Computational Geometry Lab:
FEM BASIS FUNCTIONS IN TRIANGULATIONS

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1 Introduction

This lab continues the topic of *Computational Geometry.* Having studied triangles and how triangles are used to create triangulations of a region, we will now turn to the use of triangulations in the finite element method.

The finite element method is a procedure for approximating and solving partial differential equations. Part of the finite element method involves constructing the triangulation, a topic which is discussed in other labs. Once the triangulation is available, the finite element method uses this mesh to represent functions $f(x, y)$. The representation is *discrete*, that is, it depends on just a finite number of values, but the resulting function is defined over the entire triangulated region; with some restrictions, it can be evaluated, plotted, differentiated or integrated.

If you have ever used a finite difference method to solve differential equations, you will understand an important distinction between these two methods. The finite difference method works with values of a function at given points, but it does not try to "fill in the gaps" between the tabulated points. In contrast, the finite element method may only have exact knowledge of a function at specified points, but it builds a "model" of the function over the entire problem domain.

The key to this model building is the set of **finite element basis functions**. It is the purpose of this lab to understand how these basis functions are defined, evaluated and used to create the finite element functions.

2 Overview

Our lab will involve several complicated steps. We will start with some very small matters and gradually move to a larger picture. It may help to see how these steps are related.

So we suppose that we are given a triangulation **TRI** of some region $\mathcal{R}$. The triangulation is made up, of course, of points and triangles. We will assume there are **NP** points or "nodes", with a typical point being identified as **P** or perhaps $\mathbf{P_i}$. There are also **NT** triangles, whose vertices are chosen from the set of points, with a typical triangle being **T** or $\mathbf{T_i}$.

Let us suppose that we wish to come up with a formula for a function $f(x, y)$, with the requirement that

$$f(P_i) = f_i, \quad i = 1 \ldots NT.$$

that is, we are going to specify in advance the value of the function at every node.

Our goal is to somehow come up with a formula, or a procedure, which defines $f(x, y)$ for every point (x, y) in $\mathcal{R}$, in such a way that the function is continuous, attains the specified values at the nodes, and is

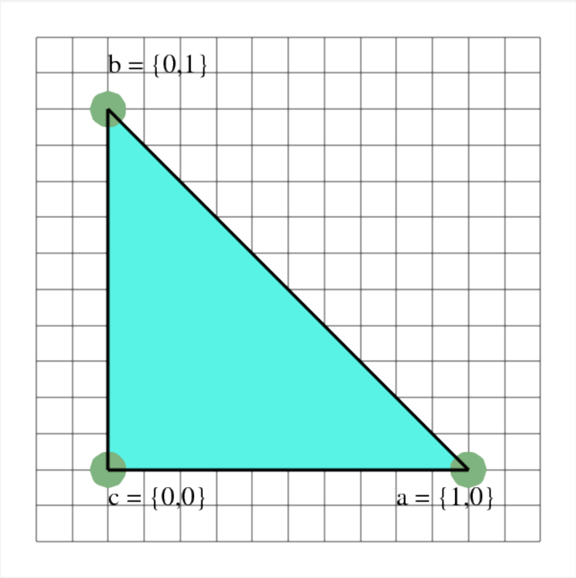

Figure 1: The Reference Triangle

relatively simple to evaluate anywhere in the region. This is an example of what is called **the interpolation problem**.

Our progress to solving the interpolation problem on a triangulation will start very simply. We will look at a "triangulation" that involves a single triangle, called the *reference triangle*. We will investigate how interpolation works in this very simple setting, and we will also "discover" the basis functions that make the answer simple to describe.

We will then transfer this formula to a general triangle. Then we will consider what happens depending on which vertex is chosen to have the value 1 under the formula.

Then we will consider the effect of setting up these formulas in *every* triangle in the triangulation simultaneously. This might seem to be a recipe for chaos. However, whenever two triangles touch, they share two vertices, and the formula we develop for each triangle will match up continuously along their common boundary (but not differentiably!).

At this point, will have all the machinery in place so that we can define the desired function $f(x, y)$ which satisfies the conditions we specified.

3 The Reference Triangle

Our task is complicated, but we have to start somewhere. Instead of an entire triangulation, we'll consider a single triangle. Instead of an arbitrary triangle, we'll work with the "reference triangle", whose definition is simply **Tref**={ a, b, c } = { {1,0}, {0,1}, {0,0} }.

Now suppose we want to define a function $f(x, y)$, defined over the entire triangle, with the properties that

$$f(a) = fa$$
$$f(b) = fb$$
$$f(c) = fc$$

It should be clear that we can probably do this with a linear function of the form

$$f(x, y) = c_1 + c_2 x + c_3 y$$

But the condition $f(c) = fc$ implies that $f(0,0) = c_1 = fc$. The condition $f(a) = fa$ then implies that $f(1,0) = fc + c_2 = fa$ which shows that $c_2 = fa - fc$, and we can similarly show that we must have $c_3 = fb - fc$.

Thus, we have solved our simple problem, because the function

$$f(x,y) = fc + (fa - fc)x + (fb - fc)y$$

has the correct values at the vertices, is defined and continuous over the entire triangle, and is simple to evaluate.

This certainly doesn't solve our real problem, but it is a helpful guide as to how we want to proceed!

4 Program #1: Interpolation in the Reference Triangle

For the reference triangle **Tref**={ a, b, c } = { {1,0}, {0,1}, {0,0} }, write a program which

- Reads three vertex values **fa**, **fb**, and **fc**;
- evaluates the interpolant function $f(x,y)$ at **a**, **b** and **c**;
- evaluates $f(x,y)$ at $\frac{a+b}{2}$, $\frac{b+c}{2}$, $\frac{c+a}{2}$ and $\frac{a+b+c}{3}$;
- evaluates $f(x,y)$ at 5 random points in the reference triangle.

For your data, use **fa**=1, **fb**=-2, and **fc**=3.

5 Basis Functions for the Reference Triangle

When we guessed that our function would be linear, we wrote out a symbolic formula. We could regard that formula as a combination of the **basis functions 1**, **x** and **y**. Any linear (actually, affine) function in the plane can be represented as such a combination.

However, there are many equivalent sets of basis functions. Notice how the formula for our solution uses the prescribed value fc several times. What if we rearranged this formula so that each prescribed value showed up exactly once. We'd get something like this:

$$f(x,y) = fa \cdot x + fb \cdot y + fc \cdot (1 - x - y)$$

Now if we look at this formula, we can regard it as using a slightly different set of basis functions than before. Let's actually rename each basis function:

$$\begin{aligned}\phi_a(x,y) =& x\\ \phi_b(x,y) =& y\\ \phi_c(x,y) =& 1 - x - y\end{aligned}$$

This set of basis functions has some useful properties:

- basis function $\phi_a(x,y)$ is 1 at vertex a, and 0 at b and c, with similar statements for the other two basis functions (*the Lagrange basis property*);
- along the triangle's edge {b,c}, basis function $\phi_a(x,y)$ is exactly 0, with similar statements for the other two basis functions;
- at any point in the triangle, the value of each basis function is between 0 and 1; for points strictly inside the triangle, the basis function is strictly between 0 and 1;

- at any point (x, y), the sum of the three basis functions is exactly 1;
- at any point (x, y), the sum of the derivatives of the three basis functions is exactly 0 (which follows from the previous statement).

It should be clear now that if we have any triangle on which we want to solve the interpolation problem, we can write down the solution immediately if we can find a set of basis functions with the Lagrange basis property!

Using our basis function notation, the solution to the interpolation problem has the very nice form:

$$f(x, y) = fa \cdot \phi_a(x, y) + fb \cdot \phi_b(x, y) + fc \cdot \phi_c(x, y)$$

A form like this is useful because it's easy to remember, it suggests what the relationship is between all the terms, and it makes it easy to guess what the corresponding formula might be in more general circumstances (a different triangle, or a problem in 3D involving tetrahedrons!).

6 Program #2: Basis Functions for the Reference Triangle

For the basis functions ϕ_a, ϕ_b and ϕ_c in the reference triangle, write a program which

- evaluates the basis functions at each vertex;
- evaluates the basis functions at $\frac{a+b}{2}$, $\frac{b+c}{2}$, $\frac{c+a}{2}$ and $\frac{a+b+c}{3}$;
- evaluates the sum of the basis functions at 5 random points in the reference triangle.

7 Basis Functions for a General Triangle

Now we are ready to consider the interpolation problem on a general triangle, of the form **T**={a,b,c}. It should be clear that our best hope will be to find a set of basis functions for this general triangle that work like the ones we found in the reference triangle.

It's actually possible to figure out the formulas for the basis functions based on the simple properties we know about them. Let us find a formula for $\phi_a(x, y)$. We are assuming that $\phi_a(x, y)$ is a linear (affine) function. Since ϕ_a is 0 at nodes **b** and **c**, it must also be zero at all points (x, y) on the line between these two nodes.

Now just imagine drawing the line between **b** and **c** and then picking any point $p = (p.x, p.y)$ on that line. The slope of the line is the same whether we use **b** and **c** or **b** and the new point **p**:

$$\frac{c.y - b.y}{c.x - b.x} = \frac{p.y - b.y}{p.x - b.x}$$

(It's customary to write the slope relationship this way. Should either denominator be zero, we could eliminate the fractions, and have a valid, if less familiar, formula.)

If we subtract one side from the other, and call the result $g(x, y)$, we have that:

$$g(x, y) = (x - b.x)(c.y - b.y) - (c.x - b.x)(y - b.y)$$

We know that $g(x, y) = 0$ for those points on the line between **b** and **c**. Assuming our triangle is not degenerate, then $g(a.x, a.y)$ must be *nonzero* (because **a** does *not* lie on the line between **b** and **c**!) So $g(x, y)$ is almost the basis function $\phi_a(x, y)$, since it's zero at vertices **b** and **c** and nonzero at vertex **a**. Now if we simply divide this function by its value at vertex **a**, the new function is 1 at vertex **a**, so it satisfies all three conditions. Our basis function formula is:

$$\phi_a(x, y) = \frac{g(x, y)}{g(a.x, a.y)} = \frac{(x - b.x)(c.y - b.y) - (c.x - b.x)(y - b.y)}{(a.x - b.x)(c.y - b.y) - (c.x - b.x)(a.y - b.y)}$$

Now let's make sure this formula does what we want. It's very easy to see that $\phi_a(a.x, a.y) = 1$. Can you also see that $\phi_a(b.x, b.y) = \phi_x(c.x, c.y) = 0$?

Thus we have constructed the formula for the affine function $\phi_a(x, y)$. Simple substitutions will produce the corresponding formulas for $\phi_b(x, y)$ and $\phi_c(x, y)$.

8 Program #3: Basis Functions for a General Triangle

Write a program which

- reads the definition **{a,b,c}** of a general triangle **T**;
- evaluates the basis functions at each vertex;
- evaluates the basis functions at $\frac{a+b}{2}$, $\frac{b+c}{2}$, $\frac{c+a}{2}$ and $\frac{a+b+c}{3}$;
- evaluates the sum of the basis functions at 5 random points in the reference triangle.

For your general triangle **T**, use the triangle **Tex5** defined by {{4,1},{3,5},{0,2}}.

9 Basis Functions for a General Triangle by Determinants

It can be shown that a formula for the basis functions can be found as the ratio of two determinants. The determinant in the denominator is essentially the area of the triangle. The determinant in the numerator is formed by replacing the first two elements of column 1, 2 or 3 by the x and y values of the point **p** where basis function ϕ_a, ϕ_b or ϕ_c is to be evaluated. The result is an expression for the area of the triangle formed by the point **p** and two of the vertices. Thus, to evaluate $\phi_a(x, y)$, we write:

$$\phi_a(x,y) = \frac{\begin{vmatrix} p_x & b_x & c_x \\ p_y & b_y & c_y \\ 1 & 1 & 1 \end{vmatrix}}{\begin{vmatrix} a_x & b_x & c_x \\ b_y & b_y & c_y \\ 1 & 1 & 1 \end{vmatrix}}$$

with corresponding formulas for $\phi_b(x, y)$ and $\phi_c(x, y)$.

You should be able to evaluate this determinant, and compare it to the formula we derived above. You should also be able to verify that, for the simple case of the reference triangle, these formulas give us the basis functions **x**, **y**, and **1-x-y**.

10 Basis Functions for a General Triangle by Mapping

A third way to determine the basis functions for a general triangle relies on the idea of a mapping $\psi_{Tref,T}(r, s)$ from the reference triangle**Tref** to the general triangle {a,b,c}. To evaluate the basis function $\phi_a(x, y)$ in the general triangle, we map (x, y) back to the corresponding point (r, s) in the reference triangle, and evaluate $\phi_a(r, s)$ there. Now in the reference triangle, the basis functions are **r**, **s** and **1-r-s**. But the inverse map from the general triangle to the reference triangle gives us (r, s), and it's trivial to evaluate $1 - r - s$, so if we can find the inverse point (r, s) we're done.

The mapping from **Tref** to **T** can be written as

$$\psi_{Tref,T}\begin{pmatrix} r \\ s \end{pmatrix} = A \cdot \begin{pmatrix} r \\ s \end{pmatrix} + \begin{pmatrix} c_x \\ c_y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

So the inverse mapping from **T** back to **Tref** can be written as

$$\psi^{-1}_{Tref,T}\begin{pmatrix} x \\ y \end{pmatrix} = A^{-1}\cdot\begin{pmatrix} x-c_x \\ y-c_y \end{pmatrix} = \begin{pmatrix} r \\ s \end{pmatrix}$$

But in the lab on mapping triangles, we have:

$$\det A = (a_x - c_x)(b_y - c_y) - (a_y - c_y)(b_x - c_x)$$

The inverse of A can then be written as

$$A^{-1} = \frac{1}{\det A}\begin{pmatrix} (b_y - c_y) & -(b_x - c_x) \\ -(a_y - c_y) & (a_x - c_x) \end{pmatrix}$$

so

$$\phi_a x, y = r(x,y) = \frac{(b_y - c_y) * (x - c_x) - (b_x - c_x) * (y - c_y)}{(a_x - c_x)(b_y - c_y) - (a_y - c_y)(b_x - c_x)}$$

and

$$\phi_b(x,y) = s(x,y) = \frac{-(a_y - c_y) * (x - c_x) + (a_x - c_x) * (y - c_y)}{(a_x - c_x)(b_y - c_y) - (a_y - c_y)(b_x - c_x)}$$

and from r and s we can work out $1 - r - s$ as well.

We have considered the basis function problem from the perspective of mapping, and come up with the same formulas that we got using linear functions, and using determinants! This is not wasted effort. It's important to be able to see this problem in three different ways, and to understand the relationships between the many ways of describing this problem.

11 Moving to the Triangulation

Now let's consider our full interpolation problem, which was posed on a triangulation. How do we propose to handle this? Suppose we are given a point (x, y), so that we have to compute the value $f(x, y)$ of our interpolation function. How do we proceed?

First, of course, we must determine the triangle that contains (x, y). Of course, a few special cases need to be considered. If (x, y) occurs in *no* triangle, then it falls outside the triangulated region, and we will simply return a zero value or an error condition. After all, we were only asked to interpolate over the triangulated region. If (x, y) occurs in *several* triangles, then we can simply choose one of the triangles. Of course, we must verify that the result will be the same no matter which triangle we choose - in other words, our definition of $f(x, y)$ is continuous.

Once we have located the triangle $\mathbf{T_i}$ containing (x, y), then we need to retrieve the vertices {a,b,c} of the triangle. Actually, we will need to retrieve the *indices* of these vertices as they appear in the list of nodes, because that will allow us to retrieve the corresponding function values.

In other words, triangle $\mathbf{T_i} = \{i_a,\ i_b,\ i_c\}$. Then we can use these indices as keys to both the NODE_XY array and the NODE_VALUE arrays.

Now let us return to the question of continuity. Suppose (x, y) occurs in multiple triangles.

If the point is a vertex in one triangle, it must be a vertex in all the triangles, because we do not allow "hanging nodes" in our triangulations. Therefore, (x, y) is not just a vertex of the triangle, but also a node of the triangulation, and has a corresponding node index i. The definition of $f(x, y)$ at a vertex with node index i is f_i, so our result is the same no matter which triangle we choose.

If (x, y) occurs in multiple triangles, but is not a vertex, then it must be a point shared by exactly two triangles with a common edge. But, as we have seen, on any edge of the triangle, the only two nonzero basis functions are those associated with the endpoints of the edge. The value of the function is the linear interpolant of the values at the two endpoints. No matter which triangle we choose, the endpoint values will be the same, and hence the value of the linear interpolant at (x, y) will be the same.

Thus, the function $f(x, y)$ is well defined. It is continuous because it is a linear function over each triangle, where any triangles have a common point, the function definitions coincide. Therefore, the function $f(x, y)$ is a (continuous) piecewise linear function over the triangulated region.

12 The Support of One Basis Function

13 Program #6: Finite Element Functions

Write a program which accepts three triangle vertices **Va, Vb, Vc** a set of three values associate with the vertices, **Wa, Wb, Wc** and a point **P**.

For the given point **P**, generate the barycentric coordinates $(\xi_a(P), \xi_b(P), \xi_c(P)$. Evaluate $f(P)$, the linear function which has the values **Wa, Wb, Wc** at the points **Va, Vb, Vc**.

Some simple checks include the following:

- setting **Wa, Wb, Wc** to (1,0,0) should mean $f(P) = \xi_a(P)$;
- setting **P = Va** should result in $f(P) = Va$;
- setting **P = (Va+Vb+Vc)/3** should result in $f(P) = (Wa + Wb + Wc)/3$;