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Research Article

Mathematical Methods in the Applied Sciences

Received 30 November 2009 Published online in Wiley InterScience

(www.interscience.wiley.com) DOI: 10.1002/mma.1279
MOS subject classification: 65 N 55

Superconvergence analysis of FEMs for the Stokes–Darcy system

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We consider a superconvergence analysis for quadratic finite element approximations of the Stokes–Darcy system. The superclose property of an extra half order is proven for uniform triangular meshes. Based on the result of the superclose property, global superconvergence is derived by applying a postprocessing technique. In addition, some numerical examples are presented to demonstrate our theoretical results. Copyright © 2009 John Wiley & Sons, Ltd.

Keywords: Stokes and Darcy equations; superconvergence; Hood–Taylor element

1. Introduction

Recently, many researches focus on the Stokes–Darcy system, which arises in modeling the interaction between surface water and groundwater, well-reservoir coupling in petroleum engineering, and computational fuel cell dynamics; see [1–13] and the references therein. Among these works, different topics are touched on: mathematical justifications for the interface boundary condition [1], robust finite element constructions [2, 12], least-squares formulations [10], well-posedness and convergence of finite element methods [4, 3], locally conservative coupling [7], mortar discretization [8], two-grid method [11], and domain decomposition methods [6, 9]. In this paper, we discuss the superconvergence of the Stokes–Darcy system.

The superconvergence analysis of finite element methods has been studied for almost 40 years; see, e.g., the early papers [14, 15]. It is well known that superconvergence analysis is a powerful tool for improving computational accuracy and efficiency. Using the superconvergence technique and suitable postprocessing, the order of the convergence can be improved from one half to two orders, depending on the setting. Thus, computational accuracy and efficiency can be improved greatly with relatively few additional computing costs. It has been shown that the superconvergence analysis is valid not only for the standard finite element schemes for elliptic partial differential equations, but also can be extended to nonconforming finite element methods, mixed finite elements, the Stokes equations, parabolic equations, integral equations, and integral–differential equations. Related works can be found in, among other papers, [16–22], and the references cited therein.

In this paper, we prove the superclose property of the Hood–Taylor element on uniform isosceles right-triangle meshes for the Stokes equations and the Stokes–Darcy system. Based on the superclose property and a postprocessing technique, global superconvergence is obtained; the convergence order is improved from $O(h^2)$ to $O(h^{2.5})$ with respect to the H^1-norm. For the L^2-norm, the convergence order is improved from $O(h^3)$ to $O(h^{3.5})$. It should be pointed that although there are many superconvergence results for the Stokes equations (see, e.g. [23–27]), to our knowledge, there are no such superconvergence results for the Hood–Taylor element for the Stokes equations. Therefore, the superconvergence results provided in this paper are new not only for the Stokes–Darcy system, but also for the Stokes equations. In order to prove the superconvergence for the Hood–Taylor element, we use a new technique to prove the weak error estimate between the pressure p and its L^2-projection; see Section 2. This technique has not previously been applied in the standard superconvergence analysis.

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Contract/grant sponsor: National Basic Research Program; contract/grant number: 2005CB321701, 010CB731505
Contract/grant sponsor: National Talents Training Base for Basic research and Teaching of Natural Science of China; contract/grant number: J0730103
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The paper is organized as follows. In Section 2, we provide the superconvergence analysis for the Stokes equations. The superconvergence property of the Hood–Taylor element on uniform isosceles right-triangle meshes is proven. In Section 3, we extend the results of Section 2 to the Stokes–Darcy system. In Section 4, some numerical examples are presented to illustrate the theoretical results.

2. Superconvergence analysis for the Stokes equations

Consider the Stokes system

$$-\Delta \boldsymbol{u}+\nabla p=\boldsymbol{f} \quad \text{in } \Omega, \tag{1}$$

$$\mathrm{div}\boldsymbol{u}=0 \quad \text{in } \Omega,$$

$$\boldsymbol{u}=0 \quad \text{on } \partial\Omega, \tag{2}$$

and $\int_\Omega p\,\mathrm{d}x=0$ so that p is uniquely determined. A variational formulation is given by: find $(\boldsymbol{u},p)\in \boldsymbol{V}\times Q$ such that

$$a(\boldsymbol{u},\boldsymbol{v})+b(\boldsymbol{v},p)=(\boldsymbol{f},\boldsymbol{v}), \quad \forall \boldsymbol{v}\in \boldsymbol{V}, \tag{3}$$

$$b(\boldsymbol{u},q)=0, \quad \forall q\in Q, \tag{4}$$

where $\boldsymbol{V}=(H_0^1(\Omega))^2$, $Q=L_0^2(\Omega)=\{q\in L^2(\Omega):\int_\Omega q=0\}$, and

$$a(\boldsymbol{u},\boldsymbol{v})=\int_\Omega \nabla\boldsymbol{u}:\nabla\boldsymbol{v}\,\mathrm{d}x, \quad b(\boldsymbol{v},q)=-\int_\Omega q\,\mathrm{div}\boldsymbol{v}\,\mathrm{d}x, \quad (\boldsymbol{f},\boldsymbol{v})=\int_\Omega \boldsymbol{f}\boldsymbol{v}\,\mathrm{d}x.$$

Finite element approximations of (3)–(4) are defined by: find $(\boldsymbol{u}_h,p_h)\in \boldsymbol{V}_0^h\times Q_0^h$ such that

$$a(\boldsymbol{u}_h,\boldsymbol{v}_h)+b(\boldsymbol{v}_h,p_h)=(\boldsymbol{f},\boldsymbol{v}_h), \quad \forall \boldsymbol{v}_h\in \boldsymbol{V}_0^h, \tag{5}$$

$$b(\boldsymbol{u}_h,q_h)=0, \quad \forall q_h\in Q_0^h, \tag{6}$$

where $\boldsymbol{V}_0^h\times Q_0^h\subset \boldsymbol{V}\times Q$ denotes a finite element space. In this section, we use the Hood–Taylor element, i.e.

$$V^h=\{v\in C(\Omega):v|_\tau\in P_2(\tau),\ \forall\tau\in T^h\}, \quad \boldsymbol{V}^h=(V^h)^2,$$

$$Q^h=\{q\in C(\Omega):q|_\tau\in P_1(\tau),\ \forall\tau\in T^h\},$$

where P_i denotes the space of polynomials of order i and T^h denotes the mesh on the domain Ω. In this section, we let T^h be a uniform isosceles right-triangular mesh. Moreover, set $V_0^h=V^h\cap H_0^1(\Omega)$, $\boldsymbol{V}_0^h=(V_0^h)^2$, and $Q_0^h=Q^h\cap L_0^2(\Omega)$. It has been proved (see, e.g. [28]) that, for the Hood–Taylor element, the LBB condition

$$\inf_{\substack{(\boldsymbol{v},q)\in \boldsymbol{V}_0^h\times Q_0^h\\ (\boldsymbol{v},q)\neq(\boldsymbol{\theta},0)}}\ \sup_{\substack{(\boldsymbol{w},r)\in \boldsymbol{V}_0^h\times Q_0^h\\ (\boldsymbol{w},r)\neq(\boldsymbol{\theta},0)}} \frac{A((\boldsymbol{v},q);(\boldsymbol{w},r))}{(\|\boldsymbol{v}\|_{1,\Omega}+\|q\|_{0,\Omega})(\|\boldsymbol{w}\|_{1,\Omega}+\|r\|_{0,\Omega})}\geqslant c \tag{7}$$

holds, where $\boldsymbol{\theta}=(0,0)$,

$$A((\boldsymbol{v},q);(\boldsymbol{w},r))=a(\boldsymbol{v},\boldsymbol{w})+b(\boldsymbol{w},q)-b(\boldsymbol{v},r).$$

Let Π_h denote the interpolation operator from $C(\Omega)$ to V^h such that, for all $v\in C(\Omega)$,

$$\Pi_h v(z_i)=v(z_i), \quad \int_{l_i}(\Pi_h v-v)\,\mathrm{d}s=0, \quad i=1,2,3,$$

where z_i and l_i, $i=1,2,3$, denote the vertices and edges of a finite element, respectively. For this interpolation operator, the following weak estimates have been proved in [18].

Lemma 2.1
Let $\boldsymbol{V}^h$ denote the continuous quadratic triangular finite element space on a uniform isosceles right-triangle mesh T^h and let the interpolation operator Π_h be defined as above. Assume that $\boldsymbol{u}\in(H_0^1(\Omega)\cap H^4(\Omega))^2$. Then, we have that (see [18, pp. 231–237])

$$|a(\boldsymbol{u}-\Pi_h\boldsymbol{u},\boldsymbol{v}_h)|\leqslant Ch^3\|\boldsymbol{u}\|_{4,\Omega}\|\boldsymbol{v}_h\|_{1,\Omega}, \quad \forall \boldsymbol{v}_h\in \boldsymbol{V}_0^h, \tag{8}$$

and

$$|b(\boldsymbol{u}-\Pi_h\boldsymbol{u},q_h)|\leqslant Ch^{2.5}\|\boldsymbol{u}\|_{4,\Omega}\|q_h\|_{0,\Omega}, \quad \forall q_h\in Q^h. \tag{9}$$

Remark 2.1
For Lemma 2.1, when the condition $\boldsymbol{v}_h \in \boldsymbol{V}_0^h$ is weakened to $\boldsymbol{v}_h \in \boldsymbol{V}^h$, the estimate (8) should be replaced by

$$|a(\Pi_h \boldsymbol{u} - \boldsymbol{u}, \boldsymbol{v}_h)| \leqslant Ch^{2.5} \|\boldsymbol{u}\|_{4,\Omega} \|\boldsymbol{v}_h\|_{1,\Omega}, \quad \forall \boldsymbol{v}_h \in \boldsymbol{V}^h. \tag{10}$$

Furthermore, from the proof of the above results in [18, 19], it can be shown that the estimates (10) and (9) can be, respectively, replaced by

$$|a(\boldsymbol{u} - \Pi_h \boldsymbol{u}, \boldsymbol{v}_h)| \leqslant Ch^{3.5} \|\boldsymbol{u}\|_{4,\Omega} \|\boldsymbol{v}_h\|_{2,\Omega}, \quad \forall \boldsymbol{v}_h \in \boldsymbol{V}^h, \tag{11}$$

and

$$|b(\boldsymbol{u} - \Pi_h \boldsymbol{u}, q_h)| \leqslant Ch^{3.5} \|\boldsymbol{u}\|_{4,\Omega} \|q_h\|_{1,\Omega}, \quad \forall q_h \in Q^h. \tag{12}$$

Next, let us consider the weak error estimate of the L^2-projection based on the integral identities provided in [18]. The result of this lemma is the key result for the superconvergence analysis in this paper; the technique used in the proof of this lemma is different with that used in standard superconvergence analyses.

Lemma 2.2
Let P_h denote the L^2-projection operator from $L^2(\Omega)$ to the continuous piecewise linear finite element space Q^h on a uniform isosceles right-triangle mesh T^h. Assume that $p \in H^3(\Omega)$. Then, we have

$$|b(\boldsymbol{v}_h, p - P_h p)| \leqslant Ch^{2.5} \|p\|_{3,\Omega} \|\boldsymbol{v}_h\|_{1,\Omega}, \quad \forall \boldsymbol{v}_h \in \boldsymbol{V}^h. \tag{13}$$

Proof
Let I_h denote the standard piecewise linear Lagrange interpolant on Q^h. It has been proved that [18, pp. 235-237]

$$b(\boldsymbol{v}_h, p - I_h p) = -(p - I_h p, \operatorname{div} \boldsymbol{v}_h) = \frac{h^2}{12}(p_{xx} + p_{yy} - p_{xy}, \operatorname{div} \boldsymbol{v}_h) + O(h^{2.5}) \|p\|_{3,\Omega} \|\boldsymbol{v}_h\|_{1,\Omega}, \quad \forall \boldsymbol{v}_h \in \boldsymbol{V}^h, \tag{14}$$

where $(\cdot,\cdot)$ denotes the standard L^2-inner product. By the same way, it can be proved that

$$(p - I_h p, w_h) = \frac{h^2}{12}(p_{xy} - p_{xx} - p_{yy}, w_h) + O(h^{2.5}) \|p\|_{3,\Omega} \|w_h\|_{0,\Omega}, \quad \forall w_h \in Q^h. \tag{15}$$

Let P_h denote the L^2-projection operator from $L^2(\Omega)$ to Q^h, such that

$$(p - P_h p, w_h) = 0, \quad \forall w_h \in Q^h. \tag{16}$$

Then, it follows from (15) and (16) that, for all $w_h \in Q^h$,

$$\begin{aligned}(I_h p - P_h p, w_h) = (I_h p - p, w_h) &= -\frac{h^2}{12}(p_{xy} - p_{xx} - p_{yy}, w_h) + O(h^{2.5}) \|p\|_{3,\Omega} \|w_h\|_{0,\Omega} \\ &= -\frac{h^2}{12}(P_h(p_{xy} - p_{xx} - p_{yy}), w_h) + O(h^{2.5}) \|p\|_{3,\Omega} \|w_h\|_{0,\Omega}.\end{aligned} \tag{17}$$

Setting

$$w_h = I_h p - P_h p + \frac{h^2}{12} P_h(p_{xy} - p_{xx} - p_{yy}),$$

the identity (17) implies that

$$\left\| I_h p - P_h p + \frac{h^2}{12} P_h(p_{xy} - p_{xx} - p_{yy}) \right\|_{0,\Omega} \leqslant Ch^{2.5} \|p\|_{3,\Omega}. \tag{18}$$

Moreover, it is easy to see that

$$b(\boldsymbol{v}_h, I_h p - P_h p) = -(I_h p - P_h p, \operatorname{div} \boldsymbol{v}_h) = -\left(I_h p - P_h p + \frac{h^2}{12} P_h(p_{xy} - p_{xx} - p_{yy}), \operatorname{div} \boldsymbol{v}_h \right) + \left(\frac{h^2}{12} P_h(p_{xy} - p_{xx} - p_{yy}), \operatorname{div} \boldsymbol{v}_h \right), \tag{19}$$

and it follows from (18) that

$$\left(I_h p - P_h p + \frac{h^2}{12} P_h(p_{xy} - p_{xx} - p_{yy}), \operatorname{div} \boldsymbol{v}_h \right) \leqslant \left\| I_h p - P_h p + \frac{h^2}{12} P_h(p_{xy} - p_{xx} - p_{yy}) \right\|_{0,\Omega} \|\operatorname{div} \boldsymbol{v}_h\|_{0,\Omega} \leqslant Ch^{2.5} \|p\|_{3,\Omega} \|\operatorname{div} \boldsymbol{v}_h\|_{0,\Omega}. \tag{20}$$

Summing up, it follows from (14), (19), and (20) that

$$
\begin{aligned}
b(\boldsymbol{v}_h, p-P_hp) &= b(\boldsymbol{v}_h, p-I_hp)+b(\boldsymbol{v}_h, I_hp-P_hp)\\
&= \frac{h^2}{12}(p_{xx}+p_{yy}-p_{xy}, \mathrm{div}\boldsymbol{v}_h)+O(h^{2.5})\|p_3\|_{3,\Omega}\|\boldsymbol{v}_h\|_{1,\Omega}-\left(I_hp-P_hp+\frac{h^2}{12}P_h(p_{xy}-p_{xx}-p_{yy}), \mathrm{div}\boldsymbol{v}_h\right)\\
&\quad+\frac{h^2}{12}(P_h(p_{xy}-p_{xx}-p_{yy}), \mathrm{div}\boldsymbol{v}_h)\\
&= \frac{h^2}{12}(P_h(p_{xy}-p_{xx}-p_{yy})-(p_{xy}-p_{xx}-p_{yy}), \mathrm{div}\boldsymbol{v}_h)+O(h^{2.5})\|p\|_{3,\Omega}\|\boldsymbol{v}_h\|_{1,\Omega}+O(h^{2.5})\|p\|_{3,\Omega}\|\mathrm{div}\boldsymbol{v}_h\|_{0,\Omega}\\
&\leqslant Ch^3\|p_{xy}-p_{xx}-p_{yy}\|_{1,\Omega}\|\mathrm{div}\boldsymbol{v}_h\|_{0,\Omega}+Ch^{2.5}\|p\|_{3,\Omega}\|\boldsymbol{v}_h\|_{1,\Omega}\\
&\leqslant Ch^{2.5}\|p\|_{3,\Omega}\|\boldsymbol{v}_h\|_{1,\Omega}.
\end{aligned}\tag{21}
$$

This proves (13). □

Theorem 1
Let $(\boldsymbol{u}, p)$ and $(\boldsymbol{u}_h, p_h)$ denote the solutions of (3)–(4) and (5)–(6), respectively. Assume that $\boldsymbol{u}\in(H^4(\Omega))^2$, $p\in H^3(\Omega)$, and T^h is a uniform isosceles right-triangle mesh. Then, we have that

$$\|\Pi_h\boldsymbol{u}-\boldsymbol{u}_h\|_{1,\Omega}+\|P_hp-p_h\|_{0,\Omega}\leqslant Ch^{2.5}(\|\boldsymbol{u}\|_{4,\Omega}+\|p\|_{3,\Omega}), \tag{22}$$

where Π_h and P_h, respectively, denote the interpolation operator and L^2-projection operator defined in this section.

Proof
Note that $\Pi_h\boldsymbol{u}\in\boldsymbol{V}_0^h$ and $P_hp\in Q_0^h$. The LBB condition (7) implies that there exists $(\boldsymbol{v}_h, q_h)\in\boldsymbol{V}_0^h\times Q_0^h$ such that

$$c(\|\Pi_h\boldsymbol{u}-\boldsymbol{u}_h\|_{1,\Omega}+\|P_hp-p_h\|_{0,\Omega})\leqslant\frac{A((\Pi_h\boldsymbol{u}-\boldsymbol{u}_h, P_hp-p_h);(\boldsymbol{v}_h, q_h))}{\|\boldsymbol{v}_h\|_{1,\Omega}+\|q_h\|_{0,\Omega}}. \tag{23}$$

Moreover, it follows from (3)–(4) and (5)–(6) that

$$
\begin{aligned}
A((\Pi_h\boldsymbol{u}-\boldsymbol{u}_h, P_hp-p_h);(\boldsymbol{v}_h, q_h)) &= a(\Pi_h\boldsymbol{u}-\boldsymbol{u}_h, \boldsymbol{v}_h)+b(\boldsymbol{v}_h, P_hp-p_h)+b(\Pi_h\boldsymbol{u}-\boldsymbol{u}_h, q_h)\\
&= a(\Pi_h\boldsymbol{u}-\boldsymbol{u}, \boldsymbol{v}_h)+b(\boldsymbol{v}_h, P_hp-p)+b(\Pi_h\boldsymbol{u}-\boldsymbol{u}, q_h).
\end{aligned}\tag{24}
$$

Thus, (22) follows from (23), (24), and Lemmas 2.1 and 2.2. □

Theorem 2
Let $(\boldsymbol{u}, p)$ and $(\boldsymbol{u}_h, p_h)$ denote the solutions of (3)–(4) and (5)–(6), respectively. Assume that $\boldsymbol{u}\in(H^4(\Omega))^2$, $p\in H^3(\Omega)$, T^h is a uniform isosceles right-triangle mesh, and Ω is convex. Then, we have that

$$\|\Pi_h\boldsymbol{u}-\boldsymbol{u}_h\|_{0,\Omega}\leqslant Ch^{3.5}(\|\boldsymbol{u}\|_{4,\Omega}+\|p\|_{3,\Omega}), \tag{25}$$

where Π_h denotes the interpolation operator defined in this section.

Proof
For any function $\boldsymbol{\phi}\in(L^2(\Omega))^2$, let $(\boldsymbol{\psi}, \rho)\in\boldsymbol{V}\times Q$ denote the solution of the following auxiliary equations:

$$a(\boldsymbol{\psi}, \boldsymbol{v})+b(\boldsymbol{v}, \rho) = (\boldsymbol{\phi}, \boldsymbol{v}), \quad \forall\boldsymbol{v}\in\boldsymbol{V}, \tag{26}$$

$$b(\boldsymbol{\psi}, q) = 0, \quad \forall q\in Q. \tag{27}$$

Then, we have (see, e.g. [28]) that

$$\|\boldsymbol{\psi}\|_{2,\Omega}+\|\rho\|_{1,\Omega}\leqslant C\|\boldsymbol{\phi}\|_{0,\Omega}. \tag{28}$$

Set $\boldsymbol{\phi}=\Pi_h\boldsymbol{u}-\boldsymbol{u}_h$, $\theta=P_hp-p_h$, and let $\boldsymbol{v}=\boldsymbol{\phi}$ and $q=\theta$ in (26)–(27). We have that

$$
\begin{aligned}
\|\boldsymbol{\phi}\|_{0,\Omega}^2 &= a(\boldsymbol{\psi}, \boldsymbol{\phi})+b(\boldsymbol{\phi}, \rho)+b(\boldsymbol{\psi}, \theta)\\
&= a(\boldsymbol{\phi}, \boldsymbol{\psi}-\Pi_h\boldsymbol{\psi})+b(\boldsymbol{\phi}, \rho-P_h\rho)+b(\boldsymbol{\psi}-\Pi_h\boldsymbol{\psi}, \theta)+a(\boldsymbol{\phi}, \Pi_h\boldsymbol{\psi})+b(\Pi_h\boldsymbol{\psi}, \theta)+b(\boldsymbol{\phi}, P_h\rho).
\end{aligned}\tag{29}
$$

From Theorem 1, standard interpolation error estimates (see, e.g. [28]), and (28), we can deduce that

$$
\begin{aligned}
&|a(\boldsymbol{\phi},\boldsymbol{\psi}-\Pi_h\boldsymbol{\psi})+b(\boldsymbol{\phi},\rho-P_h\rho)+b(\boldsymbol{\psi}-\Pi_h\boldsymbol{\psi},\theta)|\\
&\quad\leqslant C(\|\boldsymbol{\phi}\|_{1,\Omega}+\|\theta\|_{0,\Omega})(\|\boldsymbol{\psi}-\Pi_h\boldsymbol{\psi}\|_{1,\Omega}+\|\rho-P_h\rho\|_{0,\Omega})\\
&\quad\leqslant Ch^{2.5}(\|\boldsymbol{u}\|_{4,\Omega}+\|p\|_{3,\Omega})h(\|\boldsymbol{\psi}\|_{2,\Omega}+\|\rho\|_{1,\Omega})\\
&\quad\leqslant Ch^{3.5}(\|\boldsymbol{u}\|_{4,\Omega}+\|p\|_{3,\Omega})\|\boldsymbol{\phi}\|_{0,\Omega}.
\end{aligned}
\tag{30}
$$

Moreover, it follows from (3)–(4) and (5)–(6) that

$$
\begin{aligned}
a(\boldsymbol{\phi},\Pi_h\boldsymbol{\psi})+b(\Pi_h\boldsymbol{\psi},\theta)+b(\boldsymbol{\phi},P_h\rho)&=a(\Pi_h\boldsymbol{u}-\boldsymbol{u}_h,\Pi_h\boldsymbol{\psi})+b(\Pi_h\boldsymbol{\psi},P_hp-p_h)+b(\Pi_h\boldsymbol{u}-\boldsymbol{u}_h,P_h\rho)\\
&=a(\Pi_h\boldsymbol{u}-\boldsymbol{u},\Pi_h\boldsymbol{\psi})+b(\Pi_h\boldsymbol{\psi},P_hp-p)+b(\Pi_h\boldsymbol{u}-\boldsymbol{u},P_h\rho).
\end{aligned}
\tag{31}
$$

The weak error estimates (11)–(12) and the *a priori* estimate (28) imply that

$$
\begin{aligned}
|a(\Pi_h\boldsymbol{u}-\boldsymbol{u},\Pi_h\boldsymbol{\psi})+b(\Pi_h\boldsymbol{u}-\boldsymbol{u},P_h\rho)|&\leqslant Ch^{3.5}\|\boldsymbol{u}\|_{4,\Omega}(\|\Pi_h\boldsymbol{\psi}\|_{2,\Omega}+\|P_h\rho\|_{1,\Omega})\\
&\leqslant Ch^{3.5}\|\boldsymbol{u}\|_{4,\Omega}(\|\boldsymbol{\psi}\|_{2,\Omega}+\|\rho\|_{1,\Omega})\\
&\leqslant Ch^{3.5}\|\boldsymbol{u}\|_{4,\Omega}\|\boldsymbol{\phi}\|_{0,\Omega}.
\end{aligned}
\tag{32}
$$

Then the only issue left is to estimate the term $b(\Pi_h\boldsymbol{\psi},P_hp-p)$. Note that

$$
\begin{aligned}
b(\Pi_h\boldsymbol{\psi},P_hp-p)&=-(P_hp-p,\operatorname{div}(\Pi_h\boldsymbol{\psi}))\\
&=-(P_hp-p,\operatorname{div}(\Pi_h\boldsymbol{\psi})-P_h(\operatorname{div}\boldsymbol{\psi})).
\end{aligned}
\tag{33}
$$

Similar to Lemma 2.2, it can be proved that

$$
|(P_hp-p,\operatorname{div}(\Pi_h\boldsymbol{\psi})-P_h(\operatorname{div}\boldsymbol{\psi}))|\leqslant Ch^{2.5}\|p\|_{3,\Omega}\|\operatorname{div}(\Pi_h\boldsymbol{\psi})-P_h(\operatorname{div}\boldsymbol{\psi})\|_{0,\Omega}+Ch^{3.5}\|p\|_{3,\Omega}\|\operatorname{div}(\Pi_h\boldsymbol{\psi})-P_h(\operatorname{div}\boldsymbol{\psi})\|_{1,\Omega}. \tag{34}
$$

Moreover, it is easy to see that

$$
\|\operatorname{div}(\Pi_h\boldsymbol{\psi})-P_h(\operatorname{div}\boldsymbol{\psi})\|_{0,\Omega}\leqslant\|\operatorname{div}(\Pi_h\boldsymbol{\psi})-\operatorname{div}\boldsymbol{\psi}\|_{0,\Omega}+\|\operatorname{div}\boldsymbol{\psi}-P_h(\operatorname{div}\boldsymbol{\psi})\|_{0,\Omega}\leqslant Ch\|\boldsymbol{\psi}\|_{2,\Omega}\leqslant Ch\|\boldsymbol{\phi}\|_{0,\Omega}, \tag{35}
$$

and

$$
\|\operatorname{div}(\Pi_h\boldsymbol{\psi})-P_h(\operatorname{div}\boldsymbol{\psi})\|_{1,\Omega}\leqslant C\|\boldsymbol{\psi}\|_{2,\Omega}\leqslant C\|\boldsymbol{\phi}\|_{0,\Omega}. \tag{36}
$$

Summing up, it follows from (33)–(36) that

$$
|b(\Pi_h\boldsymbol{\psi},P_hp-p)|\leqslant Ch^{3.5}\|p\|_{3,\Omega}\|\boldsymbol{\phi}\|_{0,\Omega}. \tag{37}
$$

Thus, (25) can be proved from (29)–(32) and (37). □

In order to obtain the global superconvergence based on the superconvergence analysis in Theorems 1 and 2, we introduce the following interpolation postprocessing operator Π^*_{2h} (see, e.g. [18, 19]). Let T^{2h} denote a uniform isosceles right-triangle mesh with mesh size $2h$ and let T^h be constructed by dividing every element τ in T^{2h} (we refer to it as a macro-element) into four equal elements e^i_τ, $i=1,2,3,4$, in T^h. Then, for any function $w\in C(\Omega)$, define the interpolation postprocessing operator Π^*_{2h} to be such that $\Pi^*_{2h}w$ is a polynomial of order 4 on the macro element τ and

$$
\Pi^*_{2h}w(z_k)=w(z_k),\quad k=1,\dots,6,\qquad \int_{l_j}(\Pi^*_{2h}w-w)\,\mathrm{d}s=0,\quad j=1,\dots,9,
$$

where z_k, $k=1,\dots,6$, and l_j, $j=1,\dots,9$, denote the vertices and edges of the elements e^i_τ, $i=1,2,3,4$, respectively. Using the above interpolation postprocessing operator, we can obtain the following global superconvergence results.

Theorem 3
Let $(\boldsymbol{u},p)$ and $(\boldsymbol{u}_h,p_h)$ be the solutions of (3)–(4) and (5)–(6), respectively. Let Π^*_{2h} be the interpolation postprocessing operator defined above. Assume that all the hypotheses of Theorems 1 and 2 are valid. Then, we have that

$$
\|\Pi^*_{2h}\boldsymbol{u}_h-\boldsymbol{u}\|_{1,\Omega}\leqslant Ch^{2.5}(\|\boldsymbol{u}\|_{4,\Omega}+\|p\|_{3,\Omega}). \tag{38}
$$

Moreover, if Ω is convex, then

$$
\|\Pi^*_{2h}\boldsymbol{u}_h-\boldsymbol{u}\|_{0,\Omega}\leqslant Ch^{3.5}(\|\boldsymbol{u}\|_{4,\Omega}+\|p\|_{3,\Omega}). \tag{39}
$$

Proof

It follows from the definitions of the interpolation operators Π_h and Π_{2h}^* that

$$\Pi_{2h}^*\Pi_h w = \Pi_{2h}^* w, \quad \forall w \in C(\Omega), \tag{40}$$

$$\|\Pi_{2h}^* w - w\|_{k,\Omega} \leqslant Ch^{4-k}\|w\|_{4,\Omega}, \quad \forall w \in H^4(\Omega), \ \ k=0,1, \tag{41}$$

and for any finite element space V^h,

$$\|\Pi_{2h}^* w_h\|_{k,\Omega} \leqslant C\|w_h\|_{k,\Omega}, \quad \forall w_h \in V^h, \ \ k=0,1. \tag{42}$$

Then, (40)–(42) and Theorem 1 imply that

$$\begin{aligned}\|\Pi_{2h}^*\boldsymbol{u}_h - \boldsymbol{u}\|_{1,\Omega} &\leqslant \|\Pi_{2h}^*\boldsymbol{u}_h - \Pi_{2h}^*(\Pi_h\boldsymbol{u})\|_{1,\Omega} + \|\Pi_{2h}^*(\Pi_h\boldsymbol{u}) - \Pi_{2h}^*\boldsymbol{u}\|_{1,\Omega} + \|\Pi_{2h}^*\boldsymbol{u} - \boldsymbol{u}\|_{1,\Omega}\\ &\leqslant C\|\boldsymbol{u}_h - \Pi_h\boldsymbol{u}\|_{1,\Omega} + 0 + Ch^3\|\boldsymbol{u}\|_{4,\Omega}\\ &\leqslant Ch^{2.5}(\|\boldsymbol{u}\|_{4,\Omega} + \|p\|_{3,\Omega}) + Ch^3\|\boldsymbol{u}\|_{4,\Omega}\\ &\leqslant Ch^{2.5}(\|\boldsymbol{u}\|_{4,\Omega} + \|p\|_{3,\Omega}).\end{aligned}$$

This proves (38). The estimate (39) can be proved in a similar manner. □

Remark 2.2

Note that the integral identities (10)–(12) and Lemma 2.2 can be proven without the zero boundary condition assumption, i.e. without assuming that $\boldsymbol{v} \in \boldsymbol{V}_0^h$. Then, the superconvergence results in Theorems 1–3 can be extended to the Stokes equations with the Neumann or the Robin boundary conditions without any difficulty. Also note that, according to the experiences with superconvergence analyses for the elliptic equations, for the Dirichlet boundary conditions, we would expect that the superconvergence order of the uniform piecewise quadratic finite element solution should be $O(h^3)$ for the H^1-norm and $O(h^4)$ for the L^2-norm, instead of the $O(h^{2.5})$ and $O(h^{3.5})$, respectively, that we have obtained. For the Stokes problem, we have not proven these improved results in this paper, but the numerical results given in Section 4 indicate that these improved estimates do hold.

Remark 2.3

We obtained the global superconvergence results in Theorem 3 using a postprocessing technique and the superclose results provided in Theorems 1 and 2. Unfortunately, we cannot have a similar result for the pressure p, although we have obtained the superclose result in Theorem 1. This is because that the superclose result obtained in Theorem 1 is about supercloseness between p_h and $P_h p$, and we do not know how to construct a suitable postprocessing operator to obtain the global superconvergence using the result of supercloseness, as the operator P_h is global. It can be shown that there is no superconvergence between the finite element solution and the Lagrange interplant of the exact solution from the proof of this section; the numerical examples in Section 4 confirm this observation.

3. Superconvergence analysis for the Stokes–Darcy system

In this section, we will extend the results of the last section to the Stokes–Darcy system. In the Stokes–Darcy system, the domain $\Omega = \Omega_\mathrm{p} \cup \Omega_\mathrm{f}$. In the porous media region Ω_p, the fluid velocity $\mathbf{u}_\mathrm{p}$ and the hydraulic head ϕ_p satisfy the Darcy equations

$$\mathbf{u}_\mathrm{p} = -\mathbb{K}\nabla\phi_\mathrm{p} \quad \text{and} \quad \nabla\cdot\mathbf{u}_\mathrm{p} = 0, \tag{43}$$

where $\mathbb{K}$ is the hydraulic conductivity, which is a positive-definite symmetric matrix. In the fluid region Ω_f, the fluid flow satisfies the Stokes equations

$$-\nabla\cdot\mathbb{T}(\mathbf{u}_\mathrm{f}, p_\mathrm{f}) = \mathbf{f}_\mathrm{f} \quad \text{and} \quad \nabla\cdot\mathbf{u}_\mathrm{f} = 0, \tag{44}$$

where $\mathbf{u}_\mathrm{f}$ is the fluid velocity, p_f is the kinematic pressure, $\mathbf{f}_\mathrm{f}$ is the external body force, v the kinematic viscosity of the fluid, $\mathbb{T}(\mathbf{u}_\mathrm{f}, p_\mathrm{f}) = 2v\mathbb{D}(\mathbf{u}_\mathrm{f}) - p_\mathrm{f}\mathbb{I}$ is the stress tensor, and $\mathbb{D}(\mathbf{u}_\mathrm{f}) = \frac{1}{2}(\nabla\mathbf{u}_\mathrm{f} + \nabla^\mathrm{T}\mathbf{u}_\mathrm{f})$ is the deformation tensor.

Along the interface $\Gamma = \bar{\Omega}_\mathrm{p} \cap \bar{\Omega}_\mathrm{f}$, the Beavers–Joseph–Saffman–Jones interface boundary condition is imposed:

$$\mathbf{u}_\mathrm{f}\cdot\mathbf{n}_\mathrm{f} = -\mathbf{u}_\mathrm{p}\cdot\mathbf{n}_\mathrm{p}, \tag{45}$$

$$-\tau_\mathrm{f}\cdot(\mathbb{T}(\mathbf{u}_\mathrm{f}, p_\mathrm{f})\cdot\mathbf{n}_\mathrm{f}) = \alpha\tau_\mathrm{f}\cdot\mathbf{u}_\mathrm{f}, \tag{46}$$

$$-\mathbf{n}_\mathrm{f}\cdot(\mathbb{T}(\mathbf{u}_\mathrm{f}, p_\mathrm{f})\cdot\mathbf{n}_\mathrm{f}) = g\phi_\mathrm{p}, \tag{47}$$

where τ_f is the tangential direction on Γ and g, α, and v are positive constants.

The weak form of the Stokes–Darcy system is as follows:

$$a_{\mathrm{f}}(\boldsymbol{u}_{\mathrm{f}},\boldsymbol{v}_{\mathrm{f}})+b_{\mathrm{f}}(\boldsymbol{v}_{\mathrm{f}},p_{\mathrm{f}})+ga_{\mathrm{p}}(\phi_{\mathrm{p}},\psi_{\mathrm{p}})+\langle g\phi_{\mathrm{p}},\boldsymbol{v}_{\mathrm{f}}\cdot\boldsymbol{n}_{\mathrm{f}}\rangle-\langle g\boldsymbol{u}_{\mathrm{f}}\cdot\boldsymbol{n}_{\mathrm{f}},\psi_{\mathrm{p}}\rangle+\alpha\langle P_\tau\boldsymbol{u}_{\mathrm{f}},P_\tau\boldsymbol{v}_{\mathrm{f}}\rangle=(\boldsymbol{f},\boldsymbol{v}_{\mathrm{f}})_{\Omega_{\mathrm{f}}},\quad\forall\boldsymbol{v}_{\mathrm{f}}\in\boldsymbol{X}_{\mathrm{f}},\ \ \psi_{\mathrm{p}}\in X_{\mathrm{p}},\tag{48}$$

$$b_{\mathrm{f}}(\boldsymbol{u}_{\mathrm{f}},q_{\mathrm{f}})=0,\quad\forall\, q_{\mathrm{f}}\in Q_{\mathrm{f}},\tag{49}$$

where we have the spaces

$$\begin{aligned}\boldsymbol{X}_{\mathrm{f}}&=\{\boldsymbol{v}_{\mathrm{f}}\in(H^1(\Omega_{\mathrm{f}}))^2\,|\,\boldsymbol{v}_{\mathrm{f}}=0\text{ on }\partial\Omega_{\mathrm{f}}\backslash\Gamma\},\\ Q_{\mathrm{f}}&=L^2(\Omega_{\mathrm{f}}),\\ X_{\mathrm{p}}&=\{\psi_{\mathrm{p}}\in H^1(\Omega_{\mathrm{p}})\,|\,\psi_{\mathrm{p}}=0\text{ on }\partial\Omega_{\mathrm{p}}\backslash\Gamma\}\end{aligned}$$

and the bilinear forms

$$\begin{aligned}a_{\mathrm{p}}(\phi_{\mathrm{p}},\psi_{\mathrm{p}})&=(\mathbb{K}\nabla\phi_{\mathrm{p}},\nabla\psi_{\mathrm{p}})_{\Omega_{\mathrm{p}}},\\ a_{\mathrm{f}}(\boldsymbol{u}_{\mathrm{f}},\boldsymbol{v}_{\mathrm{f}})&=2\nu(\mathbb{D}(\boldsymbol{u}_{\mathrm{f}}),\mathbb{D}(\boldsymbol{v}_{\mathrm{f}}))_{\Omega_{\mathrm{f}}},\\ b_{\mathrm{f}}(\boldsymbol{v}_{\mathrm{f}},q)&=-(\nabla\cdot\boldsymbol{v}_{\mathrm{f}},q)_{\Omega_{\mathrm{f}}},\end{aligned}$$

and where $(\cdot,\cdot)_{\Omega_{\mathrm{f}}}$ and $(\cdot,\cdot)_{\Omega_{\mathrm{p}}}$ denote the L^2-inner products on the domains Ω_{f} and Ω_{p}, respectively, $\langle\cdot,\cdot\rangle$ the L^2-inner product on the interface boundary Γ, and P_τ denotes the projection onto the tangent space on Γ, i.e. $P_\tau\boldsymbol{u}=(\boldsymbol{u}\cdot\tau_{\mathrm{f}})\tau_{\mathrm{f}}$. Let

$$\tilde{A}((\boldsymbol{u},p,\phi);(\boldsymbol{v},q,\psi))=a_{\mathrm{f}}(\boldsymbol{u},\boldsymbol{v})+b_{\mathrm{f}}(\boldsymbol{v},p)+ga_{\mathrm{p}}(\phi,\psi)+\langle g\phi,\boldsymbol{v}\cdot\boldsymbol{n}_{\mathrm{f}}\rangle-\langle g\boldsymbol{u}\cdot\boldsymbol{n}_{\mathrm{f}},\psi\rangle+\alpha\langle P_\tau\boldsymbol{u},P_\tau\boldsymbol{v}\rangle-b_{\mathrm{f}}(\boldsymbol{u},q),$$
$$\forall(\boldsymbol{u},p,\phi),(\boldsymbol{v},q,\psi)\in\boldsymbol{X}_{\mathrm{f}}\times Q_{\mathrm{f}}\times X_{\mathrm{p}}.$$

Then, the system (48)–(49) can be rewritten to

$$\tilde{A}((\boldsymbol{u},p,\phi);(\boldsymbol{v},q,\psi))=(\boldsymbol{f},\boldsymbol{v})_{\Omega_{\mathrm{f}}},\quad\forall(\boldsymbol{v},q,\psi)\in\boldsymbol{X}_{\mathrm{f}}\times Q_{\mathrm{f}}\times X_{\mathrm{p}}.\tag{50}$$

A finite element discretization of (50) is to seek $(\boldsymbol{u}^h,p^h,\phi^h)\in\boldsymbol{X}_{\mathrm{f}}^h\times Q_{\mathrm{f}}^h\times X_{\mathrm{p}}^h$ such that

$$\tilde{A}((\boldsymbol{u}^h,p^h,\phi^h);(\boldsymbol{v}^h,q^h,\psi^h))=(\boldsymbol{f},\boldsymbol{v}^h)_{\Omega_{\mathrm{f}}},\quad\forall(\boldsymbol{v}^h,q^h,\psi^h)\in\boldsymbol{X}_{\mathrm{f}}^h\times Q_{\mathrm{f}}^h\times X_{\mathrm{p}}^h,\tag{51}$$

where $\boldsymbol{X}_{\mathrm{f}}^h\times Q_{\mathrm{f}}^h\times X_{\mathrm{p}}^h\subset\boldsymbol{X}_{\mathrm{f}}\times Q_{\mathrm{f}}\times X_{\mathrm{p}}$ is a finite element space. In this section, we set $\boldsymbol{X}_{\mathrm{f}}^h\times Q_{\mathrm{f}}^h$ to be the Hood–Taylor element on T_{f}^h, and X_{p}^h to be the standard continuous quadratic finite element space on T_{p}^h, where T_{f}^h and T_{p}^h are uniform isosceles right-triangle meshes on Ω_{f} and Ω_{p}, respectively. Comparing with the finite element space defined in Section 2, $\boldsymbol{X}_{\mathrm{f}}^h=\boldsymbol{V}^h$ and $Q_{\mathrm{f}}^h=Q^h$ on the domain Ω_{f}, and $X_{\mathrm{p}}^h=V^h$ on the domain Ω_{p}, with the zero boundary condition on the boundaries $\Omega_{\mathrm{f}}\backslash\Gamma$ and $\Omega_{\mathrm{p}}\backslash\Gamma$.

Theorem 4
Let $(\boldsymbol{u},p,\phi)$ and $(\boldsymbol{u}^h,p^h,\phi^h)$ denote the solutions of (50) and (51), respectively. Assume that $\boldsymbol{u}\in(H^4(\Omega_{\mathrm{f}}))^2$, $p\in H^3(\Omega_{\mathrm{f}})$, $\phi\in H^4(\Omega_{\mathrm{p}})$, and T_{f}^h, T_{p}^h are uniform isosceles right-triangle meshes. Then, we have that

$$\begin{aligned}&\|\Pi_h\boldsymbol{u}-\boldsymbol{u}^h\|_{1,\Omega_{\mathrm{f}}}+\|P_hp-p^h\|_{0,\Omega_{\mathrm{f}}}+\|\Pi_h\phi-\phi^h\|_{1,\Omega_{\mathrm{p}}}\\ &\quad\leqslant Ch^{2.5}(\|\boldsymbol{u}\|_{4,\Omega_{\mathrm{f}}}+\|p\|_{3,\Omega_{\mathrm{f}}}+\|\phi\|_{4,\Omega_{\mathrm{p}}}),\end{aligned}\tag{52}$$

where Π_h and P_h are defined in Section 2.

Proof
Let $\boldsymbol{v}^h=\Pi_h\boldsymbol{u}-\boldsymbol{u}^h$, $q^h=P_hp-p^h$, and $\psi^h=\Pi_h\phi-\phi^h$. Then, it is easy to see that $(\boldsymbol{v}^h,q^h,\psi^h)\in\boldsymbol{X}_{\mathrm{f}}^h\times Q_{\mathrm{f}}^h\times X_{\mathrm{p}}^h$, and

$$\tilde{A}((\boldsymbol{v}^h,q^h,\psi^h);(\boldsymbol{v}^h,q^h,\psi^h))\geqslant c(\|\boldsymbol{v}^h\|_{1,\Omega_{\mathrm{f}}}^2+\|\psi^h\|_{1,\Omega_{\mathrm{p}}}^2).\tag{53}$$

Note that $(\boldsymbol{u},p,\phi)$ and $(\boldsymbol{u}^h,p^h,\phi^h)$ are the solutions of (50) and (51), respectively. We have that

$$\begin{aligned}\tilde{A}((\boldsymbol{v}^h,q^h,\psi^h);(\boldsymbol{v}^h,q^h,\psi^h))&=\tilde{A}((\Pi_h\boldsymbol{u}-\boldsymbol{u}^h,P_hp-p^h,\Pi_h\phi-\phi^h);(\boldsymbol{v}^h,q^h,\psi^h))\\ &=\tilde{A}((\Pi_h\boldsymbol{u}-\boldsymbol{u},P_hp-p,\Pi_h\phi-\phi);(\boldsymbol{v}^h,q^h,\psi^h))\\ &=a_{\mathrm{f}}(\Pi_h\boldsymbol{u}-\boldsymbol{u},\boldsymbol{v}^h)+b_{\mathrm{f}}(\boldsymbol{v}^h,P_hp-p)+ga_{\mathrm{p}}(\Pi_h\phi-\phi,\psi^h)+\langle g(\Pi_h\phi-\phi),\boldsymbol{v}^h\cdot\boldsymbol{n}_{\mathrm{f}}\rangle-\langle g(\Pi_h\boldsymbol{u}-\boldsymbol{u})\cdot\boldsymbol{n}_{\mathrm{f}},\psi^h\rangle\\ &\quad+\alpha\langle P_\tau(\Pi_h\boldsymbol{u}-\boldsymbol{u}),P_\tau\boldsymbol{v}^h\rangle-b_{\mathrm{f}}(\Pi_h\boldsymbol{u}-\boldsymbol{u},q^h).\end{aligned}\tag{54}$$

Similar to the last section, it can be shown from Lemmas 2.1 and 2.2 that

$$|a_{\mathrm{f}}(\Pi_h \boldsymbol{u}-\boldsymbol{u},\boldsymbol{v}^h)| \leqslant Ch^{2.5}\|\boldsymbol{u}\|_{4,\Omega_{\mathrm{f}}}\|\boldsymbol{v}^h\|_{1,\Omega_{\mathrm{f}}}, \tag{55}$$

$$|b_{\mathrm{f}}(\boldsymbol{v}^h, P_h p-p)| \leqslant Ch^{2.5}\|p\|_{3,\Omega_{\mathrm{f}}}\|\boldsymbol{v}^h\|_{1,\Omega_{\mathrm{f}}}, \tag{56}$$

and

$$|b_{\mathrm{f}}(\Pi_h \boldsymbol{u}-\boldsymbol{u}, q^h)| \leqslant Ch^{2.5}\|\boldsymbol{u}\|_{4,\Omega_{\mathrm{f}}}\|q^h\|_{0,\Omega_{\mathrm{f}}}. \tag{57}$$

Using the standard superconvergence results (see, e.g. [18, 29]), we have that

$$|ga_{\mathrm{p}}(\Pi_h\phi-\phi,\psi^h)| \leqslant Ch^{2.5}\|\phi\|_{4,\Omega_{\mathrm{p}}}\|\psi^h\|_{1,\Omega_{\mathrm{p}}}. \tag{58}$$

Moreover, we can use standard error estimates and the trace theorem to prove that

$$|\langle g(\Pi_h\phi-\phi),\boldsymbol{v}^h\cdot\boldsymbol{n}_{\mathrm{f}}\rangle| \leqslant C\|\Pi_h\phi-\phi\|_{0,\Gamma}\|\boldsymbol{v}^h\|_{0,\Gamma} \leqslant Ch^3\|\phi\|_{3,\Gamma}\|\boldsymbol{v}^h\|_{0,\Gamma} \leqslant Ch^3\|\phi\|_{4,\Omega_{\mathrm{p}}}\|\boldsymbol{v}^h\|_{1,\Omega_{\mathrm{f}}}. \tag{59}$$

Similarly, it can be shown that

$$|\langle g(\Pi_h\boldsymbol{u}-\boldsymbol{u})\cdot\boldsymbol{n}_{\mathrm{f}},\psi^h\rangle| \leqslant Ch^3\|\boldsymbol{u}\|_{4,\Omega_{\mathrm{f}}}\|\psi^h\|_{1,\Omega_{\mathrm{p}}}, \tag{60}$$

and

$$|\alpha\langle P_\tau(\Pi_h\boldsymbol{u}-\boldsymbol{u}),P_\tau\boldsymbol{v}^h\rangle| \leqslant Ch^3\|\boldsymbol{u}\|_{4,\Omega_{\mathrm{f}}}\|\boldsymbol{v}^h\|_{1,\Omega_{\mathrm{f}}}. \tag{61}$$

Thus, it follows from (53)–(61) that

$$\|\boldsymbol{v}^h\|^2_{1,\Omega_{\mathrm{f}}}+\|\psi^h\|^2_{1,\Omega_{\mathrm{p}}} \leqslant C(\delta)h^5(\|\boldsymbol{u}\|^2_{4,\Omega_{\mathrm{f}}}+\|p\|^2_{3,\Omega_{\mathrm{f}}}+\|\phi\|^2_{4,\Omega_{\mathrm{p}}})+C\delta(\|\boldsymbol{v}^h\|^2_{1,\Omega_{\mathrm{f}}}+\|q^h\|^2_{0,\Omega_{\mathrm{f}}}+\|\psi^h\|^2_{1,\Omega_{\mathrm{p}}}), \tag{62}$$

where δ is an arbitrary small positive number.

Note that the Hood–Taylor element satisfies the inf–sup condition, i.e. for above q^h, there exists a vector function $\boldsymbol{w}^h\in\boldsymbol{X}^h_{\mathrm{f}}$ such that

$$\|q^h\|_{0,\Omega_{\mathrm{f}}} \leqslant C\frac{|b(\boldsymbol{w}^h,q^h)|}{\|\boldsymbol{w}^h\|_{1,\Omega_{\mathrm{f}}}}. \tag{63}$$

Choosing $\varphi^h\in X^h_{\mathrm{p}}$ to be such that $\|\varphi^h\|_{1,\Omega_{\mathrm{p}}}\leqslant C\|\boldsymbol{w}^h\|_{1,\Omega_{\mathrm{f}}}$, we have that

$$b(\boldsymbol{w}^h,q^h)=b(\boldsymbol{w}^h,P_hp-p^h)=b(\boldsymbol{w}^h,P_hp-p)+b(\boldsymbol{w}^h,p-p^h), \tag{64}$$

and

$$\begin{aligned}
b(\boldsymbol{w}^h,p-p^h) &= a_{\mathrm{f}}(\boldsymbol{u}-\boldsymbol{u}^h,\boldsymbol{w}^h)+ga_{\mathrm{p}}(\phi-\phi^h,\varphi^h)+\langle g(\phi-\phi^h),\boldsymbol{w}^h\cdot\boldsymbol{n}_{\mathrm{f}}\rangle-\langle g(\boldsymbol{u}-\boldsymbol{u}^h)\cdot\boldsymbol{n}_{\mathrm{f}},\varphi^h\rangle+\alpha\langle P_\tau(\boldsymbol{u}-\boldsymbol{u}^h),P_\tau\boldsymbol{w}^h\rangle\\
&= a_{\mathrm{f}}(\boldsymbol{u}-\Pi_h\boldsymbol{u},\boldsymbol{w}^h)+ga_{\mathrm{p}}(\phi-\Pi_h\phi,\varphi^h)+\langle g(\phi-\Pi_h\phi),\boldsymbol{w}^h\cdot\boldsymbol{n}_{\mathrm{f}}\rangle-\langle g(\boldsymbol{u}-\Pi_h\boldsymbol{u})\cdot\boldsymbol{n}_{\mathrm{f}},\varphi^h\rangle+\alpha\langle P_\tau(\boldsymbol{u}-\Pi_h\boldsymbol{u}),P_\tau\boldsymbol{w}^h\rangle\\
&\quad +a_{\mathrm{f}}(\Pi_h\boldsymbol{u}-\boldsymbol{u}^h,\boldsymbol{w}^h)+ga_{\mathrm{p}}(\Pi_h\phi-\phi^h,\varphi^h)+\langle g(\Pi_h\phi-\phi^h),\boldsymbol{w}^h\cdot\boldsymbol{n}_{\mathrm{f}}\rangle\\
&\quad -\langle g(\Pi_h\boldsymbol{u}-\boldsymbol{u}^h)\cdot\boldsymbol{n}_{\mathrm{f}},\varphi^h\rangle+\alpha\langle P_\tau(\Pi_h\boldsymbol{u}-\boldsymbol{u}^h),P_\tau\boldsymbol{w}^h\rangle.
\end{aligned} \tag{65}$$

Considering (64) and (65) with the superconvergence estimates (55)–(61), we can deduce that

$$\begin{aligned}
b(\boldsymbol{w}^h,q^h) &\leqslant Ch^{2.5}(\|\boldsymbol{u}\|_{4,\Omega_{\mathrm{f}}}+\|p\|_{3,\Omega_{\mathrm{f}}}+\|\phi\|_{4,\Omega_{\mathrm{p}}})(\|\boldsymbol{w}^h\|_{1,\Omega_{\mathrm{f}}}+\|\varphi^h\|_{1,\Omega_{\mathrm{p}}})\\
&\quad +C(\|\Pi_h\boldsymbol{u}-\boldsymbol{u}^h\|_{1,\Omega_{\mathrm{f}}}+\|\Pi_h\phi-\phi^h\|_{1,\Omega_{\mathrm{p}}})(\|\boldsymbol{w}^h\|_{1,\Omega_{\mathrm{f}}}+\|\varphi^h\|_{1,\Omega_{\mathrm{p}}})\\
&\leqslant C(h^{2.5}(\|\boldsymbol{u}\|_{4,\Omega_{\mathrm{f}}}+\|p\|_{3,\Omega_{\mathrm{f}}}+\|\phi\|_{4,\Omega_{\mathrm{p}}})+\|\Pi_h\boldsymbol{u}-\boldsymbol{u}^h\|_{1,\Omega_{\mathrm{f}}}+\|\Pi_h\phi-\phi^h\|_{1,\Omega_{\mathrm{p}}})\|\boldsymbol{w}^h\|_{1,\Omega_{\mathrm{f}}}.
\end{aligned} \tag{66}$$

Then (63) and (66) imply that

$$\|q^h\|_{0,\Omega_{\mathrm{f}}} \leqslant C(h^{2.5}(\|\boldsymbol{u}\|_{4,\Omega_{\mathrm{f}}}+\|p\|_{3,\Omega_{\mathrm{f}}}+\|\phi\|_{4,\Omega_{\mathrm{p}}})+\|\Pi_h\boldsymbol{u}-\boldsymbol{u}^h\|_{1,\Omega_{\mathrm{f}}}+\|\Pi_h\phi-\phi^h\|_{1,\Omega_{\mathrm{p}}}). \tag{67}$$

Substituting (67) to (62) and noting that $\boldsymbol{v}^h=\Pi_h\boldsymbol{u}-\boldsymbol{u}^h$ and $\psi^h=\Pi_h\phi-\phi^h$, we have that

$$\|\boldsymbol{v}^h\|^2_{1,\Omega_{\mathrm{f}}}+\|\psi^h\|^2_{1,\Omega_{\mathrm{p}}} \leqslant C(\delta)h^5(\|\boldsymbol{u}\|^2_{4,\Omega_{\mathrm{f}}}+\|p\|^2_{3,\Omega_{\mathrm{f}}}+\|\phi\|^2_{4,\Omega_{\mathrm{p}}})+C\delta h^5(\|\boldsymbol{u}\|^2_{4,\Omega_{\mathrm{f}}}+\|p\|^2_{3,\Omega_{\mathrm{f}}}+\|\phi\|^2_{4,\Omega_{\mathrm{p}}})+C\delta(\|\boldsymbol{v}^h\|^2_{1,\Omega_{\mathrm{f}}}+\|\psi^h\|^2_{1,\Omega_{\mathrm{p}}}).$$

Let $\delta=1/2C$. Then,

$$\|\boldsymbol{v}^h\|^2_{1,\Omega_{\mathrm{f}}}+\|\psi^h\|^2_{1,\Omega_{\mathrm{p}}} \leqslant Ch^5(\|\boldsymbol{u}\|^2_{4,\Omega_{\mathrm{f}}}+\|p\|^2_{3,\Omega_{\mathrm{f}}}+\|\phi\|^2_{4,\Omega_{\mathrm{p}}}). \tag{68}$$

Moreover, (67) and (68) imply that

$$\|q^h\|_{0,\Omega_f} \leqslant Ch^{2.5}(\|\boldsymbol{u}\|_{4,\Omega_f} + \|p\|_{3,\Omega_f} + \|\phi\|_{4,\Omega_p}). \tag{69}$$

Then (52) follows from (68) and (69). □

Similar to Theorem 2, we can deduce the following superconvergence result with respect to the L^2-norm.

Theorem 5

Let $(\boldsymbol{u}, p, \phi)$ and $(\boldsymbol{u}^h, p^h, \phi^h)$ denote the solutions of (50) and (51), respectively. Assume that $\boldsymbol{u} \in (H^4(\Omega_f))^2$, $p \in H^3(\Omega_f)$, $\phi \in H^4(\Omega_p)$, and T_f^h, T_p^h are uniform isosceles right-triangle meshes. Moreover, let Ω_f and Ω_p be such that, for all $\boldsymbol{v} \in (L^2(\Omega_f))^2$ and $\psi \in L^2(\Omega_p)$, there exists a triple $(\boldsymbol{w}, q, \varphi) \in \boldsymbol{X}_f \times Q_f \times X_p$ satisfying the following auxiliary equations:

$$\tilde{A}((\boldsymbol{\mu}, r, \rho); (\boldsymbol{w}, q, \varphi)) = (\boldsymbol{v}, \boldsymbol{\mu})_{\Omega_f} + (\psi, \rho)_{\Omega_p}, \quad \forall (\boldsymbol{\mu}, r, \rho) \in \boldsymbol{X}_f \times Q_f \times X_p \tag{70}$$

and

$$\|\boldsymbol{w}\|_{2,\Omega_f} + \|q\|_{1,\Omega_f} + \|\varphi\|_{2,\Omega_p} \leqslant C(\|\boldsymbol{v}\|_{0,\Omega_f} + \|\psi\|_{0,\Omega_p}). \tag{71}$$

Then, we have that

$$\|\Pi_h \boldsymbol{u} - \boldsymbol{u}^h\|_{0,\Omega_f} + \|\Pi_h \phi - \phi^h\|_{0,\Omega_p} \leqslant Ch^{3.5}(\|\boldsymbol{u}\|_{4,\Omega_f} + \|p\|_{3,\Omega_f} + \|\phi\|_{4,\Omega_p}), \tag{72}$$

where Π_h is defined in Section 2.

Proof

Let $\boldsymbol{v} = \Pi_h \boldsymbol{u} - \boldsymbol{u}^h$, $\theta = P_h p - p^h$, $\psi = \Pi_h \phi - \phi^h$, and let $\boldsymbol{\mu} = \boldsymbol{v}$, $r = \theta$, $\rho = \psi$ in (70). We have that

$$\begin{aligned} \|\boldsymbol{v}\|^2_{0,\Omega_f} + \|\psi\|^2_{0,\Omega_p} &= \tilde{A}((\boldsymbol{v}, \theta, \psi); (\boldsymbol{w}, q, \varphi)) \\ &= \tilde{A}((\boldsymbol{v}, \theta, \psi); (\boldsymbol{w} - \Pi_h \boldsymbol{w}, q - P_h q, \varphi - \Pi_h \varphi)) + \tilde{A}((\boldsymbol{v}, \theta, \psi); (\Pi_h \boldsymbol{w}, P_h q, \Pi_h \varphi)). \end{aligned} \tag{73}$$

Similar to Theorem 2, it follows from Theorem 4, standard interpolation error estimates , and (71) that

$$\begin{aligned} |\tilde{A}((\boldsymbol{v}, \theta, \psi); (\boldsymbol{w} - \Pi_h \boldsymbol{w}, q - P_h q, \varphi - \Pi_h \varphi))| &\leqslant C(\|\boldsymbol{v}\|_{1,\Omega_f} + \|\theta\|_{0,\Omega_f} + \|\psi\|_{1,\Omega_p})(\|\boldsymbol{w} - \Pi_h \boldsymbol{w}\|_{1,\Omega_f} + \|q - P_h q\|_{0,\Omega_f} + \|\varphi - \Pi_h \varphi\|_{0,\Omega_p}) \\ &\leqslant Ch^{2.5}(\|\boldsymbol{u}\|_{4,\Omega_f} + \|p\|_{3,\Omega_f} + \|\phi\|_{4,\Omega_p}) h (\|\boldsymbol{w}\|_{2,\Omega_f} + \|q\|_{1,\Omega_f} + \|\varphi\|_{2,\Omega_p}) \\ &\leqslant Ch^{3.5}(\|\boldsymbol{u}\|_{4,\Omega_f} + \|p\|_{3,\Omega_f} + \|\phi\|_{4,\Omega_p})(\|\boldsymbol{v}\|_{0,\Omega_f} + \|\psi\|_{0,\Omega_p}). \end{aligned} \tag{74}$$

Moreover, it follows from (50) and (51) that

$$\begin{aligned} \tilde{A}((\boldsymbol{v}, \theta, \psi); (\Pi_h \boldsymbol{w}, P_h q, \Pi_h \varphi)) &= \tilde{A}((\Pi_h \boldsymbol{u} - \boldsymbol{u}^h, P_h p - p^h, \Pi_h \phi - \phi^h); (\Pi_h \boldsymbol{w}, P_h q, \Pi_h \varphi)) \\ &= \tilde{A}((\Pi_h \boldsymbol{u} - \boldsymbol{u}, P_h p - p, \Pi_h \phi - \phi); (\Pi_h \boldsymbol{w}, P_h q, \Pi_h \varphi)). \end{aligned} \tag{75}$$

Similar to Theorems 2 and 4, it can be proved that

$$\begin{aligned} &|\tilde{A}((\Pi_h \boldsymbol{u} - \boldsymbol{u}, P_h p - p, \Pi_h \phi - \phi); (\Pi_h \boldsymbol{w}, P_h q, \Pi_h \varphi))| \\ &\quad \leqslant Ch^{3.5}(\|\boldsymbol{u}\|_{4,\Omega_f} + \|p\|_{3,\Omega_f} + \|\phi\|_{4,\Omega_p})(\|\Pi_h \boldsymbol{w}\|_{2,\Omega_f} + \|P_h q\|_{1,\Omega_f} + \|\Pi_h \varphi\|_{2,\Omega_p}) \\ &\quad \leqslant Ch^{3.5}(\|\boldsymbol{u}\|_{4,\Omega_f} + \|p\|_{3,\Omega_f} + \|\phi\|_{4,\Omega_p})(\|\boldsymbol{v}\|_{0,\Omega_f} + \|\psi\|_{0,\Omega_p}). \end{aligned} \tag{76}$$

Thus, (72) is proved from (73)–(76). □

Similar to Theorem 3, we can have the following global superconvergence result from Theorems 4, 5 and the interpolation postprocessing interpolation operator Π^*_{2h} introduced in Section 2.

Theorem 3.1

Let $(\boldsymbol{u}, p, \phi)$ and $(\boldsymbol{u}^h, p^h, \phi^h)$ denote the solutions of (50) and (51), respectively. Let Π^*_{2h} denote the interpolation postprocessing operator defined in Section 2. Assume that $\boldsymbol{u} \in (H^4(\Omega_f))^2$, $p \in H^3(\Omega_f)$, $\phi \in H^4(\Omega_p)$, and T_f^h, T_p^h are uniform isosceles right-triangle meshes. We then have that

$$\|\Pi^*_{2h} \boldsymbol{u}^h - \boldsymbol{u}\|_{1,\Omega_f} + \|\Pi^*_{2h} \phi^h - \phi\|_{1,\Omega_p} \leqslant Ch^{2.5}(\|\boldsymbol{u}\|_{4,\Omega_f} + \|p\|_{3,\Omega_f} + \|\phi\|_{4,\Omega_p}). \tag{77}$$

Moreover, assume that all the hypotheses of Theorem 5 hold. Then,

$$\|\Pi^*_{2h} \boldsymbol{u}^h - \boldsymbol{u}\|_{0,\Omega_f} + \|\Pi^*_{2h} \phi^h - \phi\|_{0,\Omega_p} \leqslant Ch^{3.5}(\|\boldsymbol{u}\|_{4,\Omega_f} + \|p\|_{3,\Omega_f} + \|\phi\|_{4,\Omega_p}). \tag{78}$$

Remark 3.1
In Sections 2 and 3, superconvergence results are proven for the quadratic finite element approximation on uniform isosceles right-triangle meshes. Using standard but more complicated superconvergence analyses techniques, the condition on the mesh can be weakened to uniform mesh, almost uniform mesh, or mildly structured grids; see, e.g. [16, 18, 19, 22, 30, 31].

4. Numerical results

In this section, we provide some numerical examples to illustrate our theoretical results. In order to do the computations more conveniently, we define another interpolation operator π_h: for all $v \in C(\Omega)$, let $\pi_h v \in V^h$ be such that

$$\pi_h v(z_i) = v(z_i), \quad i = 1, \ldots, 6, \tag{79}$$

where z_i are the vertices of the element ($i = 1, 2, 3$) and the midpoints of the edges of the element ($i = 4, 5, 6$), respectively. It is clear that it is more convenient to use π_h instead of Π_h in the computation of norms. Moreover, the difference between $\pi_h u$ and $\Pi_h u$ has a higher-order error compared with the standard error if u is smooth enough, so that the error incurred by using $\pi_h u$ instead of $\Pi_h u$ can be ignored.

In the following three examples, the domain (Ω in Examples 1 and 2) and subdomains (Ω_f and Ω_p in Example 3) are first divided by n^2 squares, then every square is subdivided into two triangles. We set $n = 2, 4, 8, 16, 32, 64$ and the mesh size $h \propto 1/n$.

Example 1 (*Darcy Problem*)
Let $\Omega = [0, 1] \times [0, 1]$, and ϕ denote the solution of

$$-\Delta\phi + \phi = f.$$

Two types of boundary conditions are used, one is the Dirichlet boundary condition and the another is mixed Dirichlet–Robin boundary condition. In the first case, the Dirichlet boundary condition $\phi = 0$ is imposed on $\partial\Omega$ and the exact solution is given by $\phi = \sin(\pi x)\sin(\pi y)$; In the second case, the Robin boundary condition $\partial\phi/\partial\mathbf{n} + \phi = G$ is imposed on the boundary segment $\Gamma = \{0 \leqslant x \leqslant 1, y = 0\}$ and the Dirichlet boundary condition $\phi = 0$ is imposed on $\partial\Omega \backslash \Gamma$; the exact solution ϕ is taken as $\phi = \sin(\pi x)\sin(1 - y)$.

The relative errors $\|\phi_h - \pi_h\phi\|_{0,\Omega}/\|\pi_h\phi\|_{0,\Omega}$, $\|\nabla(\phi_h - \pi_h\phi)\|_{0,\Omega}/\|\nabla\pi_h\phi\|_{0,\Omega}$, and $\|\Pi^*_{2h}\phi_h - \phi\|_{0,\Omega}/\|\phi\|_{0,\Omega}$ are plotted in Figure 1 (the left figure for the Dirichlet boundary condition and the right figure for the mixed Dirichlet–Robin boundary condition). In order to clarify the superconvergence behavior, we compute the convergence order from the ratio of the errors on successive mesh refinements:

$$\text{convergence order} = \frac{\text{error on the mesh size } h = 1/n}{\text{error on the mesh size } h = 1/2n}. \tag{80}$$

The convergence orders are listed in Tables I and II; it is clear that almost half-order accuracy is lost in the mixed Dirichlet–Robin boundary condition.

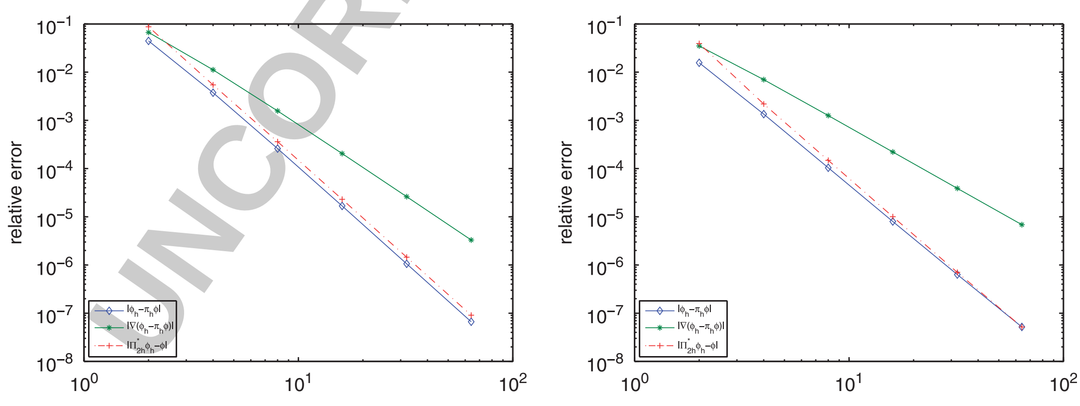

Figure 1. Log–log relative error figures of Darcy problems: Dirichlet boundary condition (left) and mixed Dirichlet–Robin boundary condition (right).

Table I. Convergence orders of the Darcy problem (Dirichlet boundary condition).

$\|\phi_h - \pi_h\phi\|_0$	3.5698	3.8481	3.9527	3.9854	3.9953
$\|\nabla(\phi_h - \pi_h\phi)\|_0$	2.5788	2.8363	2.9351	2.9725	2.9876
$\|\Pi^*_{2h}\phi_h - \phi\|_0$	4.0052	3.9093	3.9722	3.9919	3.9948

Table II. Convergence orders of the Darcy problem (mixed Dirichlet–Robin boundary condition).

$\|\phi_h-\pi_h\phi\|_0$	3.5384	3.6961	3.7036	3.6589	3.6078
$\|\nabla(\phi_h-\pi_h\phi)\|_0$	2.3211	2.4751	2.5083	2.5102	2.5068
$\|\Pi^*_{2h}\phi_h-\phi\|_0$	4.1639	3.8845	3.8809	3.8363	3.7648

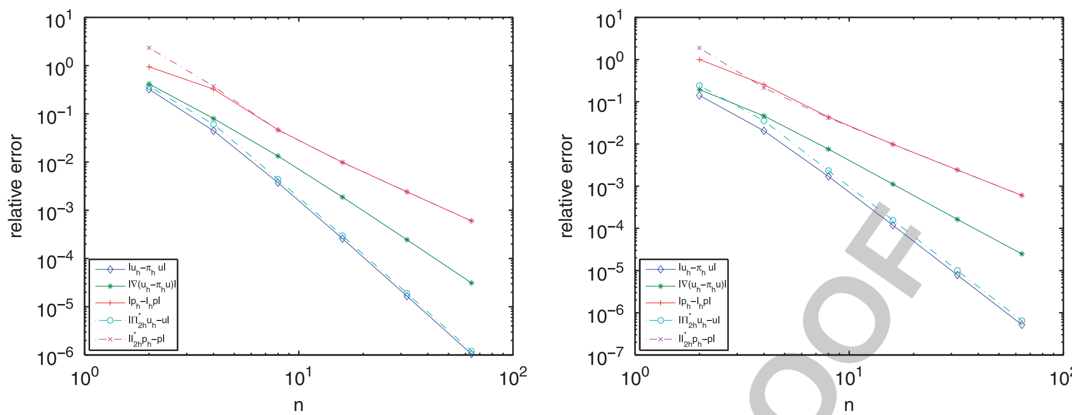

Figure 2. Log–log relative error figures of Stokes problems: Dirichlet boundary condition (left) and mixed Dirichlet–Robin boundary condition (right).

Example 2 (*Stokes Problem*)
Let $\Omega=[0,1]\times[0,1]$ and $\boldsymbol{u}$ and p denote the solution of the Stokes problem

$$-\Delta\boldsymbol{u}+\nabla p=\boldsymbol{f} \quad \text{in } \Omega, \tag{81}$$

$$\operatorname{div}\boldsymbol{u}=0 \quad \text{in } \Omega. \tag{82}$$

The Dirichlet and the mixed Dirichlet–Robin boundary conditions are also considered here. In the first case, we have the Dirichlet boundary condition $\boldsymbol{u}=\boldsymbol{0}$ on $\partial\Omega$ and the exact solution is given by $u_1=-\sin^2(\pi x)\sin(2\pi y)$, $u_2=\sin(2\pi x)\sin^2(\pi y)$, $p=\sin(\pi(x+y))$. In the second case, the Robin boundary condition $n\cdot T(u,p)\cdot n+u=G$ is imposed on the boundary segment $\Gamma=\{0\leqslant x\leqslant 1, y=0\}$ and the zero Dirichlet boundary condition on the remaining part of the boundary $\partial\Omega\backslash\Gamma$; the exact solution is given by $u_1=\sin^2(\pi x)\sin^2(\pi y), u_2=\frac{1}{4}(\sin 2\pi y-2\pi y+2\pi)\sin 2\pi x, p=\sin\pi(x+y)$.

For both cases, the relative errors $\|\boldsymbol{u}_h-\pi_h\boldsymbol{u}\|_0/\|\pi_h\boldsymbol{u}\|_0$, $\|\nabla(\boldsymbol{u}_h-\pi_h\boldsymbol{u})\|_0/\|\nabla\pi_h\boldsymbol{u}\|_0$, $\|p_h-I_hp\|_0/\|I_hp\|_0$, $\|\Pi^*_{2h}\boldsymbol{u}_h-\boldsymbol{u}\|_0/\|\boldsymbol{u}\|_0$, and $\|I^*_{2h}p_h-p\|_0/\|p\|_0$ are shown in Figure 2 (the left figure for the Dirichlet boundary condition and the right figure for the mixed Dirichlet–Robin boundary condition). Here I^*_{2h} is another interpolation postprocessing operator, which generates, from the solution p_h, a polynomial of order two on the macro-element; see, e.g. [18, 19].

The convergence orders are presented in Tables III and IV. It is clear that the finite element solution $\boldsymbol{u}_h$ has the superconvergence, whereas the solution p_h does not have.

Example 3 (*Stokes–Darcy problem*)
Let $\Omega_{\rm p}=(0,\pi)\times(-1,0)$, $\Omega_{\rm f}=(0,\pi)\times(0,1)$, $\Gamma=\{0\leqslant x\leqslant\pi, y=0\}$, $g=1$, $v=1$, and $\mathbb{K}=\mathbb{I}$ in the Stokes–Darcy system. Note that along the interface boundary Γ, $\mathbf{n}_{\rm f}=(0,-1), \boldsymbol{\tau}_{\rm f}=(1,0)$, and $\mathbf{n}_{\rm p}=(0,1)$ so that

$$\mathbb{T}(\mathbf{u}_{\rm f},p_{\rm f})\cdot\mathbf{n}_{\rm f}=-\left(\frac{\partial\mathbf{u}_{\rm f,1}}{\partial y}+\frac{\partial\mathbf{u}_{\rm f,2}}{\partial x},2\frac{\partial\mathbf{u}_{\rm f,2}}{\partial y}-p_{\rm f}\right),\quad \boldsymbol{\tau}_{\rm f}\cdot\mathbf{u}_{\rm f}=\mathbf{u}_{\rm f,1},$$

and the interface conditions become

$$-\mathbf{u}_{\rm f,2}=\frac{\partial\phi_{\rm p}}{\partial y}, \tag{83}$$

$$\frac{\partial\mathbf{u}_{\rm f,1}}{\partial y}+\frac{\partial\mathbf{u}_{\rm f,2}}{\partial x}=\alpha\mathbf{u}_{\rm f,1}, \tag{84}$$

$$2\frac{\partial\mathbf{u}_{\rm f,2}}{\partial y}-p_{\rm f}=\phi_{\rm p}. \tag{85}$$

We set $v(y)=-2+\pi^{-2}\sin^2(\pi y)$ and the exact solutions

$$u_{\rm f,1}=v'(y)\cos x,\quad u_{\rm f,2}=v(y)\sin x,\quad p_{\rm f}=\sin x\sin y,\quad \phi_{\rm p}=({\rm e}^y-{\rm e}^{-y})\sin x$$

satisfy the Beavers–Joseph–Saffman–Jones interface boundary condition with any α.

Table III. Convergence orders of Stokes problem (Dirichlet boundary condition).

$\|\boldsymbol{u}_h-\pi_h\boldsymbol{u}\|_0$	2.8645	3.5753	3.8533	3.9573	3.9880
$\|\nabla(\boldsymbol{u}_h-\pi_h\boldsymbol{u})\|_0$	2.3791	2.5844	2.8302	2.9384	2.9763
$\|p_h-I_hp\|_0$	1.5384	2.7977	2.2478	2.0272	2.0019
$\|\Pi_{2h}^*\boldsymbol{u}-u\|_0$	2.6148	3.7943	3.8836	3.9620	3.9879
$\|I_{2h}^*p_h-p\|_0$	2.6464	3.0167	2.2326	2.0253	2.0025

Table IV. Convergence orders of Stokes problem (mixed Dirichlet–Robin boundary condition).

$\|\boldsymbol{u}_h-\pi_h\boldsymbol{u}\|_0$	2.7891	3.5888	3.8440	3.9192	3.9141
$\|\nabla(\boldsymbol{u}_h-\pi_h\boldsymbol{u})\|_0$	2.0804	2.6100	2.7573	2.7734	2.7188
$\|p_h-I_hp\|_0$	1.9823	2.5395	2.1508	2.0240	2.0038
$\|\Pi_{2h}^*\boldsymbol{u}-\boldsymbol{u}\|_0$	2.7706	3.9184	3.9029	3.9567	3.9623
$\|I_{2h}^*p_h-p\|_0$	3.1520	2.3547	2.0839	2.0150	2.0031

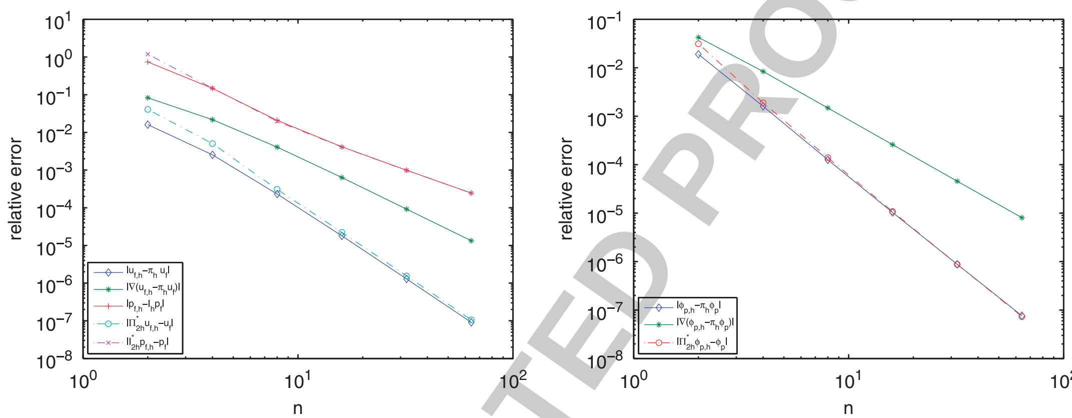

Figure 3. Log–log relative error figures of Stokes–Darcy problems.

Table V. Convergence orders of Stokes–Darcy problem.

$\|\boldsymbol{u}_{\mathrm{f},h}-\pi_h\boldsymbol{u}_\mathrm{f}\|_0$	2.6591	3.4622	3.6815	3.8014	3.8220
$\|\nabla(\boldsymbol{u}_{\mathrm{f},h}-\pi_h\boldsymbol{u}_\mathrm{f})\|_0$	1.9296	2.4202	2.6751	2.7894	2.7864
$\|p_{\mathrm{f},h}-I_hp_\mathrm{f}\|_0$	2.3312	2.8184	2.3322	2.0679	2.0139
$\|\Pi_{2h}^*\boldsymbol{u}_{\mathrm{f},h}-\boldsymbol{u}_\mathrm{f}\|_0$	3.0080	4.0330	3.7953	3.8395	3.8636
$\|I_{2h}^*p_{\mathrm{f},h}-p_\mathrm{f}\|_0$	2.9927	2.9195	2.2730	2.0477	2.0096
$\|\phi_{\mathrm{p},h}-\pi_h\phi_\mathrm{p}\|_0$	3.5691	3.6656	3.6077	3.5588	3.5305
$\|\nabla(\phi_{\mathrm{p},h}-\pi_h\phi_\mathrm{p})\|_0$	2.3363	2.5018	2.5142	2.5078	2.5035
$\|\Pi_{2h}^*\phi_{\mathrm{p},h}-\phi_\mathrm{p}\|_0$	4.0479	3.7586	3.6957	3.6289	3.5759

The relative errors of $\|\boldsymbol{u}_{\mathrm{f},h}-\pi_h\boldsymbol{u}_\mathrm{f}\|_0/\|\pi_h\boldsymbol{u}_\mathrm{f}\|_0$, $\|\nabla(\boldsymbol{u}_{\mathrm{f},h}-\pi_h\boldsymbol{u}_\mathrm{f})\|_0/\|\nabla\pi_h\boldsymbol{u}_\mathrm{f}\|_0$, $\|p_{\mathrm{f},h}-I_hp_\mathrm{f}\|_0/\|I_hp_\mathrm{f}\|_0$, $\|\Pi_{2h}^*\boldsymbol{u}_{\mathrm{f},h}-\boldsymbol{u}_\mathrm{f}\|_0/\|\boldsymbol{u}_\mathrm{f}\|_0$, and $\|I_{2h}^*p_{\mathrm{f},h}-p_\mathrm{f}\|_0/\|p_\mathrm{f}\|_0$ are shown in left figure of Figure 3, and the relative errors $\|\phi_{\mathrm{p},h}-\pi_h\phi_\mathrm{p}\|_0/\|\pi_h\phi_\mathrm{p}\|_0$, $\|\nabla(\phi_{\mathrm{p},h}-\pi_h\phi_\mathrm{p})\|_0/\|\nabla\pi_h\phi_\mathrm{p}\|_0$, and $\|\Pi_{2h}^*\phi_{\mathrm{p},h}-\phi_\mathrm{p}\|_0/\|\phi_\mathrm{p}\|_0$ are plotted in the right figure of Figure 3.

The convergence orders are listed in Table V; the superconvergence of the solutions $\mathbf{u}_{\mathrm{f},h}$ and $\phi_{\mathrm{p},h}$ can be observed, which confirms our theoretical results.

Acknowledgements

W. Chen was supported by the Chinese MOE under the 111 project grant, the National Basic Research Program under grant number 2005CB321701, P. Chen was supported in part by National Talents Training Base for Basic research and Teaching of Natural Science of China (J0730103), M. Gunzburger was supported in part by the CMG program of the National Sciences Foundation under grant number DMS-0620035, and Yan was supported by the National Natural Science Foundation of China (No. 10771211) and the National Basic Research Program under the Grants 2005CB321701 and 2010CB731505.

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