Numerical Experiments in Supersonic Boundary Layer Stability

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Abstract

The three-dimensional time-dependent compressible Navier-Stokes equations are numerically solved by a Fourier-Chebyshev collocation method to study the stability of supersonic flows over a flat plate. Several direct simulations carried out in this study suggest the existence of a secondary instability which might provide a route to transition. The interaction of the modes involved in this secondary instability is possibly amenable to a Floquet type analysis. Pertinent differences between this instability and the analogous incompressible K-type secondary instability are pointed out. Some preliminary results of a 2-D direct simulation of the nonlinear evolution of a second mode perturbation wave are also discussed

1 Nomenclature

C_p	: specific heat at constant pressure
C_v	: specific heat at constant volume
$\mathbf{I}$	: identity matrix
L_x, L_z	: streamwise and spanwise periods
M	: Mach number
p	: pressure
Pr	: Prandtl number
R	: Rankine
Re	: Reynolds number
t	: time
T	: temperature
u, v, w	: streamwise, normal and spanwise velocity
u_ϕ	: streamwise phase velocity
$\mathbf{u}$	: velocity vector
x, y, w	: streamwise, normal and spanwise directions
x_0	: point along plate were stability is studied
y_{max}	: maximum vertical coordinate
$y_{1/2}$	: half the nodes lie in $y < y_{1/2}$
α	: streamwise wavenumber

β : spanwise wavenumber

δ^* : displacement thickness

η : Chebyshev node distribution

ϵ : perturbation amplitude

E : integrated perturbation kinetic energy in y

F : non-dimensional frequency

$\mathcal{F}$: forcing term

γ : ratio of specific heats

ρ : density

μ : viscosity

ν : kinematic viscosity

ω : linear growth rate or vorticity

Φ : dissipation function

ψ : wave angle

σ : viscous stress tensor

θ : phase angle

subscripts

i : imaginary part

j, l : Fourier modes

∞ : free-stream

m : mean value

r : real part

x, y, z : streamwise, spanwise and normal components

$2D$: two-dimensional quantity

$3D$: three-dimensional quantity

superscripts

T : transpose

$\hat{}$: Fourier component

$'$: perturbation variable

2 Introduction

Recent advances in supersonic and hypersonic aerospace technology have led to a renewed interest in the stability and transition to turbulence of high speed flows, of which very little is known. The last 30 years have witnessed intermittent attempts to understand some of the fundamental routes of transition to turbulence. While a fairly comprehensive picture of the initial stages leading to the breakdown of an incompressible laminar boundary layer has emerged (under certain controlled conditions) [1], the nonlinear effects responsible for transition at high speeds are still very much a mystery. However, some of the known mechanisms operative at low Mach numbers will possibly still be present (in some modified form) in high speed flows. One such mechanism studied here is the secondary instability of the fundamental type.

Lees and Lin [2] were the first to study the linear stability of compressible flows in 1946, albeit in the inviscid limit. Their study was restricted to low Mach numbers and the conclusions drawn only apply to the first unstable mode. Mack unraveled some of the intricacies among the many possible modes present when the full stability equations are considered (including viscous and conductivity terms). From 1960 to the present time, Mack has devoted himself to a thorough numerical investigation of the linear stability characteristics of compressible flows at both subsonic and supersonic Mach numbers. An extensive review of compressible linear theory appears in his 1982 AGARD report [3]. Out of this research came the discovery of the higher unstable modes for Mach numbers above 2.2. The importance of the higher modes stems from their high spatial and temporal amplification rates and their different response to wall conditions. For example, wall cooling stabilizes first mode waves, while second mode waves are destabilized. At Mach numbers greater than 7, cone experiments have confirmed that the second mode is primarily responsible for the observed instabilities [4, 5]. Recently, Malik has shown that suction and favorable pressure gradients stabilize the second mode [6].

The relationship between the inflection point in the mean boundary layer velocity profile and instability has its counterpart in compressible flows. The inflection point now becomes the generalized inflection point which takes into account the mean density variation across the boundary-layer. The existence of the generalized inflection point is a sufficient condition for the existence of an amplified disturbance if the relative Mach number is subsonic everywhere, and if it occurs above the point where the real part of the phase velocity is greater than $1 - M_\infty^{-1}$ [3] where M_∞ is the free-stream Mach number.

However, there are also unstable waves unrelated to inflectional profiles, which depend on the existence of local regions of relative supersonic flow between the wall and the sonic point. The linear stability characteristics of compressible flows are fully described in some of Mack's articles [3, 7, 8].

To date, there are only a handful of experiments that have studied the detailed flat plate linear stability characteristics [9, 10, 5]. Laufer and Vrebalovich studied the linear stability of a Mach 2.2 flow on a flat plate, and theirs is perhaps the most complete (if not the only) flat plate experiment to date at supersonic velocities which provides information about the amplitude distributions across the boundary-layer. Kendall [10] performed flat plate stability experiments for Mach numbers ranging from 2.2 to 8.5, and observed that at the Mach 4.5 data provided the best agreement with linear theory.

Not only are hot wire measurements difficult to perform and to interpret correctly [11], but the stability characteristics of the boundary layer are affected by free-stream disturbances emanating from the wind tunnel walls. Since the disturbances scale as M_∞^2, it becomes increasingly difficult to obtain good, quantitative data at the higher speeds, except in specially designed wind tunnels. Experiments performed in the Mach 3.5 quiet tunnel at NASA Langley Research Center have already resolved some controversy between cone and flat plate results [12].

Among the few numerical codes geared towards the analysis of compressible linear stability, the best known are those of Mack [13] and Malik [14]. The former calculates spatial and temporal amplification rates, while the latter is strictly restricted to temporal analysis; a Gaster transformation is then applied to calculate the spatial growth rates. More recently, Macaraeg et al. [15], Malik [16] have developed spectral linear stability codes. The work by Erlebacher and Hussaini [24] was the first in which time accurate non-linear simulations of supersonic boundary-layer flows were presented. Between linear stability theory and direct simulation lies a large void which can be filled by several types of weakly nonlinear theories. El-Hady [17] and Nayfeh [18] have extended Herbert's secondary stability theory to compressible flow. At present, both theories can only treat instabilities of the subharmonic type. El-Hady [17] has only calculated secondary amplification rates to Mach 2.2 for the first mode. Nayfeh [18] on the other hand has studied the secondary subharmonic instability of first and second modes up to Mach 4.5. In all cases, the primary flow is two-dimensional. These theories require a lot of care if accuracy is desired since the primary instability wave and its first two derivatives must be known accurately if spurious secondary instability modes are to be avoided. Cross checks between several secondary instability formulations are necessary before the results can be fully accepted. The relative complexity of fundamental secondary instability theories compared to the subharmonic one is possibly the reason why it has not yet been studied.

In this paper, the non-stationary, three-dimensional compressible Navier-Stokes equations are numerically solved, for wall-bounded flows, to study the possible existence of a fundamental secondary instability at Mach 4.5. Mach 4.5 was chosen for two reasons. First, as mentionned above Kendall's experimental results agree best with linear theory at this Mach number. Second, the second mode temporal amplification rate is maximal at Mach 4.5, and while in a spatial code, the experimental evidence implies that the second mode only starts dominating the flow characteristics above Mach 7, it is more than likely that the qualitative physics obtained from an isolated mode at Mach 4.5 will provide the correct qualitative picture. The higher growth of the second mode at Mach 4.5 will also reduce the cost of already expensive simulations.

The evolution of a triad of first mode waves superimposed on a parallel mean boundary-layer profile is temporally followed to the initial stages of nonlinearity. Numerical experiments are conducted to confirm that the instability possesses some of the properties of a secondary instability (as defined for incompressible flows). Several characteristics of this instability which differ from the usual incompressible K-type breakdown are discussed. At Mach 4.5, the second mode is dominant in the temporal linear theory. It is more than likely that the instabilities seen in experiments will be of the 2nd mode type. This mode has properties of acoustic waves (stationary waves in a bounded region of relative supersonic flow). As a first step towards the study of its secondary instability properties, the existence of a two-dimensional saturated state is investigated. One 20 period computer simulation at $M_i nfty = 4.5$ indicates the existence of a saturated state. The observed nonlinearities due to compressibility effects are discussed.

3 Equations

The compressible Navier-Stokes equations for the stability of parallel flows are, in dimensionless form,

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0 \tag{1}$$

$$\frac{\partial (\rho \mathbf{u})}{\partial t} + \nabla \cdot (\rho \mathbf{u}\mathbf{u}) = \frac{1}{Re} \nabla \cdot \sigma \tag{2}$$

$$\frac{\partial p}{\partial t} + \mathbf{u} \cdot \nabla p + \gamma p \nabla \cdot \mathbf{u} = \frac{1}{Re Pr M_\infty^2} \nabla \cdot (\mu \nabla T) + (\gamma - 1)\Phi \tag{3}$$

where

$$\sigma = 2\mu \left[\frac{1}{2}(\nabla \mathbf{u} + \nabla \mathbf{u}^T) - \frac{1}{3}(\nabla \cdot \mathbf{u})\mathbf{I} \right] \tag{4}$$

is the viscous stress and

$$\Phi = \frac{1}{2}(\nabla \mathbf{u} + \nabla \mathbf{u}^T) : \sigma \tag{5}$$

is the viscous dissipation function. The density, velocity vector, temperature and presure are denoted by ρ, $\mathbf{u}$, T and p respectively. The equation of state is

$$\gamma M_\infty^2 p = \rho T, \tag{6}$$

with $\gamma = C_p/C_v$, the ratio of specific heats. The coordinate system is (x, y, z)=(streamwise, normal, spanwise) direction. Lengths are non-dimensionalized with respect to the displacement thickness δ^*. Reference values for velocity, temperature, viscosity, and density are their free-stream values (indicated by a subscript ∞). Pressure is normalized with respect to the total head $\rho_\infty U_\infty^2$. The Prandtl number, Pr, is assumed equal to 0.7, and Sutherland's law is prescribed for the viscosity, μ. The Reynolds number, Re is based on δ^*.

In incompressible boundary-layer flows, it has been observed that the evolution to laminar breakdown for ribbon-excited Tollmien-Schlichting (TS) waves occupies a space of several streamwise wavelengths [19]. An adequate resolution of the boundary layer in all three directions over this distance (up to the first and second spikes) would exceed the present capability of today's most powerful computers. The situation is worse for compressible flows where the instability mechanisms are often attenuated. The validity of the parallel approximation has been successfully demonstrated in incompressible theories [20] and numerical simulations [21]. It remains to be established whether the parallel flow approximation will yield useful qualitative information for highly compressible flows. The computational domain is restricted to one period in the streamwise distance. Periodicity is also assumed in the spanwise direction. A properly posed initial-value problem can be set up, and qualitative comparisons can be made with experimental data by relating time evolution (in the simulations) to the streamwise x directions in the experiments. For consistency, the mean parallel boundary layer flow must be a stationary solution to the Navier-Stokes equations; for this purpose source terms must be added to equations (1)-(3). Specifically, the source term in the momentum equation is

$$\mathcal{F}_{\rho u} = \frac{1}{Re}\frac{\partial}{\partial y}(\mu\frac{\partial U_m}{\partial y}) \tag{7}$$

while in the energy equation

$$\mathcal{F}_p = \frac{1}{RePrM_\infty^2}\frac{\partial}{\partial y}(\mu\frac{\partial T_m}{\partial y}) + \frac{1}{Re}[(\frac{\partial U_m^2}{\partial y})^2] \tag{8}$$

These source terms are a function of the mean streamwise velocity and temperature profile $U_m(y)$, and $T_m(y)$. In obtaining these relations, the mean pressure is kept constant across the boundary layer. The mean velocity has only a streamwise non-zero component.

4 Algorithm

The Navier-Stokes equations (1)-(3) with source terms (7)-(8) are solved in conservative form under the parallel flow assumption. Periodicity in the streamwise and spanwise directions permit a Fourier representation of the primitive variables u, v, w, p and ρ. The periods of the physical domain in the streamwise and spanwise directions (L_x and L_z) are related to the wave numbers α and β by $L_x = 2\pi/\alpha$ and $L_z = 2\pi/\beta$. In the absence of discontinuities in the solution, spectral collocation methods offer a high degree of accuracy along with very low dispersion and dissipation errors [22]. In the direction normal to the plate, Chebyshev collocation is the natural choice. The physical domain $(0, y_{max})$ in the direction normal to the plate is mapped onto the computational domain with an algebraic mapping

$$y = \frac{y_{1/2} y_{max}(1+\eta)}{y_{max} - \eta(y_{max} - 2y_{1/2})} \qquad -1 \leq \eta \leq 1. \tag{9}$$

One half the nodes are located between $y = 0$ and $y = y_{1/2}$. In all the first mode simulations presented in this paper, $y_{1/2}$ is set to 2 and $y_{max} = 30$. Due to the strong non-linearities near the generalized inflection point in the second mode wave structure, a hyperbolic tangent mapping is imposed prior to the algebraic stretching. The physical domain $y \in [0, y_{max}]$ in the direction normal to the plate is mapped onto the computational domain $\eta \in [-1, 1]$ through a convolution of two mappings. First, a hyperbolic tangent transformation takes η to an intermediate variable ψ according to

$$\psi + t_\epsilon \tanh\left(\frac{\psi - \psi_0}{\Delta\psi}\right) = \frac{\eta - \eta_0}{\Delta\eta}. \tag{10}$$

In this equation, the hyperbolic tangent concentrates nodes about ψ_0, and $\Delta\psi$ measures the width of the concentration region. t_ϵ measures the degree of influence of the hyperbolic tangent term. If $t_\epsilon = 0$, there is no redistribution of nodes at this level. The algebraic (Eq. (9)) stretching then maps ψ onto the physical variable $y \in [0, y_{max}]$. The user specifies the following: 1) t_ϵ, 2) y_0, the physical location about which the concentration occurs, 3) Δy_0, the concentration width in physical space, 4) y_{max}, the maximum vertical extent of the physical domain, and 5) $y_{1/2}$. With both mappings in hand, $\eta_0, \Delta\eta$ ψ_0 and $\Delta\psi$ are readily determined [23]. All second mode runs were performed with $y_{max} = 15$, $y_{1/2} = 1$, $t_\epsilon = 0.8$, $y_0 = 1.2$ and $\Delta y_0 = 0.4$.

5 Initial and Boundary Conditions

Perturbation waves are superimposed on a 2-D mean flow in a manner consistent with linear parallel stability theory. The mean flow velocity profiles are generated from the solution to the similar compressible boundary-layer equations with zero pressure gradient and zero heat transfer at the wall. To impose a parallel mean flow, the vertical component of velocity is neglected. Forcing terms result from the condition that the mean flow must satisfy the discretized flow equations (1)-(3). These additional terms are the forcing terms given earlier, but with a discretization consistent with the spectral discretization.

Upon the mean flow is superimposed one perturbation wave which is a 2-D analogue of the 2-D Tollmien-Schlichting (TS) wave (hereafter referred to as TS waves) and two skewed TS waves of streamwise wavenumber α and spanwise wavenumbers $\pm\beta$. This insures that the spanwise velocity vanishes in two spanwise planes (normal to the spanwise direction) separated by $L_z/2$. In incompressible flows, these planes are distinguished by the structure of the normal velocity component v. The peak plane is distinguished by a predominantly upward component of v, while the valley plane is characterized by v predominantly downward. The spanwise structures of the flow become very concentrated about the peak plane. For incompressible flows, the classical experiments of Klebanoff et al. [25] and Kovasznay et al. [19] indicate that the laminar breakdown process takes place in the vicinity of the peak plane. In supersonic flows however, the numerical simulations performed indicate that the distinction between peak and valley planes is not clear cut as it is for low subsonic Mach numbers, and it is, therefore, premature to draw any definite conclusions about the laminar breakdown process.

In summary, the initial conditions are

$$
\begin{aligned}
u(x,y,z) &= U_m(y) + \epsilon_{2D} u_{2D}(y)\cos(\theta_{2D}(y) + \alpha x) \\
&\quad + 2\epsilon_{3D} u_{3D}(y)\cos((\theta_{3D}(y) + \alpha x))\cos(\beta z), \\
v(x,y,z) &= \epsilon_{2D} u_{2D}(y)\cos(\theta_{2D}(y) + \alpha x) \\
&\quad - 2\epsilon_{3D} u_{3D}(y)\sin((\theta_{3D}(y) + \alpha x))\sin(\beta z), \\
w(x,y,z) &= \epsilon_{2D} u_{2D}(y)\cos(\theta_{2D}(y) + \alpha x) \\
&\quad + 2\epsilon_{3D} u_{3D}(y)\cos((\theta_{3D}(y) + \alpha x))\cos(\beta z), \\
\rho(x,y,z) &= \rho_m(y)\epsilon_{2D} u_{2D}(y)\cos(\theta_{2D}(y) + \alpha x) \\
&\quad + 2\epsilon_{3D} u_{3D}(y)\cos((\theta_{3D}(y) + \alpha x))\cos(\beta z), \\
p(x,y,z) &= p_m(y)\epsilon_{2D} u_{2D}(y)\cos(\theta_{2D}(y) + \alpha x) \\
&\quad + 2\epsilon_{3D} u_{3D}(y)\cos((\theta_{3D}(y) + \alpha x))\cos(\beta z).
\end{aligned}
\tag{11}
$$

Two-dimensional simulations are conducted with $\epsilon_{3D} = 0$. The 2-D and 3-D TS waves are eigenfunctions of the linearized Navier-Stokes equations and were calculated with the linear compressible stability code COSAL [14]. The eigenfunctions used in the second mode computations were obtained from SPECLS [15]. The angles θ_{2D} and θ_{3D} are the phase angles of the complex TS waves. The maximum amplitudes of u_{2D} and u_{3D} are unity. Therefore, ϵ_{2D}

and ϵ_{3D} are perturbation wave amplitudes.

All variables are periodic in the streamwise and spanwise directions. No-slip conditions are applied at the adiabatic wall. In the far-field, all the variables are frozen at their initial values, which is permitted because the perturbation variables are vanishingly small. (y_{max} is sufficiently large that the primitive variable perturbations are negligible.) The far-field conditions on velocity and temperature for the second mode calculations are determined by a zero normal stress condition [23].

6 Results

First Mode

In this paper, all first mode simulations correspond to $M_\infty = 4.5$, $T_\infty = 110°R$, $Re_{\delta^*} = 10000$, $\alpha = 0.6$ and $\beta = 1.032$. The linear eigenvalues that correspond to the above parameters are ($\omega = \omega_r + \imath\omega_i$)

$$\begin{aligned} \omega_{2D} &= 0.5012 + 0.00204i \\ \omega_{3D} &= 0.4977 + 0.01980i. \end{aligned} \tag{12}$$

The initial perturbation waves are assumed to have the form

$$u_{pert}(x,y,z) = u'(y)e^{i(\alpha x + \beta z - \omega_r t) + \omega_i t} \tag{13}$$

so that the ω_r is the wave frequency and ω_i the growth rate. The wave is unstable if $\omega_i > 0$. Based on the 2-D wave frequency, the temporal period is 12.54 in non-dimensional time units. In what follows, time is always expressed in units of 2-D wave periods. The 2-D and 3-D streamwise phase velocities are approximately equal to each other. Based on an approximate value of $\omega_r = 0.5$, the streamwise phase velocity is $u_{\phi 2D} = 0.83$ which corresponds to a critical layer at $y_{cr} = 0.98$. The mean profile has a generalized inflection point [3] at $y_{cr} = 1.02$. The mean temperature across the boundary layer varies by more than a factor of 4. This is in sharp contrast with the slowly varying mean temperature profiles at low subsonic Mach numbers. Figures 1-2 compare the 2-D and 3-D eigenfunction profiles used as initial conditions in the numerical simulations, against representative profiles at Mach 0.5. At Mach 0.5, the flow parameters are $Re_{\delta^*} = 1100$, $T_\infty = 520°R$, $\alpha = 0.25$ and $\beta = 0.209$ (wave angle $\psi = 40°$). This set of figures shows that the temperature fluctuations at Mach 4.5 are several times larger than the corresponding velocity fluctuations. This is in direct contrast to the low Mach number case (Fig. 1) where the temperature fluctuations are quite low. Another effect of high Mach number is the outward push of the critical layer towards the free-stream. At $M_\infty = 4.5$, $y_{cr} = 1.02$ while at $M_\infty = 0.5$, $y_{cr} = 0.61$. Since the lengths are expressed in terms of δ^*, it is clear that the critical layer at $M_\infty = 4.5$ is much farther away from the wall. At $M_\infty = 4.5$, $\delta^* = 10.5\sqrt{\nu_\infty x/U_\infty}$, while at $M_\infty = 0.5$, $\delta^* \approx 1.7\sqrt{\nu_\infty x/U_\infty}$. Here, ν_∞ is the free-stream kinematic viscosity. Plots of second mode eigenfunctions can be found in [23].

Four numerical simulations (S1 to S4) were performed on the Numerical Aerodynamic Simulation Cray 2 at NASA Ames Research Center, to determine whether a primary first mode instability at $M_\infty = 4.5$ sets the stage for a secondary instability. Case S1 considers $\epsilon_{2D} = 0.017$ and $\epsilon_{3D} = 0.004$.

The energy content E_{jl} in Fourier mode (j,l) is defined by

$$E_{jl} = \int_0^{y_{max}} \rho_m |\hat{\mathbf{u}}_{jl}|^2 dy \tag{14}$$

where $\hat{\mathbf{u}}_{jl}$ is the Fourier coefficient of $\mathbf{u}$ in the double Fourier expansion

$$\vec{u}(x,y,z) = \sum_{j=-N_x/2}^{N_x/2-1} \sum_{l=-N_z/2}^{N_z/2-1} \hat{\mathbf{u}}_{jl}(y) e^{i(jx+lz)} \tag{15}$$

of the velocity vector. E_{jl} is normalized with respect to

$$\tfrac{1}{2}\int \rho_m (u^2+v^2+w^2) dxdydz \tag{16}$$

at $t = 0$. When the flow is incompressible, ρ_m is a constant, and E_{jl} is the integrated (along y) perturbation kinetic energy in the (j,l) Fourier mode. However, in the compressible regime, this description is not exact. A more exact definition of the (j,l) perturbation kinetic energy level should be

$$E_{jl} = \int_0^{y_{max}} \left[\rho_m |\hat{\mathbf{u}}_{jl}|^2 + \frac{1}{2} U_m (\hat{\rho}_{jl}\hat{u}_{-j,-l} + \hat{\rho}_{jl}\hat{u}_{-j,-l})\right] dy \tag{17}$$

which is more suitable for density fluctuations large with respect to the velocity fluctuations, but still small with respect to the free-stream mean flow levels. However, the form defined by Eq. (17) is used in the diagnostics, mostly because we are interested in the growth rate characteristics of the fourier components, not the actual exchange of energy between modes.

Figure 3 tracks the perturbation kinetic energy in several Fourier modes for nearly 9 wave periods. After a linear interval of almost 3 wave periods, the (1,1) mode becomes stable. The rise of the (0,1) spanwise mode which is possibly the interaction between the (1,0) and the (-1,1) modes. The (0,1) overtakes the (1,1) mode at $t = 5$ at which point the 3-D wave starts losing energy. The 2-D mode grows at a constant rate for the full duration of the simulation. At $t = 9$, the simulation was stopped because of loss of resolution in the normal direction. At this point, the grid size was $24 \times 24 \times 65$. Based on these results, one might prematurely conclude that the secondary instability is absent. However, unless the simulation is pursued further, the question cannot be put to rest. Unfortunately, higher resolution in y renders the explicit simulation prohibitively costly. Therefore, a second simulation, S2, identical to S1 except for initial perturbation amplitudes three times those of S1 was conducted.

In S2, $\epsilon_{2D} = 0.054$ and $\epsilon_{3D} = 0.012$. A criticism might be leveled against the choice of such a high 3-D amplitude. However, Fig. 4 clearly shows that the evolution of the various modes in S2 for $0 < t < 3.5$ is qualitatively similar to the range $0 < t < 9$ in S1. In this range, the (1,0) and (1,1) modes increase linearly and the (0,2) mode overtakes the (1,1) mode, at which point, the (1,1) mode starts losing energy (t=2.7). One point of difference is the stabilization of the 2-D mode in S2 which probably loses its energy to (0,1) by interacting with the decreasing $(-1, 1)$ mode. It is quite possible that the structure of the 2-D mode at the time the secondary instability effect takes hold, would be qualitatively different had the initial 3-D amplitude been more realistic. The choice was made to achieve realistic results in a reasonable number of Cray 2 CPU hours (≈ 100). When the (0,1) mode reaches a threshold amplitude, its interaction with the (1,0) mode triggers the sudden growth of (1,1). This growth is accompanied by a sudden decrease in (0,2) which could be losing its energy to (1,1) through the interaction

$$(0, 2) + (1, -1) \rightarrow (1, 1). \tag{18}$$

The interpretation of the above events as a secondary instability is confirmed in S3 where $\epsilon_{2D} = 0$ and ϵ_{3D} retains its value from S2. It seems clear from Fig. 5 that the (0,2) mode results from the interaction of (1,1) and (1,-1). Modes (0,1) and (1,1) are outside the plot limits. As the (0,2) mode gains in intensity ($t > 4$), it interacts with (1,-1) whose growth rate is increased. This interaction also explains the decrease in the growth rate of (0,2). Of course the direction of energy flow from one mode to another can only be ascertained from an analysis involving modal equations of the Landau type. At $t \approx 7$, the (0,2) mode overtakes the (0,1) mode which leads to an increase of spanwise vorticity structure in the valley plane. Until the development of a weakly non-linear or Floquet theory, the above assumptions on the wave interaction mechanisms offer the simplest working hypothesis. With sufficient computer resources, some of these hypotheses can be tested by artificially suppressing specific modes in the numerical simulation. This is done in S4 (Fig. 6) when, starting at $t = 4$, the (0,1) mode is set to zero at each time step. A comparison of Figs. 4 and 6 shows that the growth rate of the (1,1) mode after the onset of the secondary instability has been attenuated. If the (0,1) spanwise mode had been suppressed from the onset of the simulation, the attenuation would probably be stronger. The onset of the secondary instability is also delayed. It begins at $t = 4.3$ in S1, and at $t = 4.6$ in S4. These effects indicate that the (0,1) mode feeds, either directly or indirectly, energy into the 3-D wave. As seen from S3, the (0,2) mode might also play a role. Therefore, more experiments of the S4 type are desirable.

To further substantiate the presence of the secondary instability, an additional run is conducted with 2-D and 3-D wave amplitudes levels respectively set to 3.5% and 0.5%. To reduce the cost of the simulations to a few Cray 2 CPU hours, we compute on a coarse grid of $8 \times 8 \times 49$ and partially dealiase the primitive variables with a one half rule. (A one half rule is necessary to properly dealias the cubic non-linearities in the momentum equation.) The initial conditions are generated with a spectral linear stability code which incorporate the hyperbololic tangent mapping described earlier. The parameters used for this mapping are the same as mentionned above. Convergence of the solution on this coarse grid is checked by repeating the simulation on a $12 \times 12 \times 49$ grid and plotting the energy harmonics for the $(1,1)$, $(0,1)$, $(0,2)$, and $(1,0)$ modes (Fig. 7). Identical results are obtained to within a couple of percent. Certainly the conclusions drawn from the energy time plots are valid. The similar trends between these results and those of S2 are evident.

Plots of spanwise vorticity of S2 at t=8 are presented in Fig. 8 and compared with similar plots (Fig. 9) from the secondary instability stage of an incompressible simulation performed with the initial conditions of Kovasznay et al. [19]. The necessarily different parameter ranges and regimes preclude a direct comparison of features except in the most qualitative sense. Both figures are plotted to a height of $2\delta^*$, but it must be remembered that at M=0, $\delta^* \approx 1.72\delta$, while at under the present supersonic conditions, $\delta^* \approx 10.5\delta$ so that the structures are much further removed from the wall in the supersonic regime. In both regimes, vorticity is produced in the critical layer region and at the wall. In the compressible case, the initial critical layer lies approximately at one displacement thickness, whereas it is much closer to the wall at $M = 0$ ($y_{crit} = 0.6\delta^*$). Consequently, the peak plane vorticity structure in Fig. 8 is far removed from the wall, and does not seem to have gone through the severe stretching which occurs at $M = 0$. (The vertical scales have been stretched by a factor of 4 in Fig. 8 and by a factor of 2 in Fig. 9.) The concentration of spanwise vorticity remains localized at the wall in the valley plane. On the other hand, the spanwise vorticity is more spread out in the vertical direction in the supersonic case. The cause for this phenomenon is unknown and under investigation.

Second Mode

Direct simulations of the nonlinear evolution of a single 2-D second mode perturbation wave were also performed. The objective was to gain some insight into the structure of the saturated stated (if it exists) of an unstable 2-D wave. This could provide a clue into potential instability mechanisms. Details of this work can be found in [23]. However, some of the main features of the non-linear numerical simulation are described below. The self-similar mean flow is identical to that in S1-S4, but the wave parameters are chosen to maximize the temporal growth rate. The parameters are $M_\infty = 4.5$, $Re_{\delta^*} = 8000$, $\alpha = 2.25$ and $\beta = 0.0$. The linear growth rate is

$$\omega = 2.04674 + 0.02149\,\iota \qquad (19)$$

One non-linear simulation tracked the 2nd mode wave for 25 TS periods. The initial streamwise velocity amplitude (ϵ_{2D}) is 2% which leads to a density amplitude of 9.2%. Figures 10a-b show the growth rate and frequency of the fundamental streamwise velocity, u_{10} as a function of y at $t = 0, 5, 10, 15, 20$ TS periods. (The growth rate and the frequency are obtained from the coefficient of $e^{\iota\alpha x}$ in the Fourier expansion of u calculated at two successive time steps.) The major feature to note is the sharp decrease in the local growth rate near the critical layer ($y_{cr} = 1.2$). After a slight initial rise, the growth rate starts decreasing which indicates an asymptotic evolution towards a saturated state for the (1,0) mode (zero growth rate for all y). The frequency, and therefore the phase velocity is decreasing slightly towards zero, albeit at a slower rate than the growth rate (Fig. 10a). The growth rate and frequency initially show a marked change near the wall. and the cause is not yet understood. One hypothesis is advanced. Typically, numerically generated growth rates are obtained from the integrated energy, which wipes out any oscillations due to inconsistent or inaccurate initial conditions. The mean flow and the initial profiles were obtained spectrally, with two different codes and grids, and spectral interpolation was performed to transfer information from one grid to another. Because the mean flow was computed on a smaller grid than were the fundamental profiles, the interpolation generated slight oscillations in the second derivative of the mean flow. This in turn affects the local growth profiles. It is obvious however from the figure that the initial slight inconsistencies are washed away at later times.

In Figs. 11-12, the fundamental and first harmonic components u and ρ are plotted. The maximum amplitudes of first harmonic waves are approximately an order of magnitude smaller than that of the fundamentals. This suggests that a weakly non-linear asymptotic theory should be sufficient to predict the time evolution of a single unstable two-dimensional mode, hopefully revealing the structure of its saturated state. Numerical experiments strongly suggest that the strong nonlinearities observed in the critical layer region are the contributions of the cubic nonlinearities (in the momentum equations) to the amplitude of u_{10}. To support this hypothesis, a second calculation was performed with the same initial conditions. However, all streamwise modes above the fundamental were set to zero after each time step (More precisely they were set to zero at each stage of the 3rd order Runga-Kutta algorithm). Growth rate curves at $t = 10$ are virtually identical to that shown in Fig. 10a. This indicates that the higher streamwise modes only play a secondary role in the initial evolution towards a saturated state.

To understand better the origin of this critical layer structure, Fourier transform the streamwse momentum equation in the x direction, and only consider the terms that contribute to the fundamental mode. Neglecting the viscous terms, this leads to the equation

$$\rho_0 u_{1t} + \iota\alpha(\rho_0 u_0 u_1 + p_1) + \rho_0(v_1 u_{0y} + v_0 u_{1y}) \iota\alpha\rho_{-1} u_1^2 + \rho_{-1} v_1 u_{1y} + \rho_1 (v_1 u_{-} \tag{20}$$

if u_1 is assumed to vary as $e^{-\iota\omega t}$ (where ω is a function of y and t), the instantaneous growth rate and frequency are obtained through after dividing this equation by $\rho_0 u_1$:

$$-\iota\omega = linear\ terms + cubic\ terms \tag{21}$$

where

$$linear\ terms = \iota\alpha(u_0 + \frac{p_1}{u_1}) + \frac{v_1 u_{0y} + v_0 u_{1y}}{u_1} \tag{22}$$

and

$$cubic\ terms = \iota\alpha\frac{\rho_{-1} u_1}{\rho_0} + \frac{\rho_{-1} v_1 u_{1y} + \rho_1 (v_1 u_{-1y} + v_{-1} u_{1y})}{\rho_0 u_1} \tag{23}$$

The growth rate is the real part of Eq. (21). The linear terms contribute a zeroth order growth rate, while the growth rate due to the cubic nonlinearities is quadratic in the amplitude of the fundamental mode. In figures 13a-b, the growth rate ω_r is plotted as a function of the normal coordinate y. Figure 13a plots the combination of linear terms, excluding the viscous interaction, which explains the sharp increase in growth near the origin and the structure at the critical layer, which are removed by the viscous effects. If the viscous effects were included, and the wave amplitude sufficiently small, Fig. 13a would be a line of constant growth rate. Cubic nonlinearities are plotted in fig. 13b. One notes the absence of growth rate everywhere, except in the critical layer and in a thin layer near the wall. The nonlinearities near the wall can be explained by consideration of the viscous terms. The viscosity is a function of temperature and becomes, after Taylor expansion,

$$\mu(T) \approx \mu_0 + \mu_{0T} T_1 + \tfrac{1}{2} T_{-1} T_1 \mu_{0TT} + \tfrac{1}{2} T_{-1} T_1^2 \mu_{0TTT}. \tag{24}$$

Derivatives of the mean viscosity μ_0 with respect to the mean temperature are denoted by the subscripts T. Since the viscous stress terms are quadratic in the viscosity and the primitive variables, only the mean and fundamental components of viscosity need be retained. The terms that contribute to the linear equations (in the fundamental amplitude) are

$$\mu_0 + \mu_{0T} T_1 \tag{25}$$

while

$$\tfrac{1}{2} T_{-1} T_1 \mu_{0TT} + \tfrac{1}{2} T_{-1} T_1^2 \mu_{0TTT} \tag{26}$$

contributes to the cubic nonlinearities. The cubic nonlinearities drop out if μ is independent of temperature. More detailed mapping of these various terms is under investigation

Finally, several terms in the compressible vorticity production equation

$$\frac{\partial \vec{\omega}}{\partial t} + \vec{v}\cdot\nabla\vec{\omega} = \vec{\omega}\cdot\nabla\vec{v} - \vec{\omega}\nabla\cdot\vec{v} - \nabla\rho^{-1}\times\nabla p + 2\mu\nabla\times\left[\rho^{-1}\nabla\cdot(\vec{\vec{\tau}} - \tfrac{1}{3}\nabla\cdot\vec{v}\,\vec{\vec{I}})\right], \tag{27}$$

where $\vec{\omega} = \nabla\times\vec{v}$, are plotted in Fig. 14 as a function of y, after integration in the streamwise direction. Terms not present in the limit of incompressibility are the dilatation term $\vec{\omega}\nabla\cdot\vec{v}$ and the baroclinic torque $\nabla\rho^{-1}\times\nabla p$. The vorticity production due to dilatation and baroclinic torque are shown in Figs. 14a-b. Production terms are obtained by multiplying the vorticity equation by $\vec{\omega}$ and integrating over the physical domain. In both plots, the production of vorticity is localized in the near wall region, and in the neighborhood of the critical layer. It is clear that dilatation and baroclinic torque effects on the production of vorticity are of the same order. However, they partially cancel each other out, with a slight predominance of the baroclinic term. The rms perturbation vorticity (based on the perturbation velocities) is shown in Fig. 15 at different time levels. The minimum perturbation vorticity is initially at the critical layer, but slowly drifts to small y at later times. Also of note, the vertical locations of the first 4 zeros of perturbation vorticity are approximately independent of time.

7 Conclusions

A three-dimensional, fully spectral code developed for the direct simulation of the wall-bounded compressible flows has been applied to determine the existence of a secondary instability mechanism for a first mode TS wave perturbed by a pair of streamwise vortices at Mach 4.5 flow on a flat plate. It was observed that the interaction between a finite-amplitude 2-D wave with a 3-D perturbation triggers a secondary instability. This instability differs in several ways from the K-type instability found in incompressible flows. Most noticeable is the shear layer development in both the peak and valley planes, which now lose their most distinctive characteristic. The spanwise vorticity is no longer a maximum in the peak plane, but in a plane parallel to it, approximately one third the distance towards the valley plane.

The interactions between the energy modes are qualitatively similar to those found in Herbert's incompressible secondary instability theory. It is the interaction of the lowest order spanwise mode with the 2-D wave which eventually triggers the onset of the instability. This is confirmed by a numerical experiment where the (0,1) mode is artificially suppressed: as expected, the (1,1) growth rate is both delayed and reduced. The qualitative similarities between this secondary instability and the K-type breakdown have led to several attempts to apply Floquet theory to compressible flows at high Mach numbers.

As was previously known, the observed instability at Mach 4.5 is found to be weaker than those found in incompressible flows. This is true even with perturbation amplitudes several times higher than those typically found in incompressible simulations.

Although at Mach 4.5, the second mode is the most unstable, the first mode secondary instability was studied to tie in with current knowledge of incompressible flows. It is possible that the observed instability is not the dominant one. Secondary instabilities associated with higher modes and acoustic disturbances may predominate. Therefore, a first study of the nonlinear evolution of an unstable 2-D 2nd mode wave was conducted. It was found that nonlinearities build up in the critical layer region, where the cubic nonlinearities due to the high density perturbations are maximum. This is accompanied by a steady decrease in the growth rate as a function of time. It is concluded that the asymptotic state is a saturated 2-D wave whose structure is still undetermined. The exact structure of this state awaits the application of weakly nonlinear theories to this type of evolution.

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9 Figure Captions

Fig. 1: Amplitude of M=0.5 eigenfunctions (w is vertical velocity)

(1a) 2-D eigenfunctions (u, v, T)

(1b) 3-D eigenfunctions (u, v, T, w)

Fig. 2: Amplitude of M=4.5 eigenfunctions (w is vertical velocity)

(2a) 2-D eigenfunctions (u, v, T)

(2b) 3-D eigenfunctions (u, v, T, w)

Fig. 3: Harmonic content of selected energy modes (Log. scale). $\epsilon_{2D} = 0.017$, $\epsilon_{3D} = 0.004$.

Fig. 4: Harmonic content of selected energy modes (Log. scale). $\epsilon_{2D} = 0.054$, $\epsilon_{3D} = 0.012$.

Fig. 5: Harmonic content of selected energy modes (Log. scale). $\epsilon_{2D} = 0.0$, $\epsilon_{3D} = 0.012$.

Fig. 6: Harmonic content of selected energy modes (Log. scale). $\epsilon_{2D} = 0.054$, $\epsilon_{3D} = 0.012$. The (0,1) spanwise mode is suppressed for $t > 4$.

Fig. 7: Harmonic content of selected energy modes (Log. scale). $\epsilon_{2D} = 0.035$, $\epsilon_{3D} = 0.005$.

Fig. 8: Spanwise vorticity (S2) in peak and valley planes at $t = 8$.

Fig. 9: Spanwise vorticity in peak and valley planes. Results from direct simulation of incompressible K-type breakdown during the secondary instability regime.

Fig. 10: Growth rate and frequency at $t = 0, 5, 10, 15, 20$ periods. Initial u amplitude of 2%.

(10a) Growth rate

(10b) Frequency

Fig. 11: Amplitude of fundamental modes at $t = 0, 5, 10, 15, 20$ periods.

(11a) $\hat{u}_1$

(11b) $\hat{\rho}_1$

Fig. 12: Amplitude of first harmonic modes at $t = 0, 5, 10, 15, 20$ periods.

(12a) $\hat{u}_2$

(12b) $\hat{\rho}_2$

Fig. 13: Growth rate at $t = 0, 5, 10, 15, 20$ periods. Initial u amplitude of 2%.

(13a) Contribution of linear terms to (0,1) growth rate

(13b) Contribution of cubic nonlinearities to (0,1) growth rate

Fig. 14: Vorticity production terms at t=0,5,10,15,20 periods.

(14a) Baroclinic torque

(14b) Dilatation term

Fig. 15: Perturbation spanwise vorticity at $t = 0, 5, 10, 15, 20$ periods.