Computational Geometry Lab:
QUADRATURE ON TRIANGLES

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1 Introduction

This lab continues the study of Computational Geometry. We are now concerned with the idea of *quadrature*, that is, the approximation of the integral of some function $f(x, y)$ over a finite two dimensional integration domain $\mathcal{R}$.

In this discussion, we will consider the problem of approximating integrals over much simpler domains. We will briefly look at intervals, and then concentrate on triangles. Once we see how to deal with triangles of any shape, we will be prepared to consider approximating integrals over general regions that we regard as a collection of triangles.

2 Quadrature Rules for an Interval

The second semester of a calculus course is devoted to developing rules to determine the integral of a function $f(x)$ over a region, especially a line segment $[a, b]$. Unfortunately, there is no general procedure that can compute the integral for any function. If a value for the integral is desired, then the only option left is to try some kind of approximation.

A *quadrature rule* is a rule for approximating an integral over a specific integration region. A typical quadrature rule for the 1 dimensional case will approximate an integral using a weighted sum of function values:

$$\int_{-1}^{1} f(x)\,dx \approx \sum_{i=1}^{n} w_i\, f(x_i)$$

Here, the number n is called the *order* or the rule, the values x_i, where the function is evaluated, are called the *abscissas*, and the values w_i are known as the *weights*. Generally, the abscissas will be expected to be elements of the interval $[-1, 1]$ (that is, we don't expect an abscissa at 1.5, for instance!) and the weights will usually all be positive values.

Another feature of most quadrature rules is that they have some level of *polynomial precision* or *exactness*. A quadrature rule has precision equal to p if its approximation to the integral of $f(x)$ is exact whenever $f(x)$ is a polynomial of degree p of less. In particular, you should see that if a rule defined on $[0, 1]$ has polynomial precision 0, the weights must sum to 1!

If a quadrature problem is defined on the interval [a,b], then the quadrature rule defined on $[-1, 1]$ can

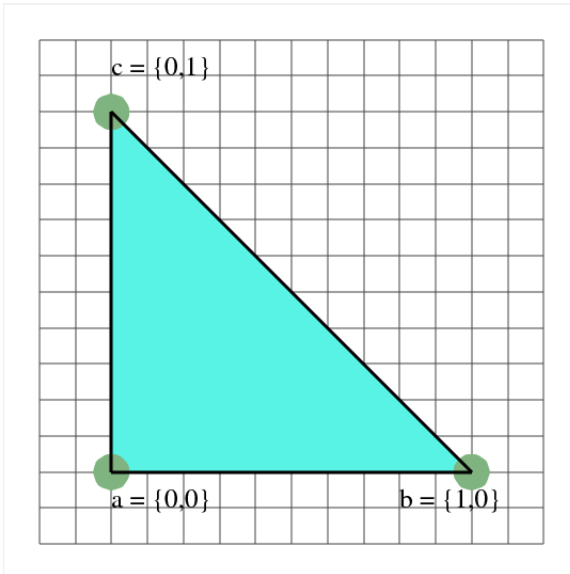

Figure 1: The unit triangle or reference triangle **Tref**.

be transformed to a corresponding rule on $[a, b]$ using the transformation

$$x_i \to X_i = \frac{(x_i - 1)a + (x_i + 1)b}{2}$$
$$w_i \to W_i = w_i \frac{\text{Area([a, b])}}{\text{Area}([-1, +1])} = w_i \frac{(b - a)}{2}$$

When we turn to the problem of approximating the integral of a function over a triangle, the same issues will arise, with some added complexity due to the geometry.

3 Quadrature over the Unit Triangle

A *quadrature rule* for the triangle is a rule for approximating an integral over triangle. A typical quadrature rule is specified for **Tref**, *the unit triangle*, also known as the *reference triangle*, whose vertices are {{0,0}, {1,0},{0,1}}. The rule has the form:

$$\int_{\text{Tref}} f(x, y)\, dx \approx \sum_{i=1}^{n} w_i\, f(x_i, y_i)$$

A quadrature rule for the unit triangle has precision p if its approximation to the integral of $f(x, y)$ is exact whenever $f(x, y)$ is a polynomial of total degree p of less. The total degree is the maximum exponent sum over all the monomial terms in a polynomial.

There is a formula for the exact integral of any monomial $x^q y^r$ over the unit triangle:

$$\int_{\text{Tref}} x^q y^r\, dx = \frac{q!r!}{(q + r + 2)!}$$

4 Program #1: Quadrature over the Unit Triangle

Here are examples of quadrature rules for the unit triangle, with the order N, precision P, weights W, and abscissas (X, Y):

Table 1: Quadrature Rules for the Unit Triangle.

N	P	W	X	Y
1	1	1.000000	0.333333	0.333333
3	2	0.333333	0.500000	0.000000
		0.333333	0.500000	0.500000
		0.333333	0.000000	0.500000
4	3	-0.562500	0.333333	0.333333
		0.520833	0.600000	0.200000
		0.520833	0.200000	0.600000
		0.520833	0.200000	0.200000
6	4	0.109951	0.816847	0.091576
		0.109951	0.091576	0.816847
		0.109951	0.091576	0.091576
		0.223381	0.108103	0.445948
		0.223381	0.445948	0.108103
		0.223381	0.445948	0.445948
7	5	0.225000	0.333333	0.333333
		0.125939	0.797427	0.101287
		0.125939	0.101287	0.797427
		0.125939	0.101287	0.101287
		0.132394	0.059716	0.470142
		0.132394	0.470142	0.059716
		0.132394	0.470142	0.470142

Write a program which applies these quadrature rules to integrate the function $f(x, y) = x^q\, y^r$ on the unit triangle **Tref**.

Your program should:

- read the order of the quadrature rule **N**;
- read the abscissas (x_i, y_i) and weights w_i of the quadrature rule;
- read the powers q and r of the integrand;
- compute the exact integral $I = \int_{\text{Tref}} x^q\, y^r\, dx\, dy$;
- compute Q, integral estimate from the quadrature rule.
- compute the error $E = \|Q - I\|$;
- print q, r, $q + r$, P, I, E.

If the precision of a rule is **P**, then the rule should be able to integrate exactly any monomial $x^q\, y^r$ for which $q + r \leq P$. Verify the precision claims for the quadrature rules.

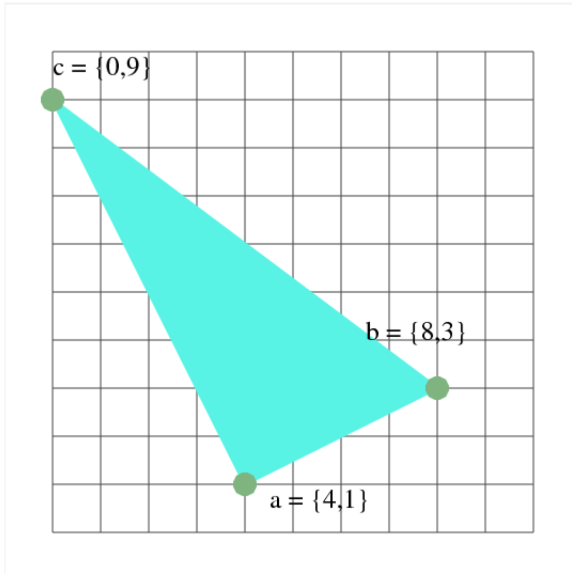

Figure 2: Example triangle #1, "Tex1".

5 Quadrature Over a General Triangle

If we are integrating over a general triangle $T = \{a, b, c\}$, then it is possible to apply a quadrature rule defined on the reference triangle, but only after we have transformed that rule to the new region:

$$(x_i, y_i) \to (X_i, Y_i) = (1 - x_i - y_i)a + x_i b + y_i c$$
$$w_i \to W_i = w_i \frac{\text{Area(T)}}{\text{Area(Tref)}} = 2\, w_i \,\text{Area(T)}$$

Consider our example triangle #1 or **"Tex1"**, whose definition is

```
{ { 4, 1 },
  { 8, 3 },
  { 0, 9 } }
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The **Tref** point (x_2, y_2)=(0.6, 0.2) in rule #2 would map to the point (X_2, Y_2) in **Tex1** as follows:

$$X_2 = (1.0 - 0.6 - 0.2) * (4) + 0.6 * (8) + 0.2 * (0) = 5.6$$
$$Y_2 = (1.0 - 0.6 - 0.2) * (1) + 0.6 * (3) + 0,2 * (9) = 3.8$$

Notice that this transformation is a linear mapping between the two triangles, is invertible (as long as T is not a degenerate triangle) and that the mapping takes (0,0) to a, (1,0) to b and (0,1) to c.

Because this is a linear mapping, any function $f(x, y)$ defined on **Tref** of total degree p will be transformed by the mapping to a function $F(X, Y)$ on T which is also of total degree p. This means that if a quadrature rule has precision p on the unit triangle, it will have the same precision on a general triangle under the linear mapping.

Table 2: Values of some test integrals.

(p,q)	$\int x^p y^q dxdy$
(0,0)	20.00
(1,0)	80.00
(0,1)	86.66
(2,0)	373.33
(1.1)	306.66
(0,2)	433.33
(2,1)	1333.33
(2,2)	5294.22

There is an inverse map from T to **Tref**:

$$X_i \to x_i = \frac{(Y_3 - Y_1)(X_i - X_1) - (X_3 - X_1)(Y_i - Y_1)}{(Y_3 - Y_1)(X_2 - X_1) - (X_3 - X_1)(Y_2 - Y_1)}$$
$$Y_i \to y_i = \frac{(X_2 - X_1)(Y_i - Y_1) - (Y_2 - Y_1)(X_i - X_1)}{(Y_3 - Y_1)(X_2 - X_1) - (X_3 - X_1)(Y_2 - Y_1)}$$

For instance, the point (X_2, Y_2)=(5.6, 3.8) in **Tex1** will be mapped to the point (x_2, y_2) in **Tref** as follows:

$$x_2 = \frac{(9-1)(5.6-4) - (0-4)(3.8-1)}{(9-1)(8-4) - (0-4)(3-1)} = 0.6$$
$$y_2 = \frac{(8-4)(3.8-1) - (3-1)(5.6-4)}{(9-1)(8-4) - (0-4)(3-1)} = 0.2$$

6 Program #2: Quadrature on the General Triangle

Write a program which applies the quadrature rules to integrate the function $f(x, y) = x^q\, y^r$ on a general triangle.

Your program should:

- read a triangle **T**;
- read the order of the quadrature rule **N**;
- read the abscissas (x_i, y_i) and weights w_i of the quadrature rule;
- read the powers **q** and **r**;
- compute transformed abscissas (X_i, Y_i) and weights W_i;
- apply the quadrature rule, and print the integral estimate;

Try your program on the **Tex1** triangle.

Since we are working in a general triangle, we don't have a formula for the exact integrals. Here are several values:

Try to approximate some of these values with your program.

7 Quadrature Over a Triangulation

Now suppose that we have a region $\mathcal{R}$ for which we have a triangulation $\mathcal{T} = \{T_i : 1 \leq i \leq N\}$, with the triangles T_i having disjoint interiors and whose union is $\mathcal{R}$. Suppose that we wish to estimate the integral

$$I(\mathcal{R}, f) = \int_{\mathcal{R}} f(x, y)\, dx\, dy$$

Since $\mathcal{R}$ is identical to the extent of $\mathcal{T}$, and since $\mathcal{T}$ is the disjoint sum of the triangles T_i, an integral over $\mathcal{R}$ is the sum of the integrals over the triangles:

$$I(\mathcal{R}, f) = \int_{\mathcal{T}} f(x, y)\, dx\, dy$$
$$= \sum_{i=1}^{N} \int_{T_i} f(x, y)\, dx\, dy = \sum_{i=1}^{N} I(T_i, f)$$

and, if we now apply a quadrature rule Q to approximate the integral over each triangle, we have:

$$I(\mathcal{R}, f) = \sum_{i=1}^{N} I(T_i, f) \approx \sum_{i=1}^{N} Q(T_i, f)$$

In other words, to approximate an integral over a triangulated region, we may use a quadrature rule to approximate the integral of the function over each triangle in the triangulation and sum the result. In our next discussion, we will illustrate this idea, and work through some examples.