Computational Geometry Lab: FINITE ELEMENTS

John Burkardt
School of Computational Science
Florida State University
http://people.sc.fsu.edu/∼burkardt/presentations/cg_lab_fem.pdf

June 7, 2009

1 Introduction

This lab continues the topic of *Computational Geometry.* Having studied triangle,s and how triangles are used to create triangulations of a region, we will now turn to the use of triangulations in the finite element method.

The finite element method is a procedure for approximating and solving partial differential equations. In order to discretize the problem so that it is suitable for computational analysis, it is first necessary to take the domain over which the partial differential equation is defined, and construct a triangulation. Using the triangulation, it is possible to define a family of functions that can be used to represent possible solutions of the partial differential equation. The finite element method then constructs a linear or nonlinear system of equations that determines which of these candidate solutions will be selected as the approximate solution.

Our interest here is only in the way that triangulations are used in the finite element method. In particular, we will look at how the triangulation gives rise to a space of functions.

2 Finite Element Basis Functions

In finite element analysis, a region is simulated by a collection of triangles, and a function defined over that region might be simulated by a piecewise linear function, whose values are known at the vertices of the triangles. Knowing vertex values is logically enough to determine the value of the function inside the triangle. But the computational details can be a little perplexing.

If we want to evaluate a linear function in a triangle, given the location of the vertices and the values there, it's easy if we can construct three *basis functions.* Basis function $\phi_a(x, y)$ will be a linear function which is associated with node **Va**, where it has the value 1; it is 0 at the other two nodes. Again, this is enough to define $\phi_a(x, y)$. Basis functions $\phi_b(x, y)$ and $\phi_c(x, y)$ are defined similarly.

If we can find an arithmetic definition of these basis functions, then our linear function $f(x, y)$ is simply a linear combination of them:

$$f(x, y) = Wa * \phi_a(x, y) + Wb * \phi_b(x, y) + Wc * \phi_c(x, y)$$

for certain values **Wa, Wb, Wc**, and so our task will be done.

So let us find a formula for $\phi_a(x, y)$. Since it is 0 at nodes Vb and Vc, it must also be zero at all points (x, y) on the line between these two nodes. But since these points lie on a line, we already know one linear relationship they satisfy:

$$\frac{Vc.y - Vb.c}{Vc.x - Vb.x} = \frac{y - Vb.y}{x - Vb.x}$$

(It's customary to write the slope relationship this way. Should either denominator be zero, we could elminate the fractions, and have a valid, if less familiar, formula.)

If we subtract one side from the other, and call the result $g(x, y)$, we have that:

$$g(x, y) = (x - Vb.x)(Vc.y - Vb.y) - (Vc.x - Vb.x)(y - Vb.y)$$

We know that $g(x, y) = 0$ for those points on the line between Vb and Vc. Assuming our triangle is not degenerate, then $g(x_i, y_i)$ is *nonzero* (because n_i does *not* lie on the line between Vb and Vc!). So $g(x, y)$ is almost a basis function, since it's zero at the right places, and nonzero at the other. But it's easy to scale a function so that a nonzero value is 1. How's this for a candidate for our basis function:

$$\phi_a(x, y) = \frac{g(x, y)}{g(Va.x, Vb.y)}$$

You should easily see that this formula is indeed zero at Vb and Vc, and sure enough, it's 1 at Va, and therefore, it represents our basis function. Of course, it would be nice to see this formula explicitly. All we have to do is substitute, to get:

$$\phi_a(x, y) = \frac{(x - Vb.x)(Vc.y - Vb.y) - (Vc.x - Vb.x)(y - Vb.y)}{(Va.x - Vb.x)(Vc.y - Vb.y) - (Vc.x - Vb.x)(Va.y - Vb.y)}$$

This is a linear function of two arguments. Putting in the values $(Va.x, Va.y)$, $(Vb.x, Vb.y)$, and $(Vc.x, Vc.y)$ gives you the values 1, 0 and 0, respectively, so we know it's right.

This means we have a way to construct and evaluate the linear function in the triangle, based on its values at the nodes.

3 Program #6: Finite Element Functions

Write a program which accepts three triangle vertices **Va, Vb, Vc** a set of three values associate with the vertices, **Wa, Wb, Wc** and a point **P**.

For the given point **P**, generate the barycentric coordinates $(\xi_a(P), \xi_b(P), \xi_c(P)$. Evaluate $f(P)$, the linear function which has the values **Wa, Wb, Wc** at the points **Va, Vb, Vc**.

Some simple checks include the following:

- setting **Wa, Wb, Wc** to (1,0,0) should mean $f(P) = \xi_a(P)$;
- setting **P = Va** should result in $f(P) = Va$;
- setting **P = (Va+Vb+Vc)/3** should result in $f(P) = (Wa + Wb + Wc)/3$;