Turning Lights Out with Linear Algebra

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The game *Lights Out*, commercially available from Tiger Electronics, consists of a 5×5 array of 25 lighted buttons; each light may be on or off. A *move* consists of pushing a single button. Doing so changes the on/off state of the light on the button pushed, and of all its vertical and horizontal neighbors. Given an initial configuration of lights which are turned on, the object is to turn out all the lights.

A complete strategy for the game can be obtained using linear algebra, requiring only knowledge of Gauss-Jordan elimination and some facts about the column and null spaces of a matrix. All calculations are done modulo 2.

We make some initial observations.

1. Pushing a button twice is equivalent to not pushing it at all. Hence, for any given configuration, we need consider only solutions in which each button is pushed no more than once.
2. The state of a button depends only on how often (whether even or odd) it and its neighbors have been pushed. Hence, the order in which the buttons are pushed is immaterial.

We will represent the state of each light by an element of $\mathbb{Z}_2$, the field of integers modulo 2; 1 for on, and 0 for off. We will denote the state of the light in the ith row and jth column by $b_{i,j}$, an element of $\mathbb{Z}_2$, and the entire array by a 25×1 column vector $\vec{b}$, with entries ordered as follows:

$$\vec{b} = (b_{1,1}, b_{1,2}, \cdots, b_{1,5}, b_{2,1}, \cdots b_{5,5})^T$$

(T stands for transpose). We will call such a vector a *configuration* of the array.

Pressing a button changes the configuration vector by adding to $\vec{b}$ a vector that has 1's at the location of the button and its neighbors and 0's elsewhere. The order of pushing buttons makes no differences, so we may represent a *strategy* by another 25×1 column vector $\vec{x}$, where $x_{i,j}$ is 1 if the (i,j) button is to be pushed, and 0 otherwise.

If we start with all the lights out and configuration $\vec{b}$ is obtained by strategy $\vec{x}$, then

$$b_{1,1} = x_{1,1} + x_{1,2} + x_{2,1},$$

$$b_{1,2} = x_{1,1} + x_{1,2} + x_{1,3} + x_{2,2},$$

$$b_{1,3} = x_{1,2} + x_{1,3} + x_{1,4} + x_{2,3}.$$

More generally, it is straightforward to check that the result $\vec{b}$ of the strategy $\vec{x}$ is the matrix product $A\vec{x} = \vec{b}$, where A is the 25×25 matrix:

$$A = \begin{pmatrix} B & I & O & O & O \\ I & B & I & O & O \\ O & I & B & I & O \\ O & O & I & B & I \\ O & O & O & I & B \end{pmatrix}$$

here I is the 5×5 identity matrix, O is the 5×5 matrix of all zeros, and B is the matrix

$$B = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

Note that B is a symmetric matrix, and so A is symmetric too.

Given an arbitrary configuration $\vec{b}$, we will say that $\vec{b}$ is *winnable* if there exists a strategy $\vec{x}$ to turn out all the lights in $\vec{b}$. The key observation is as follows:

> *If a set of buttons is pushed to create a configuration, then starting with that configuration and pressing the same set of buttons will turn the lights out.*

That is, to find a strategy to turn out all the lights in $\vec{b}$, we need to solve $\vec{b} = A\vec{x}$. Thus, a configuration $\vec{b}$ is winnable if and only if it belongs to the column space of the matrix A; we denote the latter by $\mathrm{Col}(A)$.

To analyze $\mathrm{Col}(A)$, we perform Gauss-Jordan elimination on A. This would be tedious to perform by hand, but is easier using any computer algebra system capable of handling matrices with entries from $\mathbb{Z}_2$; *Maple* or *Mathematica* will do the job. Gauss-Jordan will yield $RA = E$, where E is the Gauss-Jordan echelon form, and R is the product of the elementary matrices which perform the reducing row operations. The matrices R and E are rather formidable, and not particularly illuminating. We will not display them here but invite the reader to calculate them using a favorite computer algebra system.

Having done this calculation, we see that the matrix E is of rank 23, with two free variables $x_{5,4}$ and $x_{5,5}$ in the last two columns. Indeed, the last two columns of E are

$$(0,1,1,1,0,1,0,1,0,1,1,1,0,1,1,1,0,1,0,1,0,1,1,0,0)^T$$

and

$$(1,0,1,0,1,1,0,1,0,1,0,0,0,0,0,1,0,1,0,1,1,0,1,0,0)^T.$$

Now A is a symmetric matrix, and so $\mathrm{Col}(A)$ equals the row space of A, denoted $\mathrm{Row}(A)$. But $\mathrm{Row}(A)$ is the orthogonal complement of the null space of A (denoted $\mathrm{Null}(A)$), which in turn equals $\mathrm{Null}(E)$. So, to describe $\mathrm{Col}(A)$, we need only determine a basis for $\mathrm{Null}(E)$.

Since E is in Gauss-Jordan echelon form, it is easy to find an orthogonal basis for $\mathrm{Null}(E)$ by examining the last two columns of E:

$$\vec{n}_1 = (0,1,1,1,0,1,0,1,0,1,1,1,0,1,1,1,0,1,0,1,0,1,1,1,0)^T$$

and

$$\vec{n}_2 = (1,0,1,0,1,1,0,1,0,1,0,0,0,0,0,1,0,1,0,1,1,0,1,0,1)^T.$$

Putting this together, we have the following:

THEOREM 1. *A configuration $\vec{b}$ is winnable if and only if $\vec{b}$ is perpendicular to the two vectors $\vec{n}_1$ and $\vec{n}_2$.*

Therefore, to see if a configuration is winnable, we simply compute the dot product of that configuration with $\vec{n}_1$ and $\vec{n}_2$. For example, consider the configurations below (which we have shaped as 5×5 arrays):

$$\vec{f} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix} \qquad \vec{g} = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Then $\vec{f}$ is winnable, while $\vec{g}$ is not ($\vec{g}$ is not perpendicular to $\vec{n}_2$).

Since the dimension of the null space is 2, and the scalar field is $\mathbb{Z}_2$, it follows from this theorem that of the 2^{25} possible configurations, only one-fourth of them are winnable. Furthermore, if $\vec{b}$ is a winnable configuration with winning strategy $\vec{x}$, then $\vec{x} + \vec{n}_1$, $\vec{x} + \vec{n}_2$ and $\vec{x} + \vec{n}_1 + \vec{n}_2$ are also winning strategies.

Suppose now that $\vec{b}$ is a winnable configuration. We would like to find one of the four strategies $\vec{x}$ for which $A\vec{x} = \vec{b}$. But since we need only find one solution, we may as well set the two free variables $x_{5,4}$ and $x_{5,5}$ equal to zero. In this case $\vec{x} = E\vec{x}$. So, $\vec{x} = E\vec{x} = RA\vec{x} = R\vec{b}$. Explicitly, we have a winning strategy given by $\vec{x} = R\vec{b}$. We thus have the following theorem:

THEOREM 2. *Suppose that $\vec{b}$ is a winnable configuration. Then the four winning strategies for $\vec{b}$ are*

$$R\vec{b}, \quad R\vec{b} + \vec{n}_1, \quad R\vec{b} + \vec{n}_2, \quad R\vec{b} + \vec{n}_1 + \vec{n}_2.$$

We observed above that the configuration $\vec{f}$ is winnable. To find a winning strategy, we compute $R\vec{f}$ (where we reshape $\vec{f}$ as a column vector):

$$R\vec{f} = (0,0,1,1,0,0,0,0,0,1,0,0,0,0,1,0,1,0,0,0,1,0,0,0,0)^T.$$

This theorem gives our solutions in closed, computable form. Admittedly, this computation is tedious to do by hand, preserving the game's appeal. We can do better than completing the entire computation, if we proceed algorithmically. For suppose we only compute the strategy for the first row (that is, the first five entries in the column $R\vec{b}$). We then carry out these moves; Theorem 2 says that no more moves in the first row are necessary. We then look to see if there are any lights on in the first row. The only way to turn these out, using moves in the last four rows, is to push the button immediately below each light which is on. Having now determined a strategy for the first two rows, we then move on to each successive row in the same way.

Lights Out can be generalized to an $n \times n$ array of lights. One can proceed in a manner similar to the way we solved the 5×5 case. What is interesting is the dimension of the null space of the corresponding matrices for various values of n (we call these $n^2 \times n^2$ matrices A_n); the table below summarizes the results.

Of course if the dimension of the null space is zero, every configuration is winnable and the solution unique (if no buttons are pressed more than once). We haven't spent any time trying to solve some of these larger puzzles, but they must be very difficult!

n	Dimension of Null(A_n)	n	Dimension of Null(A_n)
2	0	12	0
3	0	13	0
4	4	14	4
5	2	15	0
6	0	16	8
7	0	17	2
8	0	18	0
9	8	19	16
10	0	20	0
11	6	21	0

A further natural generalization is to consider *Lights Out* on a torus; that is, lights on the top row are considered neighbors of lights on the bottom row, and likewise for the leftmost and rightmost columns. This "wrap around" changes the matrices A_n, of course. (We leave this as an exercise for the reader.) Here are some corresponding results for the game on tori of various sizes:

n	Dimension of Null(A_n)	n	Dimension of Null(A_n)
2	0	12	16
3	4	13	0
4	0	14	0
5	8	15	12
6	8	16	0
7	0	17	16
8	0	18	8
9	4	19	0
10	16	20	32
11	0	21	4

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