# Computational Geometry Lab: TETRAHEDRONS

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http://people.sc.fsu.edu/~jburkardt/presentations/cg_lab_tetrahedrons.pdf

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## 1 Introduction

We analyzed two dimensional regions using collections of triangles. In three dimensions, the corresponding approach uses collections of tetrahedrons. Triangles and tetrahedrons are the 2D and 3D examples of the **simplex** family. The useful properties of triangles and tetrahedrons are typical of the simplex family, and so this gives an indication of how abstract N-dimensional problems can be treated as well.

This lab is intended to introduce the basic geometry and representation of tetrahedrons. We will measure the sides, faces, and internal angles of the tetrahedron. We will also define a shape quality measure, which considers whether the tetrahedron is well-shaped or flattened.

Of course, once we are comfortable with a single tetrahedron, we will want to look at collections of them, that form a mesh. We will discover in that subsequent lab that a 3D mesh of tetrahedrons is significantly harder to define and handle, compared to a triangular mesh.

## 2 Naming Things on a Tetrahedron

It will be helpful if we can come up with a simple and short way of describing a tetrahedron and various quantities associated with it.

Let's start by supposing we've chosen a name for a particular tetrahedron, which might be simply **T1** or **Tet** or **Brick**. Every tetrahedron has four vertices. We could number them, but for now it might be more convenient to identify them by letter. So the vertices will be named **a, b, c, d**. Sometimes, we may be considering more than one tetrahedron, and so we will optionally specify the "full name" of a vertex by beginning with the name of the tetrahedron that it belongs to. Thus, if we need to avoid ambiguity, we can more fully specify the second vertex of tetrahedron **T1** as **T1.b**.

Now each vertex is a point in 3D space, so it has coordinates. We can specify a particular coordinate by appending **.x** or **.y** or **.z** to the name of the vertex, as in **T1.c.z**, which refers to the z coordinate of the third vertex of tetrahedron T1.

## 3 Defining a Tetrahedron

We begin by assuming that we have defined a tetrahedron **T**, which means that we know the coordinates of its vertices **a, b, c, d**. Unlike the case of triangles, there is no way to impose an orientation on the vertices of the tetrahedron, so this is a rare instance where moving to 3D makes our life simpler.

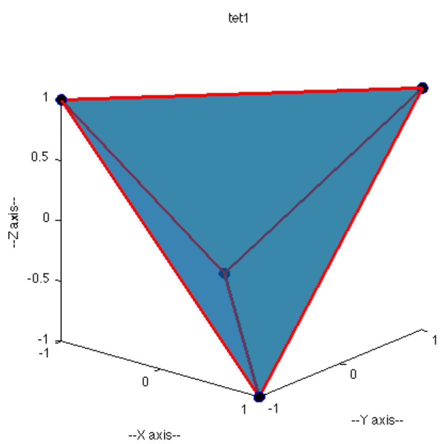


Figure 1: Tet1, Tetrahedron Example 1.

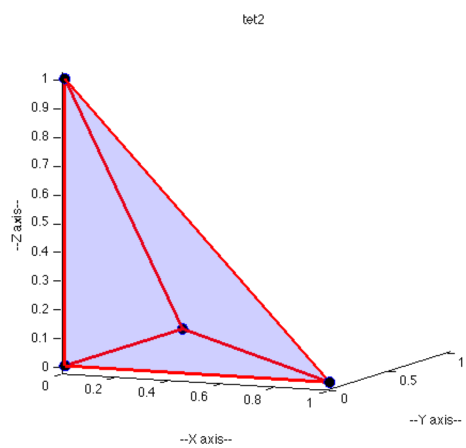


Figure 2: Tet2, Tetrahedron Example 2.

We can think of a representation of the information defining our tetrahedron as a list of the vertices, with each vertex being a list of the coordinates. We will use curly braces to collect the elements of a list. Thus, the definition of a tetrahedron might be thought of as:

```
T = {  T.a, T.b, T.c, T.d }
  = { {T.a.x, T.a.y, T.a.z},
      {T.b.x, T.b.y, T.b.z},
      {T.c.x, T.c.y, T.c.z},
      {T.d.x, T.d.y, T.d.z} }
```

For example, the example tetrahedron Tet1 in Figure 1 can be described as

```
Tet1 = { {1,1,1}, {-1,-1,1}, {-1,1,-1}, {1,-1,-1} }
```

Our second example tetrahedron Tet2 is the "reference tetrahedron". It can be described as

```
Tet2 = { {0,0,0}, {1,0,0}, {0,1,0}, {0,0,1} }
```

## 4 The Centroid of a Tetrahedron

The centroid of a tetrahedron can be thought of as the center of mass. Any plane through the centroid divides the tetrahedron into two pieces of equal volume. The centroid is just the average of the vertices:

$$\text{Centroid} = \frac{a+b+c+d}{4}$$

Here, of course, we mean that the $x$, $y$ and $z$ coordinates of the centroid are computed by averaging the corresponding coordinates of the vertices.

## 5 The Volume of a Tetrahedron

One of the most important properties of a tetrahedron is, of course, its volume.

$$\text{Volume } = \pm\frac{1}{6}\begin{vmatrix} a.x & a.y & a.z & 1 \\ b.x & b.y & b.z & 1 \\ c.x & c.y & c.z & 1 \\ d.x & d.y & d.z & 1 \end{vmatrix}$$

The reason for the plus/minus sign is that a tetrahedron is not oriented the way a triangle is, so we can reorder the vertices in any way we like. However, switching two vertices will make the determinant switch sign. So to get the (positive) volume of the tetrahedron, we must take the absolute value of this quantity!

It's easy to see that this formula is related to the similar one for triangles.

## 6 The Lengths of the Edges of a Tetrahedron

Let's get some practice with our notation by describing how to determine the lengths of the edges of the tetrahedron. It will help us in our formulas if we give names to the edges. Since each edge is defined by two vertices of the tetrahedron, appropriate names for the edges might be **Eab, Eac, Ead, Ebc, Ebd, Ecd**.

To find the length of edge **Eab**, we compute the *Euclidean length* of the vector from vertex **a** to **b**, that is:

$$\text{length(Eab)} = \sqrt{(b.x - a.x)^2 + (b.y - a.y)^2 + (b.z - a.z)^2}$$

In particular, for tetrahedron Tet1, we have

$$\text{length(Tet1.Eab)} = \sqrt{(-1-1)^2 + (-1-1)^2 + (1-1)^2} = \sqrt{8} = \approx 2.82$$

## 7 Program #1: Basic Tetrahedron Measurements

Write a program that reads a file containing the vertex coordinates of a tetrahedron and which then computes:

- the location of the centroid;
- the volume;
- the lengths of the edges,

Test your program on tetrahedron Tet1. Assume that the input is in a text file called "tet1.txt" of the form:

```
 1.0  1.0  1.0
 1.0  1.0 -1.0
 1.0 -1.0  1.0
-1.0  1.0  1.0
```

Repeat your tests on Tet2.

## 8 The Areas of the Faces of a Tetrahedron

The tetrahedron is bounded by its four triangular faces. We may wish to know the area of these faces. Although we can easily do this in 2D, the triangles are now objects in 3D. What do we do? First, it's important to realize that Heron's formula, which depends only on the lengths of the sides, could be used.

To do this in 3D, we use a formula that relies on cross products. If we consider the triangular face bounded by the tetrahedral vertices **a, b, c**, the area of that face is:

$$\begin{aligned} v_1 =& (b.y - a.y) * (c.z - a.z) - (b.z - a.z) * (c.y - a.y) \\ v_2 =& (b.z - a.z) * (c.x - a.x) - (b.x - a.x) * (c.z - a.z) \\ v_3 =& (b.x - a.x) * (c.y - a.y) - (b.y - a.y) * (c.x - a.x) \\ \text{area} =& 0.5 * \sqrt{v_1^2 + v_2^2 + v_3^2} \end{aligned}$$

## 9 The Angles on the Faces of a Tetrahedron

Since the faces of the tetrahedron are triangles, it is also natural to want to describe the shape of these faces in terms of the angles of these triangles. We are simply imagining that we take some face of the tetrahedron and lay it down on a plane and measure the angles in the usual way.

Consider the triangular face bounded by the tetrahedral vertices **a, b, c**. Denote by **Abc** the angle opposite the triangle edge **Ebc**, and let **Sbc** indicate the length of the edge **Ebc**. Then the angles of the triangle may be found by:

$$\begin{aligned} Abc &= \arccos \frac{Sab^2 + Sca^2 - Sbc^2}{2\, Sab\, Sca} \\ Aca &= \arccos \frac{Sbc^2 + Sab^2 - Sca^2}{2\, Sca\, Sab} \\ Aab &= \arccos \frac{Sca^2 + Sbc^2 - Sab^2}{2\, Sca\, Sbc} \end{aligned}$$

## 10 Program #2: The Faces

Write a program that reads a file containing the vertex coordinates of a tetrahedron and which then computes:

- the area of each faces;
- the three angles formed by the face;

Run your program on Tet1 and Tet2.

## 11 The Dihedral Angles between Pairs of Faces of a Tetrahedron

Any pair of faces of the tetrahedron have a common edge. We can think of this common edge as a hinge, and the two faces as wings of the hinge. The faces would swing open through an increasing angle, but they are not free to do so because they are part of the tetrahedron. So the two faces are frozen at some particular angle with respect to the hinge axis. This angle between a pair of faces is called a *dihedral* ("two-face") angle. A regular tetrahedon has dihedral angles equal to $\arctan 2 * \sqrt{2}$ or about 71°. Tetrahedrons with dihedral angles close to 0° or greater than 90° are usually undesirable in graphics, engineering and computational work.

For instance, the faces {a,b,c} and {a,b,d} have the common edge defined by {a,b}. Since the dihedral angle is associated with the common edge, we can use the notation $DA.ab$ to denote the dihedral angle between the pair of faces whose common edge is {a,b}.

To compute the dihedral angle $DA.ab$, we compute the unit normal vectors to each triangular face. For instance, the unit normal vector $n_{\{a,b,c\}}$ to the first face is computed from the cross product by:

$$N_{\{a,b,c\}} = (b-a) \times (c-a)$$
$$n_{\{a,b,c\}} = \frac{N_{\{a,b,c\}}}{\|N_{\{a,b,c\}}\|}$$

Computing $n_{\{a,b,d\}}$ is similar. Once we have the unit normal vectors, we compute their dot product. Therefore, the dihedral angle along edge {a,b} is the arc cosine of this value:

$$\text{DA.ab} = \arccos\left(n_{\{a,b,c\}} \cdot n_{\{a,b,d\}}\right)$$

The dihedral angles along the other edges are computed in a similar fashion.

## 12 The Solid Angles of a Tetrahedron

At each vertex of the tetrahedron, three faces come together, forming a **solid angle**. Since a solid angle is associated with a vertex of the tetrahedron, we can use the notation **SA.a** to denote the solid angle associated with vertex **a**, for instance. A solid angle is the 3D analog of the plane angles we are familiar with from geometry. Unlike a triangle, however, the solid angles of a tetrahedron do not have to add up to a constant sum.

It is still useful to measure the solid angles. One way to do this is to note that the solid angle at a vertex is related to the sum of the three dihedral angles associated with the edges incident on that vertex. Thus, the formula for solid angle **SA.a** is:

$$\text{SA.a} = \text{DA.ab} + \text{DA.ac} + \text{DA.ad} - \pi$$

and the other three solid angles may be computed in a similar way.

## 13 Program #3: Dihedral and Solid Angles

Write a program that reads a file containing the vertex coordinates of a tetrahedron and which then computes:

- the 6 dihedral angles associated with each pair of faces;
- the 4 solid angles associated with each vertex.

Run your program on Tet1 and Tet2.

## 14 The Inscribed Sphere of a Tetrahedron

The inscribed sphere or *insphere* is the largest sphere that can be contained in the tetrahedron. The center of this sphere is called the *incenter* and the radius is the *inradius*.

The insphere touches each face of the tetrahedron at a single point. These points of contact are actually the centroids of the triangular faces of the tetrahedron. Therefore, the point of contact for a face can be computed as the average of the vertices of that face, and the insphere can then be determined as the unique sphere through the four given face centroids.

For convenience, define $\alpha$, which is just a multiple of the volume:

$$\alpha = \begin{vmatrix} a.x & a.y & a.z & 1 \\ b.x & b.y & b.z & 1 \\ c.x & c.y & c.z & 1 \\ d.x & d.y & d.z & 1 \end{vmatrix}$$

and recall that the (non-unit) normal vector to the face defined by vertices {a,b,c} is

$$N_{\{a,b,c\}} = (b-a) \times (c-a)$$

If we now compute all four normal vectors, then we can determine the coordinates of the incenter **(x,y,z)** as a combination of the vertices:

$$\text{incenter}(T) = \frac{\|N_{\{a,b,c\}}\|\,d + \|N_{\{a,b,d\}}\|\,c + \|N_{\{a,c,d\}}\|\,b + \|N_{\{b,c,d\}}\|\,a}{\|N_{\{a,b,c\}}\| + \|N_{\{a,b,d\}}\| + \|N_{\{a,c,d\}}\| + \|N_{\{b,c,d\}}\|}$$

and the inradius can be computed as:

$$\text{inradius}(T) = \frac{|\alpha|}{\|N_{\{a,b,c\}}\| + \|N_{\{a,b,d\}}\| + \|N_{\{a,c,d\}}\| + \|N_{\{b,c,d\}}\|}$$

# 15 The Circumscribed Sphere of a Tetrahedron

The circumscribed sphere or *circumsphere* is the smallest sphere that contains the tetrahedron. The center of this sphere is called the *circumcenter* and the radius is the *circumradius*.

The circumcenter **(x,y,z)** satisfies the following equation:

$$\begin{vmatrix} x^2+y^2+z^2 & x & y & z & 1 \\ a.x^2+a.y^2+a.z^2 & a.x & a.y & a.z & 1 \\ b.x^2+b.y^2+b.z^2 & b.x & b.y & b.z & 1 \\ c.x^2+c.y^2+c.z^2 & c.x & c.y & c.z & 1 \\ d.x^2+d.y^2+d.z^2 & d.x & d.y & d.z & 1 \end{vmatrix} = 0$$

which looks like an augmented version of the formula for the volume.

By manipulating the determinant, we can define five useful quantities:

$$\alpha = +\begin{vmatrix} a.x & a.y & a.z & 1 \\ b.x & b.y & b.z & 1 \\ c.x & c.y & c.z & 1 \\ d.x & d.y & d.z & 1 \end{vmatrix}$$

$$\gamma = +\begin{vmatrix} a.x^2+a.y^2+a.z^2 & a.x & a.y & a.z \\ b.x^2+b.y^2+b.z^2 & b.x & b.y & b.z \\ c.x^2+c.y^2+c.z^2 & c.x & c.y & c.z \\ d.x^2+d.y^2+d.z^2 & d.x & d.y & d.z \end{vmatrix}$$

$$D_x = +\begin{vmatrix} a.x^2+a.y^2+a.z^2 & a.y & a.z & 1 \\ b.x^2+b.y^2+b.z^2 & b.y & b.z & 1 \\ c.x^2+c.y^2+c.z^2 & c.y & c.z & 1 \\ d.x^2+d.y^2+d.z^2 & d.y & d.z & 1 \end{vmatrix}$$

$$D_y = -\begin{vmatrix} a.x^2+a.y^2+a.z^2 & a.x & a.z & 1 \\ b.x^2+b.y^2+b.z^2 & b.x & b.z & 1 \\ c.x^2+c.y^2+c.z^2 & c.x & c.z & 1 \\ d.x^2+d.y^2+d.z^2 & d.x & d.z & 1 \end{vmatrix}$$

$$D_z = +\begin{vmatrix} a.x^2+a.y^2+a.z^2 & a.x & a.y & 1 \\ b.x^2+b.y^2+b.z^2 & b.x & b.y & 1 \\ c.x^2+c.y^2+c.z^2 & c.x & c.y & 1 \\ d.x^2+d.y^2+d.z^2 & d.x & d.y & 1 \end{vmatrix}$$

Using these variables, we can write the circumcenter **(x,y,z)** as:

$$\text{circumcenter}(T) = \frac{(D_x, D_y, D_z)}{2\alpha}$$

and the circumradius as:

$$\text{circumradius}(T) = \frac{\sqrt{D_x^2 + D_y^2 + D_z^2 - 4\,\alpha\,\gamma}}{2|\alpha|}$$

## 16 A Quality Measure for a Tetrahedron

When generating a mesh of tetrahedrons, the ideal shape is usually an equilateral tetrahedron. However, even in the best of cases, it is usually not possible to generate a regular mesh of equilateral tetrahedrons of the same size. When a meshing program must produce a tetrahedral mesh for a general region bounded by polygonal faces and a variety of internal obstacles, it is natural that a great many tetrahedral shapes will be produced. Since a machine is generating the tetrahedrons, it will usually be necessary for the machine to be able to evaluate their shapes, and reject those that deviate too much from the equilateral ideal. And that means that the vague notion of "close to an equilateral tetrahedron" must be replaced by a formula that can give a definite numeric evaluation of the quality of a tetrahedron's shape.

There are a number of such quality measures. One of the must useful simply compares the inradius and circumradius.

For the equilateral tetrahedron, the radiuses of these two spheres are in the ratio of 1 to 3. As a tetrahedron deviates from the ideal shape, this ratio will decrease in a smooth way, going to 0 as the tetrahedron flattens to zero volume. This gives us one way to detect and reject tetrahedrons with bad shape. We simply tell the meshing program that any tetrahedron with a quality measure less than, say 0.25, must be rejected.

For a tetrahedron **T**, we define a quality measure as follows:

$$Q(T) = 3 * \frac{\text{inradius}(T)}{\text{circumradius}(T)}$$

## 17 Program #4: Inscribed and Circumscribed Spheres

Write a program that reads a file containing the vertex coordinates of a tetrahedron and which then computes:

- the inradius;
- the circumradius.
- the quality measure **Q**.

Run your program on Tet1 and Tet2.