Theorem (Convergence Rate for Newton-Raphson Iteration) Assume that Newton-Raphson iteration produces a sequence \( (p_k)_{k=0}^{\infty} \) that converges to the root \( p \) of the function \( f(x) \).

If \( p \) is a simple root, then convergence is quadratic and
\[
|E_{k+1}| \approx \frac{|f'''(p)|}{2 |f''(p)|} (|E_k|)^2 \quad \text{for } k \text{ sufficiently large.}
\]

If \( p \) is a multiple root of order \( m \), then convergence is linear and
\[
|E_{k+1}| \approx \frac{m-1}{m} |E_k| \quad \text{for } k \text{ sufficiently large.}
\]

Proof.

Expand \( f(x) \) in a Taylor polynomial of degree \( n = 1 \), about \( x = p_k \) to get
\[
f(x) = f(p_k) + f'(p_k)(x - p_k) + \frac{1}{2!} f^{(2)}(c_k)(x - p_k)^2.
\]

Since \( p \) is a zero of \( f(x) \), set \( x = p \) in the above equation and obtain
\[
0 = f(p_k) + f'(p_k)(p - p_k) + \frac{1}{2!} f^{(2)}(c_k)(p - p_k)^2.
\]

Which can be rewritten as
\[
f(p_k) + f'(p_k)(p - p_k) = -\frac{f^{(2)}(c_k)}{2!} (p - p_k)^2.
\]

Now assume that \( f'(x) \neq 0 \) for all \( x \) near the root \( p \), and observe that \( f'(p_k) \neq 0 \), so that we can divide by it and obtain:
\[
\frac{f(p_k)}{f'(p_k)} + \frac{f'(p_k)}{f'(p_k)}(p - p_k) = -\frac{f^{(2)}(c_k)}{2 f'(p_k)} (p - p_k)^2.
\]

Rearrange the terms and simplify to get
\[ (p - p_k) + \frac{f'(p_k)}{f''(p_k)} = -\frac{f'''(c_k)}{2f''(p_k)} (p - p_k)^2. \]

The above equation can be rewritten as:

\[ p - \left( p_k - \frac{f'(p_k)}{f''(p_k)} \right) = -\frac{f'''(c_k)}{2f''(p_k)} (p - p_k)^2. \]

Now use the Newton-Raphson iteration formula and substitute it into the above equation to obtain:

\[ p - p_{k+1} = -\frac{f'''(c_k)}{2f''(p_k)} (p - p_k)^2. \]

Assuming \( f'(p_k) = f'(p) \) and \( f'''(c_k) = f'''(p) \) when \( k \) is sufficiently large yields

\[ p - p_{k+1} = -\frac{f'''(p)}{2f''(p)} (p - p_k)^2, \]

\[ E_{k+1} = -\frac{f'''(p)}{2f''(p)} (E_k)^2. \]

Now take absolute values and obtain the desired conclusion

\[ |E_{k+1}| \approx -\frac{|f'''(p)|}{|2f''(p)|} (|E_k|)^2. \]

Q. E. D.