this by playing many times. In advance, we have to move 100 spaces using steps of average size 3.5 units, so, ignoring snakes and ladders, we'd give a lower limit of about 30 steps.

Note that when we simulate this game, our behavior at each step depends on where we are. Rolling a 3 may simply move us ahead 3, or we may hit a snake or ladder.

In many other simulations, we also need to keep track of our current status before we decide what choices we have available, or what gains or losses we may pick up.

Coding Snakes and Ladders

To simulate the game, we need to create some kind of list of which squares are snakes or ladders, and where they connect to. After that, we simply do something like this:

```
square = 0
rolls = 0

while ( square < 100 )
  die = randi ( 6 )          <-- Random integer between
      1 and 6.
  rolls = rolls + 1
  square = square + die
  if ( snake_start ( square ) )
    square = snake_end ( square ) )
  elseif ( ladder_start ( square ) )
    square = ladder_end ( square )
  end
end
```

Simulate 1000 games, and do it 10 times

Trial	Average	Shortest	Longest
1	39.8	7	175
2	39.2	7	185
3	38.7	7	158
4	39.5	7	205
5	38.5	7	187
6	38.3	7	198
7	39.5	8	207
8	38.8	7	176
9	39.9	7	185
10	39.0	7	242

While the average and shortest values don't vary much, we don't get a constant result for the longest game. But that's because in reality it's unbounded.

Mortality

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Mortality

From a table of death counts we can infer

1. the population of survivors over time
2. the probability of death at a particular age bracket,
3. the probability of death by a particular age bracket,
4. the probability of living to a given age bracket and dying then.

Bracket	Pop	Deaths	PDF	CDF	This Year
[0: 1]	2423509	29138	0.0120	0.0120	0.0120
[1: 2]	2394371	1940	0.0008	0.0128	0.0008
[2: 3]	2392431	1178	0.0005	0.0133	0.0005
[3: 4]	2391253	882	0.0004	0.0137	0.0004
[4: 5]	2390371	703	0.0003	0.0140	0.0003
.....	.....	.....	.....	.....	.....
[110:111]	60	29	0.0000	1.0000	0.4833
[111:112]	31	17	0.0000	1.0000	0.5484
[112:113]	14	9	0.0000	1.0000	0.6429
[113:114]	5	4	0.0000	1.0000	0.8000
[114:115]	1	1	0.0000	1.0000	1.0000

Term Insurance

```
pay_in = 0.0;  pay_out = 0.0;  pay_age = 0;
value = 0.0;  interest_rate = 0.04;

for year_age = age : age + term - 1

  value = value * ( 1.0 + interest_rate );
  value = value + yearly_fee;

  pay_in = pay_in + yearly_fee;
  died = ( rand ( ) < total_this_year ( year_age ) );
  if ( died )
    pay_age = year_age;
    pay_out = death_benefit;
  end

  if ( died )
    break;
  end
end
```

Sample cases

Starting at age 25, annual payment of \$500 for 40 years, for term life insurance of \$100,000.

Age at Death	Prob Death This Year?	Payments	Bank 4%	Payoff
[26,27]	0.0018	1000	1020	100000
[36,37]	0.0023	6000	7513	100000
[49,50]	0.0076	12500	20823	100000
[50,51]	0.0082	13000	22156	100000
[61,62]	0.0168	18500	40851	100000
[65, ?]	0.0174	20000	47513	0

Estimating Profits and Savings

Monte Carlo Simulation can easily be adjusted to handle a variety of related problems.

For instance, for this insurance problem:

1. The term insurance function can be used to estimate the balance between insurance fees and payouts for the company.
2. The program also reveals a comparison between the average payout (which is usually 0!) the size at the end of the term of a savings account storing the insurance fees.
3. We could also use simulation to adjust the insurance fee, based on the age of the customer and the length of the term.

A Definition

In Monte Carlo Simulation, we construct an artificial model of some process that includes randomness in its input or behavior.

We run the simulation many times, trying to consider as much random variation as is reasonable.

From the results, we compute means, variances, maximums and other information that we can use to try to understand, adjust or control the original process