Homework 14  (Equality Constrained Minimization)

1. Find a rectangle of sides a and b respectively that has the greatest possible area subject to a prescribed perimeter 2 (a+b).

2. Find the dimensions x, y and z of a box to maximize the volume of the box, given a fixed amount of cardboard of surface 72 sq. ft.

3. Consider an equality constrained problem with 2 equality constraints:

\[ h_1(x) = x_1^2 + x_2^2 + x_3^2 - 3 = 0 \]
\[ h_2(x) = 2x_1 - 4x_2 + x_3^2 + 1 = 0 \]

At the feasible point:
\[ x^*_r = (1,1,1)^T \]

Calculate gradients of the constraints at \( x^*_r \) and show that they are linearly independent, so \( x^*_r \) is a regular point.

4. Consider the problem :

Minimize:
\[-(x_1x_2 + x_2x_3 + x_1x_3)\]

Subject to
\[ x_1 + x_2 + x_3 = 3 \]

Write first order conditions and you find
\[ x_1^* = x_2^* = x_3^* = 1 \]
\[ \lambda^* = 2 \]

Write the Hessian of the Lagrangian \( \nabla^2_{xx} L(x^*, \lambda^*) \):
\[
\begin{bmatrix}
0 & -1 & -1 \\
-1 & 0 & -1 \\
-1 & -1 & 0
\end{bmatrix}
\]

Show that second order sufficiency conditions apply i.e.
\[ y \in V = \{ y \mid \nabla h(x^*)^T y = 0 \} = \{ y \mid y_1 + y_2 + y_3 = 0 \} \]

With \( y \neq 0 \)

\[ y^T \nabla^2_{xx} L(x^*, \lambda^*) y > 0 \] and sufficient conditions for second order equality constrained minimum are satisfied and \( x^* \) is a strict local minimum.