Role of Wavelets in the Physical and Statistical Modelling of Complex Geological Processes

D. A. Yuen[1], G. Erlebacher[2], O. V. Vasilyev[3], D. E. Goldstein[4], and M. Fuentes[5]

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Abstract — Today wavelets are recognized to have a wide range of useful properties that allow them to treat effectively multifacet problems, such as data compression, scale-localization analysis, feature extraction, statistics, numerical simulation, visualization, and communication. Second-generation wavelets represent a recent improvement of the wavelet algorithm, allowing for greater flexibility in the spatial domain and other computational advantages. We will show how these wavelets can be employed to extract large-scale coherent structures from (1.) three-dimensional turbulent flows and (2.) high Rayleigh number thermal convection. We will discuss the concept of modelling via decomposition into coherent and incoherent fields, taking into account the effect of the incoherent field via statistical modelling. We will discuss wavelet properties and how they can be utilized and integrated in handling large-scale problems in earthquake physics and other nonlinear phenomena in the geosciences.

Key words: wavelets, second generation wavelets, physical modelling, statistical modelling, geological processes

1 Introduction

Today there is an ongoing explosion in the volume of information coming from high-resolution numerical simulations of nonlinear phenomena, from more precise instrumentation on satellite missions, and from laboratory experiments of greater precision in many areas of the geosciences and environmental disciplines. Gargantuan amounts of data are being produced by modelling of earthquakes and its concomitant manifestations of wave propagation and stress transfer, which are intrinsically nonlinear and have a multiple-scale character in both space and time. Indeed this current tsunami

[1]Minnesota Supercomputer Institute, Department of Geophysics, University of Minnesota, Minneapolis, MN. E-mail: davey@msi.umn.edu

[2]School of Computational Science and Information Technology, Department of Mathematics/Computer Science, Florida State University, Tallahassee, FL. E-mail: erlebach@csit.fsu.edu

[3]Department of Mechanical Engineering, University of Colorado, Boulder, CO. E-mail: Oleg.Vasilyev@Colorado.EDU

[4]Department of Mechanical Engineering, University of Colorado, Boulder, CO. E-mail: Daniel.E.Goldstein@Colorado.EDU

[5]Department of Statistics, North Carolina State University, Raleigh, NC. E-mail: fuentes@stat.ncsu.edu

involving massive amounts of data has precipitated an urgent need for novel tools to process data more efficiently and to unravel the the complex dynamical mechanisms buried inside this mountain of numbers.

The last decade has witnessed the development of wavelets or wavelet analysis, a new mathematical tool, which originated in seismology (GOUPILLAUD *et al.* 1984) and in a brief period has quickly spread to a whole spectrum of fields in science and engineering and has been promoted in a popular book by Burke-Hubbard (1998). Wavelets have reached a certain level of maturity as a well-defined mathematical subject with a strong interdisciplinary character (STRANG and NGUYEN 1996, MALLAT 1998, VAN DEN BERG 1999), which has certainly begun to make an impact in the geophysical community (CHAO and NAITO 1995, KUMAR and FOUFOULA-GEORGIOU 1997, VASILYEV *et al.* 1997A, BERGERON *et al.* 1999, YUEN *et al.* 2002, ALEXANDRESCU *et al.* 1995, SIMONS and HAGER 1997, CHIAO and KUO 2001, VECSEY and MATYSKA 2001). Wavelets are mathematical transformations, which allow one to tackle problems of multiple-scale character in time and space in visualization (DEVORE *et al.* 1992, MALLAT 1998), analysis (DAUBECHIES 1988, DAUBECHIES 1990, MEYER 1990), statistics (JAMESON and MIYAMA 2000, LUO and JAMESON 2002) and in the solution of nonlinear partial differential equations (ERLEBACHER *et al.* 1996, VASILYEV and PAOLUCCI 1997, **?**, **?**). Wavelets are best suited for problems with a sharp multiple-scale character and should be used in cutting-edge applications, where the edge of the envelope in the computational front is being pushed. Our research team, with members coming from different disciplines, can be called on to address some of these diverse problems in a unified fashion.

In this paper we will first describe the general properties of second generation wavelets (SWELDENS 1996, SWELDENS 1998). Then we will discuss the concept of decomposition into coherent and incoherent structures and how this idea is applied to two diverse chaotic situations, one in turbulent flows and the other in high Rayleigh number thermal convection. In particular, we will describe a wavelet approach that can be used for efficient numerical simulation of the coherent field evolution and briefly introduce statistical methods that can be used to understand and, in the future, to model the effects of the incoherent field on the evolution of the coherent structures. We will also show how thermal plumes can be efficiently extracted from a voluminous data set using the idea of wavelet thresholding (DONOHO 1993). Finally we go over the prospects of using second-generation wavelets for analysis work in other fields of geophysics.

2 General Properties of Second Generation Wavelets

Wavelet analysis is a recent numerical concept, which allows one to represent a function in terms of basis functions, localized in both space and scale (DAUBECHIES 1988, SWELDENS 1996). Good wavelet localization properties in both physical (x) and wavenumber (ξ) spaces are illustrated in Figure 1. One may think of a wavelet decomposition as a multilevel or multiresolution representation of a function, where

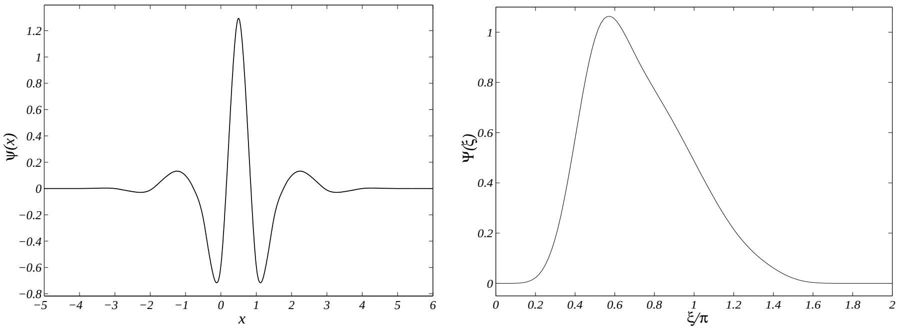


Fig. 1. Second generation wavelet of order 6, $\psi(x)$, and its Fourier transform, $\Psi(\xi)$.

each level of resolution j (except the coarsest one) consists of wavelets $\psi^j_{\mathbf{l}}$ or family of wavelets $\psi^{\mu,j}_{\mathbf{l}}$ having the same scale but located at different positions. The most general wavelet decomposition of a function $f(\mathbf{x})$ can be written as

$$f(\mathbf{x}) = \sum_{\mathbf{k}\in\mathcal{K}^0} c^0_{\mathbf{k}}\phi^0_{\mathbf{k}}(\mathbf{x}) + \sum_{j=0}^{+\infty}\sum_{\mu}\sum_{\mathbf{l}\in\mathcal{L}^{\mu,j}} d^{\mu,j}_{\mathbf{l}}\psi^{\mu,j}_{\mathbf{l}}(\mathbf{x}), \tag{1}$$

where $\phi^0_{\mathbf{k}}(\mathbf{x})$ and $\psi^{\mu,j}_{\mathbf{l}}$ are respectively n-dimensional scaling functions and wavelets of different families (μ) and levels of resolution (j). The major strength of wavelet analysis, *i.e.*, their ability to both compress and de-noise signals, now appears. For functions that contain isolated small scales on a large-scale background, most wavelet coefficients will be small, as illustrated in Figure 2. A good approximation can be retained even after discarding a large number of wavelets with small coefficients. This attractive property of wavelets allows one either to compress or to de-noise the function.

Traditionally, wavelets are constructed by the discrete (typically dyadic) dilation and translation of a single mother wavelet $\psi(x)$. This results in the construction of first generation wavelets (Daubechies 1988) that are defined either in infinite or periodic domains. It is desirable in many geophysical applications to have a larger class of wavelets that can be defined in general domains and on irregular sampling intervals. We must abandon translation/dilation relations of the first generation wavelets and instead construct the wavelets in the physical domain rather than in Fourier space. Recently, a whole new class of wavelets, currently referred to as *second generation wavelets* (Sweldens 1996, Sweldens 1998) have come to the fore. The main advantages of the second generation wavelets over the first generation wavelets include the following:

1. Second generation wavelets are constructed in a spatial domain and can be customized for complex multi-dimensional domains and irregular sampling intervals.

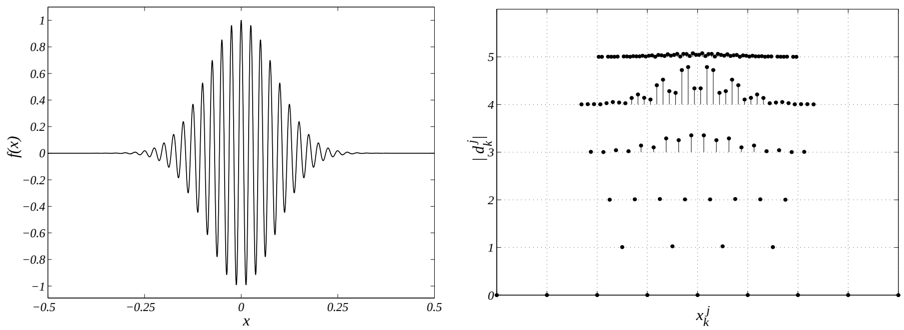


Fig. 2. Distribution of coefficients d_k^j and c_k^0 (right) of the second generation wavelet decomposition of the function $f(x) = \cos(80\pi x)e^{-64x^2}$ (left). Only coefficient whose absolute value is above 10^{-3} are shown.

2. No auxiliary memory is required and the original signal can be replaced with its wavelet transform.

Second generation wavelets have been utilized recently to construct a dynamically adaptive wavelet collocation method (Vasilyev and Bowman 2000) for the solution of both time evolution and elliptic problems. (**?**, **?**). The method employs wavelet compression as an integral part of the solution. The adaptation is achieved by retaining only those wavelets whose coefficients are greater than an *a priori* prescribed threshold. This property of the multi-level wavelet approximation allows local grid refinement up to an arbitrary small scale without a drastic increase in the number of grid points; thus high resolution computations are carried out only in those regions where sharp transitions occur. With this adaptation strategy, a solution is obtained on a near optimal grid, *i.e.*, the compression of the solution is performed at each time step.

3 Simulation and Modelling

3.1 Wavelet De-Noising

The wavelet de-noising procedure, also called wavelet-shrinkage, was originally introduced by Donoho (Donoho 1993, Donoho 1994). It can be briefly described as follows: given a function that is the superposition of a smooth function and noise, one performs a forward wavelet transform, and sets to zero the “noisy” wavelet coefficients if the square of the wavelet coefficient is less than the noise variance σ^2. This procedure, known as hard or linear thresholding, is optimal for denoising signals in the presence of Gaussian white noise because wavelet-based estimators minimize the maximal L^2-error for functions with inhomogeneous regularity. In many geophysical applications, the assumption of Gaussian noise is no longer true (Holzer and Siggia 1994, Ten *et al.* 1997). In this case, alternative nonlinear thresholding strategies,

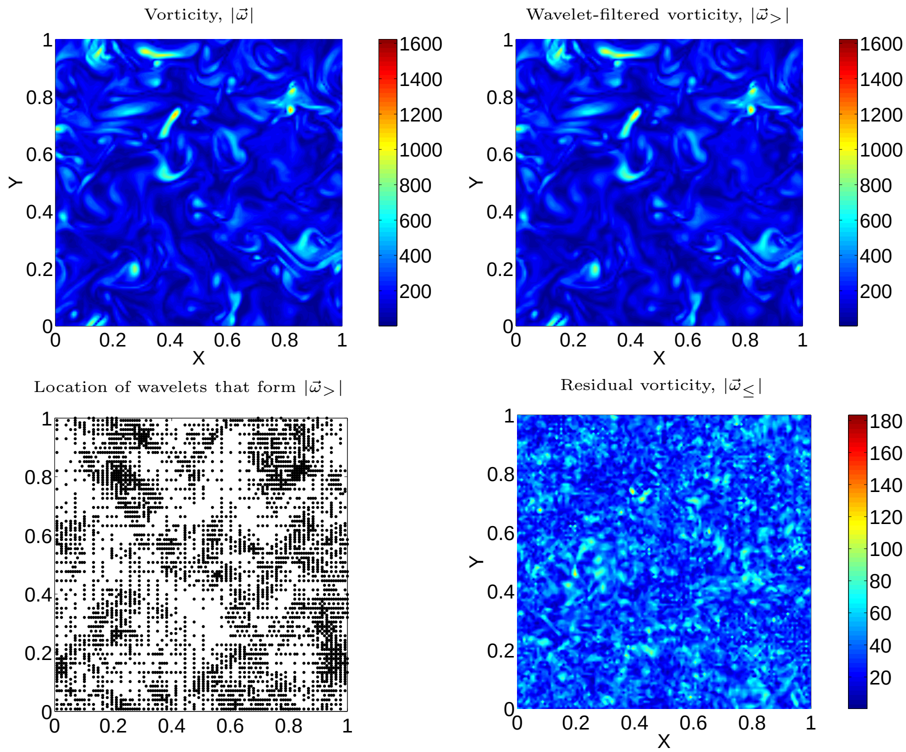


Fig. 3. Example of vorticity field decomposition (Eq. (2)) using wavelet thresholding filter for three-dimensional forced isotropic turbulence (two-dimensional slices of the three-dimensional field are shown). The locations of wavelets corresponding to coherent field are also shown. $Re_\lambda = 168$.

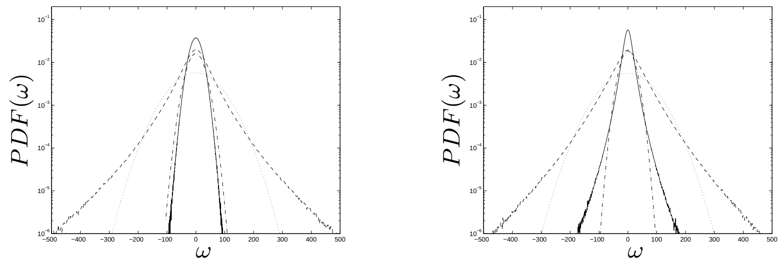
called soft thresholding, can be utilized (Donoho and Johnstone 1994). In soft thresholding, the threshold values for wavelet coefficients are scale-dependent. The most general version of soft thresholding is when the threshold value depends on both wavelet scale and wavelet location.

3.2 Coherent-Incoherent Decomposition

The wavelet de-noising property was recently used by Farge *et al.* (Farge *et al.* 1999) to suggest an approach for simulating turbulent flow, called Coherent Vortex Simulation (CVS). In CVS the turbulent vorticity field is decomposed into coherent (organized), $\vec{\omega}_>$, and incoherent (random, Gaussian), $\vec{\omega}_\leq$, fields:

$$\vec{\omega} = \vec{\omega}_> + \vec{\omega}_\leq. \tag{2}$$

This decomposition is achieved by performing a forward wavelet transform, setting to zero wavelet coefficients whose L^2 or L^∞ norm is below an *a priori* prescribed



(a) Optimal compression using wavelet filter. (b) Equivalent Fourier cutoff filter.

Fig. 4. PDF of forced isotropic turbulence field using second generation wavelet filter at 86% compression (a) and Fourier cutoff filter at the equivalent compression (b) for three-dimensional forced isotropic turbulence at $Re_\lambda = 168$. Filtered field: (– – – –), with its associated Gaussian PDF: (········). Residual field: (———), with its associated Gaussian PDF: (—·—).

threshold parameter ϵ, which can vary for different levels of resolution, followed by an inverse wavelet transform. An example of the vorticity field decomposition for three-dimensional forced homogeneous turbulence is shown in Figure 3, where two-dimensional slices are shown. Figure 3 also shows the locations of wavelets that form $|\vec{\omega}_>|$, *i.e.*, whose coefficients are above ϵ. When a non-linear wavelet thresholding filter is applied to a moderately high Reynolds number isotropic turbulence field, the residual field is close to being statistically Gaussian. This has been shown in Farge *et al.* (1999) for two-dimensional turbulent flow and by Goldstein *et al.* (2000) for three-dimensional homogeneous turbulent flow.

To demonstrate the ability of wavelet filtering to decompose the vorticity field into coherent and incoherent fields, we present the results of an *a priori* analysis of forced isotropic turbulence field at $Re_\lambda = 168$ (Jimenez and Wray 1993). In Figure 4 the Probability Density Functions (PDF) of the filtered and residual vorticity fields at optimum wavelet compression using second generation wavelets of order 6 are compared to those from a Fourier cutoff filter that retains the same number of modes (Goldstein *et al.* 2000). The difference in the Gaussianity of the residual field of the two filters can most clearly be seen in the tails. With wavelet thresholding the PDF of the residual field is clearly more Gaussian in the tails than the one resulting from the Fourier cutoff filter.

Numerical simulation of the coherent field evolution in an efficient manner requires the use of a highly adaptive numerical algorithm. A recently developed dynamically adaptive second generation wavelet collocation method (Vasilyev and Bowman 2000, **?**) is ideally suited for the solution of such problems, since the grid adaptation is based on the same criterion as in coherent structure extraction, *i.e.*, at any given time the computational grid consists of points corresponding to wavelets whose coefficients are above an optimal threshold ϵ. With this adaptation strategy a solution is obtained on a grid $G_>$ that "tracks" the coherent structures. In actuality, we would have to perform numerical simulations on a slightly bigger computational grid

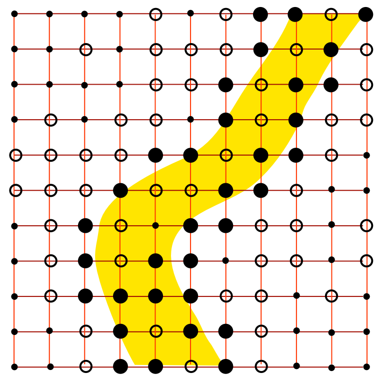


Fig. 5. Succession of grids used in coherent field simulations. Small filled dots (G_0): wavelet coefficients that are below numerical threshold ϵ_0. Large open dots ($G_{\leq}$): wavelet coefficients on adaptive grid that correspond to the incoherent field. Large filled dots ($G_{>}$) wavelet coefficients on adaptive grid defined by wavelet coefficients above $\epsilon \geq \epsilon_0$ that correspond to coherent structures. Grid $G = G_0 + G_{\leq} + G_{>}$.

that includes wavelet coefficients whose coefficients are above a numerical threshold ϵ_0, a parameter which controls the accuracy of numerical simulations. The adaptive grid structure for the coherent field simulation using wavelet collocation algorithm is illustrated in Figure 5.

3.3 Incoherent Field Statistics

In order to simulate the evolution of coherent structures, the effect of the filtered (unresolved) residual field on the resolved coherent field needs to be modelled. In theory, if the unresolved residual field is purely incoherent, then its effect on the evolution of the coherent structures can be modelled by a stochastic model. In developing such models, one first has to understand the statistical properties of the residual (incoherent) field. Classical statistics is based on processes that are stationary and isotropic in the sense that the spatial structure of the flow is independent of location. However, geological processes, such as earthquake dynamics, are inherently time-dependent and spatially heterogeneous, as schematically shown in Figure 5. Therefore statistical modelling using wavelets can be employed to address the regimes in which there is no stationarity nor spatial homogeneity. This motivation for using wavelets dates back at least as far as Cohen and Jones (1969), with their representation of a spatial field in terms of the Karhunen-Loève expansion of its covariance function, leading to representations of the form

$$Z(x) = \sum_{\nu=1}^{M} a_\nu \lambda_\nu^{1/2} \psi_\nu(x) \tag{3}$$

where $\{\psi_\nu\}$ are a fixed basis of orthogonal functions, $\{\lambda_\nu\}$ are coefficients to be estimated, and $\{a_\nu\}$ are independent standard normal random variables. Models of this form have become very widely used in geophysical sciences (Creutin and Obled 1982)). Nychka *et al.* (1999) have recently proposed models where $\{\psi_\nu\}$ are replaced by wavelet basis functions. The wavelet representation is motivated by nonstationarity and they also emphasize the computational applicability of the approach in very large systems. There is also the possibility of a mixture of the two kinds of models (Nychka and Saltzman 1998, Nychka *et al.* 1999) based on representations of the form

$$Z(x) = \sigma(x) \left\{ \rho^{1/2} Z_0(x) + \sum_{\nu=1}^{M} a_\nu \lambda_\nu^{1/2} \psi_\nu(x) \right\} \tag{4}$$

in which $Z_0(x)$ is a stationary isotropic process, ρ is a positive constant and $\sigma(x)$ is a scaling function. The idea is based on the expansion for the covariance function

$$C(x_1, x_2) = \sigma(x_1)\sigma(x_2) \left\{ \rho e^{-||x_1 - x_2||/\theta} + \sum_{\nu=1}^{M} \lambda_\nu \psi_\nu(x_1) \psi_\nu(x_2) \right\} \tag{5}$$

which permits the standard deviation to vary with location x according to a general function $\sigma(x)$, and a leading term that corresponds to a stationary isotropic model of exponential covariance type. The remaining terms depend on eigenvalues λ_ν and eigenfunction ψ_ν of the covariance operator and support various degrees of nonstationarity according to the value of the index M. This wavelet approach can be extended to non-Gaussian processes. Once understood, the statistics of the incoherent residual field can be used to develop a spatial statistical model to serve as input into the evolution equations for the coherent structures.

4 Feature extraction of thermal plumes

Similar to vortex tubes in high Reynolds number flow, thermal plumes are coherent features formed in high Rayleigh number convection (Zocchi *et al.* 1990, Moses *et al.* 1991, Yuen *et al.* 1993) and forms one of the cornerstones for the theory of hard turbulent convection (Castaing *et al.* 1989). In the Earth's mantle, where the inertia terms can be neglected in convection, thermal turbulence develops by nonlinearity due to the advection term, $\mathbf{u} \cdot \mathrm{grad}T$, in the temperature equation. This mechanism is analogous to the Reynolds stress term in the inertial flow regime. Figure 6 shows a snapshot of the temperature field in such a turbulent convective scenario at a Rayleigh number of 10^9 in a basally heated configuration in a box with an aspect-ration of $4 \times 4 \times 1$. The number of grid points used was 500^3. From the figure one discerns the presence of both connected and disconnected plumes, indicating that this flow lies in the hard-turbulent regime. The extraction of plumes under these tumultuous circumstances is a challenge for wavelets.

We have employed the second generation wavelets described above together with the wavelet-denoising procedure to extract salient features from the temperature fields

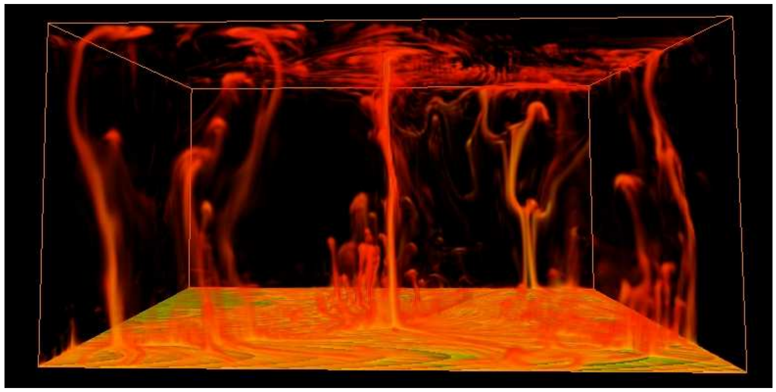


Fig. 6. Temperature field of 3-D mantle convection at a Rayleigh number of 10^9. The grid consists of 500^3 points. Finite difference and spectral methods are used in vertical and two horizontal directions respectively. The volumetric rendering covers temperatures greater than 0.7. The temperature at the bottom of the convection layer is set at $T = 1$. The top is maintained at $T = 0$. The aspect-ratio is $4 \times 4 \times 1$, with unity being the depth.

in thermal convection. We have employed a lower Rayleigh number of 10^6 for the same aspect-ratio and heating configuration as in the previous figure. The number of grid points used was 97^3. Figure 7 is a downward view of two-dimensional surfaces of the three-dimensional convection layer, where we have carried out wavelet analysis. The full reconstruction is shown at the upper left panel (Figure 7a). To its right (Figure 7b) is the view shown with the 5% largest (in magnitude) wavelet coefficients retained. We can still see similar features in the planform even when we have discarded 95% of the wavelet coefficients. The black dots shown in Figure 7c (the lower left panel) are the locations of the wavelet coefficients above the threshold of 10^{-2} in the middle of the convection layer. We can see that there is an extremely good correlation between the outlines of the convective planforms and the locations of the wavelet coefficients above the threshold value. Finally in the lower right panel (Figure 7d) we show the small scale thermal residuals left by subtracting the coherent thermal structure Figure 7b from the total reconstructed field (Figure 7a). The small-scale scars left by the coherent structure are still discernable in the residual field.

Three-dimensional aspects of the wavelet filtering and denoising are illustrated in Figure 8. The full reconstruction of the three-dimensional temperature field is shown in Figure 8a. We display in Figure 8b the locations of the points, where the wavelet threshold of 10^{-2} is exceeded, along with the thermal field constructed with the truncated set of wavelets, about 5% of the original number. This comparison shows the great efficiency of wavelets in compressing the data. The residual field (constructed from 95% of the wavelets with the smallest coefficients) associated with the incoherent thermal field is shown in Figure 8c.

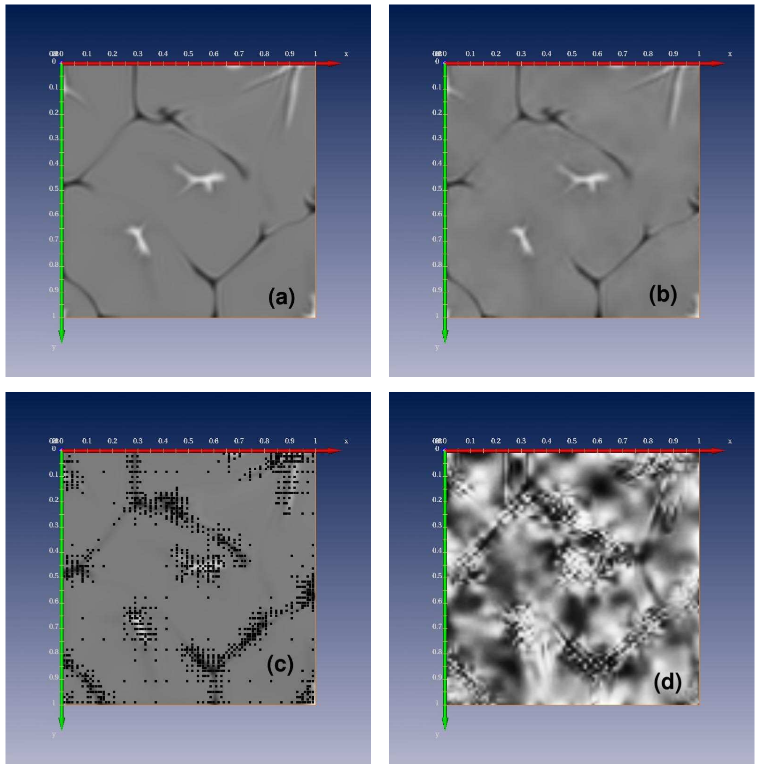


Fig. 7. Downward views of a single slice of the temperature field in 3-D convection at a Rayleigh number of 10^6 in a box of aspect ratio of $4 \times 4 \times 1$. (a) Original dataset. (b) Reconstruction of wavelet transformed dataset with the smallest 95% of the wavelet coefficient set to zero. (c) Same as (b) with the physical locations of the largest 5% of the wavelet coefficients displayed as black dots. The dots encompass a vertical zone of 4 consecutive horizontal slices. (d) Residual temperature field reconstructed from the smallest 95% of the coefficients.

5 Concluding Remarks and Perspectives

Our experience with using second generation wavelets shows that they can be employed successfully in many diverse applications in geophysics, such as the solution of nonlinear partial differential equations (Vasilyev *et al.* 1997b, Vasilyev *et al.* 1998, Vasilyev *et al.* 2001, **?**, **?**), the extraction of coherent features in mantle convection and data compression. We have set forth here the idea for a decomposition of nonlinear processes into coherent and incoherent components and new ways to model with wavelets the spatial-temporal evolution of the coherent components. This new strategy will play an important role in understanding multiscale phenomena ranging from the convoluted three-dimensional microstructures in porous media (Manwart *et al.* 2002), dilatant plasticity in shear localization processes (Bercovici 1998), all

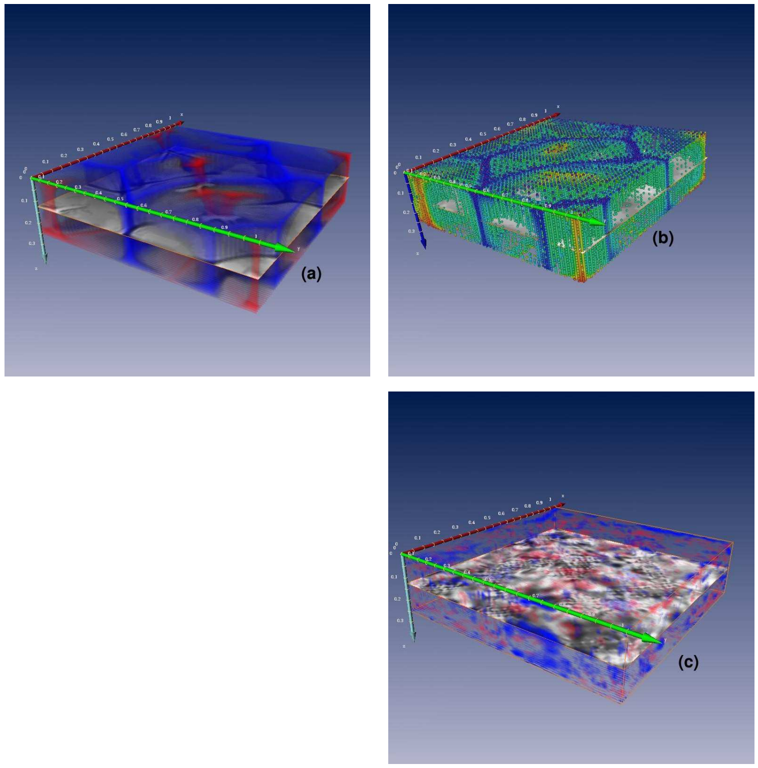


Fig. 8. Three-dimensional volume rendered views of the temperature field in 3-D convection at a Rayleigh number of 10^6 in a box of aspect ratio of 4×4. (a) Full dataset. (b) Reconstruction of wavelet transformed dataset with the smallest 95% of the wavelet coefficient set to zero. The physical locations of the non-zero wavelet coefficients are displayed as spheres colored by temperature. These spheres follow the outlies of the hot plumes and networks of cold down-wellings. (c) Residual temperature field reconstructed from the smallest 95% of the wavelet coefficients. Blue: $T \in [0.04, 0.0008]$, red: $t \in [0.003, 0.02]$.

the way to the large-scale earthquake rupturing process in solid turbulence associated with the earthquake phenomena (Kagan 1992). Multiscale methods, such as wavelets, are strongly needed to enhance the chances for new discoveries in the field of earthquake research, which is full of exotic non-linear instabilities (*e.g.* (Saleur *et al.* 1996, Ben-Zion *et al.* 1999)) quite different from those encountered in fluid mechanics (*e.g.* (Drazin and Reid 1981)).

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References

ALEXANDRESCU, M., GIBERT, D., HULOT, G., LE MOUEL, J.-L., and SARACCO, G., 1995, *Detection of geomagnetic jerks using wavelet analysis*, J. Geophys. Res., **100**, 12557–12572.

BEN-ZION, Y., DAHMEN, K., LYAKHOVSKY, V., ERTAS, D., and AGNON, A., 1999, *Self-driven mode switching of earthquake activity on a fault system*, Earth And Planetary Science Lett., **172**, 11–21.

BERCOVICI, D., 1998, *Generation of plate tectonics from lithosphere-mantle flow and void-volatile self-lubrication*, Earth Planet. Sci. Lett., **154**, 139–151.

BERGERON, S. Y., VINCENT, A. P., YUEN, D. A., TRANCHANT, B. J. S., and TCHONG, C., 1999, *Viewing seismic velocity anomalies with 3-D Continuous Gaussian wavelets*, Geophys. Res. Lett., **26**(15), 2311–2314.

BURKE-HUBBARD, B., 1998, *The World according to Wavelets*, AK. Peters, Wellesley, MA.

CASTAING, B., GUNARATNE, G., HESLOT, F., KADANOFF, L., LIBCHABER, A., THOMAE, S., WU, X., ZALESKI, S., , and ZANETTI, 1989, *Scaling of hard thermal turbulence in Rayleigh-Benard convection*, J. Fluid Mech., **204**, 1–30.

CHAO, B.-F. and NAITO, I., 1995, *Wavelet Analysis provides a new tool for studying Earth's rotation*, EOS, Transactions, American Geophys. Union, **16**, 161–165.

CHIAO, L.-Y. and KUO, B.-Y., 2001, *Multiscale seismic tomography*, Geophys. J. Int., **145**, 517–527.

COHEN, A. and JONES, R., 1969, *Regression on a random field*, Journal of the American Statistical Association, **64**.

CREUTIN, J. D. and OBLED, C., 1982, *Objective analysis and mapping techniques for rainfall fields, an objective comparison*, Water Resources Research, **18**, 413–431.

DAUBECHIES, I., 1988, *Orthonormal Bases of Compactly Supported Wavelets*, Comm. Pure and Appl. Math., **41**, 909–996.

DAUBECHIES, I., 1990, *The wavelet transform, time-frequency localization and signal analysis*, IEEE Trans. Inform. Theory, **36**, 961–1005.

DEVORE, R. A., JAWERTH, B., and LUCIER, B. J., 1992, *Image Compression Through Wavelet Transform Coding*, IEEE Trans. Inform. Theory, **38**(2), 719–746.

DONOHO, D., 1993, *Unconditional Bases are Optimal Bases for Data Compression and for Statistical Estimation.*, Appl. Comput. Harmon. Anal., **1**, 100–115.

DONOHO, D. L., 1994, *De-noising by soft-thresholding*, IEEE Trans. Inf. Theory, **41**(3), 613–627.

DONOHO, D. L. and JOHNSTONE, I. M., 1994, *Ideal Spatial Adaptation via Wavelet Shrinkage*, Biometrika, **81**, 425–455.

DRAZIN, P. and REID, W. H., 1981, *Hydrodynamic Stability*, Cambridge University Press.

ERLEBACHER, G., HUSSAINI, M. Y., and JAMESON, L. M., eds., 1996, *Wavelets: Theory and Applications*, Oxford University Press.

FARGE, M., SCHNEIDER, K., and KEVLAHAN, N., 1999, *Non-Gaussianity and Coherent Vortex Simulation for Two-Dimensional Turbulence Using an Adaptive Orthogonal Wavelet Basis*, Phys. Fluids., **11**(8), 2187–2201.

GOLDSTEIN, D. A., VASILYEV, O., WRAY, A., and ROGALLO, R., 2000, Evaluation of the use of second generation wavelets in the coherent vortex simulation approach, in *Proceedings of the 2000 Summer Program*, pp. 293–304, Center for Turbulence Research.

GOUPILLAUD, P., GROSSMAN, A., and MORLET, J., 1984, *Cyclo-Octave and Related Transforms in Seismic Signal Analysis*, Geoexploration, **23**, 85–102.

HOLZER, M. and SIGGIA, E. D., 1994, *Turbulent mixing with a passive scalar*, Phys. Fluids, **6**(5), 1820–1837.

JAMESON, L. and MIYAMA, T., 2000, *Wavelet analysis and ocean modeling: A dynamically adaptive numerical method "WOFD-AHO"*, Monthly Weather Review, **128**, 1536–1548.

JIMENEZ, J. and WRAY, A. A., 1993, *The structure of intense vorticity in isotropic turbulence*, J. Fluid Mech., **255**, 65–90.

KAGAN, Y. Y., 1992, *Seismicity: turbulence of solids*, Nonlinear Sci. Today, **2**, 1–13.

KUMAR, P. and FOUFOULA-GEORGIOU, E., 1997, *Wavelet analysis for geophysical applications*, Review Geophysics, **35**, 385–412.

Luo, J. and Jameson, L., 2002, *A wavelet-based technqiue for identifying, labeling and tracking of ocean eddies*, J. Atmos. and Ocean Tech., **19**(3), 381–390.

Mallat, S., 1998, *A Wavelet Tour of Signal Processing*, Academic Press.

Manwart, C., Aaltosalmi, U., Koponen, A., Hilfer, R., and Timonen, J., 2002, *Lattice Boltzmann and finite-difference simulations for the permeability for three-dimensional porous media*, Phys. Rev E, **66**, 016702.

Meyer, Y., 1990, *Ondelettes et Opérateurs*, Hermann, Paris.

Moses, E., Zocchi, G., Procaccia, I., and Libchaber, A., 1991, *The dynamics and interaction of laminar thermal plumes*, Europhys. Lett., **14**, 55–60.

Nychka, D. and Saltzman, N., 1998, Design of air quality networks, in *Case Studies in Environment statistics*, edited by D. Nychka, W. W. Piegorsch, and L. H. Cox, no. 132 in Lecture Notes in Statistics, Springer-Verlag, New York.

Nychka, D., Wikle, C., and Royle, J., 1999, Large spatial prediction problems and nonstationary random fields, Tech. rep., Geophysical Statistical Program, National Center for Atmospheric Research.

Saleur, H., Sammis, C., and Sornette, D., 1996, *Discrete scale invariance, complex fractal dimensions, and log-periodic fluctuations in seismicity*, J. Geophys. Res., **101**, 17,661–17,677.

Simons, M. and Hager, B. H., 1997, *Localization of the gravity field and the signature of glacial rebound*, Nature, **390**, 500–504.

Strang, G. and Nguyen, T., 1996, *Wavelets and Filter Banks*, Wellesley-Cambridge Press, Wellesley, MA.

Sweldens, W., 1996, *The lifting scheme: A custom-design construction of biorthogonal wavelets*, Appl. Comput. Harmon. Anal., **3**(2), 186–200.

Sweldens, W., 1998, *The lifting scheme: A construction of second generation wavelets*, SIAM J. Math. Anal., **29**(2), 511–546.

Ten, A., Yuen, D., Podladchikov, Y., Larsen, T., Pachepsky, E., and Malvesky, A., 1997, *Fractal features in mixing of non-Newtonian and Newtonian mantle convection*, Earth Planet. Sci. Lett., **146**, 401–414.

Van den Berg, J. C., ed., 1999, *Wavelets in Physics*, Cambridge University Press.

Vasilyev, O. V. and Bowman, C., 2000, *Second Generation Wavelet Collocation Method for the Solution of Partial Differential Equations*, J. Comp. Phys., **165**, 660–693.

Vasilyev, O. V. and Paolucci, S., 1997, *A Fast Adaptive Wavelet Collocation Algorithm for Multi-Dimensional PDEs*, J. Comput. Phys., **125**, 16–56.

Vasilyev, O. V., Yuen, D. A., and Paolucci, S., 1997, *The Solution of PDEs Using Wavelets*, Computers in Phys., **11**(5), 429–435.

Vasilyev, O. V., Yuen, D. A., and Podladchikov, Y. Y., 1997, *Applicability of Wavelet Algorithm for Geophysical Viscoelastic Flow*, Geophys. Res. Lett., **24**(23), 3097–3100.

Vasilyev, O. V., Podladchikov, Y. Y., and Yuen, D. A., 1998, *Modeling of Compaction Driven Flow in Poro-Viscoelastic Medium Using Adaptive Wavelet Collocation Method*, Geophys. Res. Lett., **25**(17), 3239–3242.

Vasilyev, O. V., Podladchikov, Y. Y., and Yuen, D. A., 2001, *Modeling of Viscoelastic Plume-Lithosphere Interaction Using Adaptive Multilevel Wavelet Collocation Method*, Geophys. J. Int., **147**(3), 579–589.

Vecsey, L. and Matyska, C., 2001, *Wavelet spectra and chaos in thermal convection modelling, submitted to Geophys*, Geophys. Res. Lett., **28**(2), 395–398.

Yuen, D. A., Hansen, U., Zhao, W., Vincent, A. P., and Malevsky, A. V., 1993, *Hard turbulent thermal convection and thermal evolution of the mantle*, J. Geophys. Res., **98**(E3), 5355–5373.

Yuen, D. A., Vincent, A. P., Kido, M. J. B., and Vecsey, L., 2002, *Geophysical applications of multidimensional filtering with wavelets*, Pure and Applied Geophysics, **159**(10), 2285–2309.

Zocchi, G., Moses, E., and A., L., 1990, *Coherent structures in turbulent convection, an experimental study*, Physica, **166**, 387–407.