A dual-weighted POD approach for 4D-Var adaptive mesh ocean modelling

F. Fang\textsuperscript{1}, C.C. Pain\textsuperscript{1}, I.M. Navon\textsuperscript{2}, M.D. Piggott\textsuperscript{1}, G.J. Gorman\textsuperscript{1}, P.A. Allison\textsuperscript{1} and A.J.H. Goddard\textsuperscript{1}

\textsuperscript{1} Applied Modelling and Computation Group
Department of Earth Science and Engineering
Imperial College London, UK

Under funding from NE/C52101X/1

\textsuperscript{2} School of Computational Science and Department of Mathematics
Florida State University
Outline of talk

• Aims and Objectives
• Proper Orthogonal Decomposition (POD) in ICOM
• Mesh adaptivity with POD
• Goal-based error measurement
• Test cases
• Conclusion
The two colours represent the different density. The heavier one sinks and runs under the lighter. Kelvin-Helmholtz billows are created at the interface due to shear.
Flow past a cylinder (Re = 100, Minimum mesh size= 0.04, Maximum mesh size =1) forward (top), POD (bottom) (20 basis functions, 41 snapshots)
Aims and Objectives

Develop a reduced order POD controller for a novel advanced mesh adaptive finite element model which includes many recent developments in ocean modelling.

In particular, our aim is to develop a new goal-based approach to:
• guide the mesh adaptivity and inversion;
• estimate error and optimise the POD bases.
What is Proper Orthogonal Decomposition (POD)?

POD- A numerical procedure that can be used to extract a basis for a model decomposition from an ensemble of signals.

-- originally proposed by Kosamibi, Loeve and Karhunen;
-- also known as Principal Components analysis (PCA) in statistics;
Empirical Orthogonal Function (EOF) in oceanography and meteorology.

The variables can be expressed as an expansion in $\Phi_k$

$$U^{POD}(t, x, y, z) = \bar{U}(x, y, z) + \sum_{m=1}^{M} \alpha_m(t) \Phi_m(x, y, z)$$

The original PDE $\rightarrow$ The reduced order ODE
POD-based reduced order model

Define the mean of variables $\mathbf{U} = (u, v, w, p)$:
$$\overline{\mathbf{U}} = \frac{1}{K} \sum_{i=1}^{K} \mathbf{U}^i$$

The spatial correlation matrix
$$H_{i,j} = \int_{\Omega} (\mathbf{U}^i - \overline{\mathbf{U}})(\mathbf{U}^j - \overline{\mathbf{U}}) d\Omega; \quad 1 \leq i, j \leq K$$

Solve the eigenvalue problem:
$$H \mathbf{v} = \lambda \mathbf{v}$$

The POD basis functions:
$$\Phi_m = \sum_{i=1}^{K} (v_m)_i (\mathbf{U}^i - \overline{\mathbf{U}}), \quad m = 1, \ldots, M$$

Energy:
$$I(M) = \frac{\sum_{i=1}^{M} \lambda_i}{\sum_{i=1}^{K} \lambda_i}$$

The variables can be expressed as an expansion in $\Phi_k$
$$\mathbf{U}^{POD}(t, x, y, z) = \overline{\mathbf{U}}(x, y, z) + \sum_{m=1}^{M} \alpha_m(t) \Phi_m(x, y, z)$$
Mesh adaptivity in the POD model

Challenge:

The snapshots can be of different length at different time levels
A functional can be defined as the model reduction error or solution which is of interest in a target region.

$$\mathcal{G}(\psi) = \int_{\Omega} f(\psi) \, dV.$$  

The functional is used to
- optimise uncertainties (inversion problem) in models;
- determine the error measure for mesh adaptivity;
- optimise the POD bases.
The goal-based error measure approach for mesh adaptivity

\[ \bar{M}_i = \frac{\gamma}{|\epsilon_i|} |\bar{H}_i| . \]

\[ \bar{M}_i^* = \frac{\gamma}{|\epsilon_i^*|} |\bar{H}_i^*| . \]

\[ \bar{M}_{ij}^{G_f} = G_f \left( \bar{M}_i, \bar{M}_i^* \right) . \]

To satisfy the goal, the minimal ellipsoid is obtained by superscribing both ellipses and used for mesh adaptivity.
A dual weighted method is developed to analyse the error of models and find an optimal POD basis.

To maximise the accuracy of the functional, a weighted diagonal Matrix is introduced to the snapshots

$$\tilde{V}_k = V_{k,i}, \ 1 \leq i \leq N$$

$$\tilde{A} = \begin{pmatrix} \tilde{\omega}_1^{\frac{1}{2}} V_1, \ldots, \tilde{\omega}_k \tilde{M}^{\frac{1}{2}} V_k, \ldots, \tilde{\omega}_K \tilde{M}^{\frac{1}{2}} V_K \end{pmatrix}$$

$$\tilde{\omega}_i^l = (\delta F_i^l)^{\frac{1}{2}}$$

$$\tilde{\omega}_i^l = (\delta F_i^l)^{\frac{1}{2}} = \left( \min \{ (\delta F_i^l)^{\text{forward}}, (\delta F_i^l)^{\text{adjoint}} \} \right)^{\frac{1}{2}}$$
Case: Gyre (Re = 400)

Computational domain: 1000 km x 1000 km

\[ \rho = 1000 \quad \beta = 1.8 \times 10^{-11} \quad \tau_0 = 0.1 \]

Assimilation period: 200 days    Time step: 3 hrs

Aim: to find an optimised mesh for the reduced forward and adjoint models

Run the reduced forward and adjoint models to find

\[ \bar{M}_i^{G_j} = G_f (\bar{M}_i, \bar{M}_i^*) . \]
Optimised adaptive mesh for the reduced order forward and adjoint models

Inversion

Vorticity
Optimised adaptive mesh for the reduced order forward and adjoint models
Case 4: Gyre – inversion of initial conditions

Objective Function used in Inversion

\[ \mathcal{S}(U^0) = \frac{1}{2} (U^0 - U_b)^T B^{-1} (U^0 - U_b) + \frac{1}{2} \sum_{n=1}^{NT} (HU^n - y_\circ^n)^T W_\circ (HU^n - y_\circ^n) \]

Spin-up period: 200 days

Simulation period: [200, 400] day

Time step: 6hrs

Pseudo-observations are u and v, which are available at t= 300 and 350 days.
Conclusion

The advantages of the POD model developed here over existing POD approaches are the ability to:

• Introduce adaptive meshes into the POD model;
• The goal-based error measure approach developed here can be used to
  (1) guide the mesh adaptivity and inversion;
  (2) optimise the POD bases (weight the snapshots).
Case: Gyre – inversion of initial conditions

Full model

Inversion model
The goal-based error measure approach for mesh adaptivity

\[ A\Psi - b = 0 \]

\[ A^T\Psi_{\text{exact}} - \frac{\partial \bar{\Omega}}{\partial \Psi} = 0. \]

\[ \bar{M}_i = \frac{\gamma}{|\bar{\varepsilon}_i|} \left| \bar{H}_i \right|. \]

\[ \bar{M}^*_i = \frac{\gamma}{|\bar{\varepsilon}^*_i|} \left| \bar{H}^*_i \right|. \]

\[ \bar{H}_i = \frac{1}{\sum_{n=1}^{NT} \sum_{i=1}^{M} |\lambda_{i,n}^l|} \sum_{n=1}^{NT} \sum_{l=1}^{M} |\lambda_{i,n}^l| \left| H_{i,n}^l \right| \]

\[ \bar{H}^* = \frac{1}{\sum_{n=1}^{NT} \sum_{l=1}^{M} |\lambda_{i,n}^l|} \sum_{n=1}^{NT} \sum_{l=1}^{M} |\lambda_{i,n}^l| \left| H_{i,n}^* \right|. \]

\[ |\bar{r}_i| = \left| \sum_{j \neq i} A_{i,j} \Psi_j + A_{i,i} \hat{\Psi}_i - b_i \right| \]

\[ \bar{\varepsilon}_i = \frac{\delta \bar{\Omega}}{\sum_{n=1}^{NT} \sum_{l=1}^{M} |\lambda_{i,n}^l|} \]

\[ \bar{\varepsilon}^*_i = \frac{\delta \bar{\Omega}}{\sum_{n=1}^{NT} \sum_{l=1}^{M} |\lambda_{i,n}^l|} \]