Chapter 5

Simple Examples on Rectangular Domains

In this chapter we consider simple elliptic boundary value problems in rectangular domains in $\mathbb{R}^2$ or $\mathbb{R}^3$; our prototype example is the Poisson equation but we also briefly consider the biharmonic equation and the Helmholtz equation. Similar to our exposition of the two-point boundary value problem in Chapter **??**, we consider the implementation of different boundary conditions for our prototype equation. Much of this chapter is a straightforward extension of the analysis presented in the previous chapter for the two-point boundary value problem. However, a few important differences are evident.

For the finite element approximation of these elliptic boundary value problems, we only consider approximating with finite elements spaces which are obtained by taking tensor products of one-dimensional finite element spaces. In Chapter **??** we consider the general problem of determining finite element spaces on polygonal domains and in a later chapter we consider isoparametric finite elements for curved domains.

5.1 The Poisson equation with homogeneous Dirichlet boundary data

In this section we consider Poisson's equation defined in a bounded domain in $\mathbb{R}^2$ or $\mathbb{R}^3$ with homogeneous Dirichlet boundary data. We let $\vec{x}$ denote a point in $\mathbb{R}^2$ or $\mathbb{R}^3$. Specifically, we let Ω be an open, connected, bounded set in $\mathbb{R}^2$ or $\mathbb{R}^3$ and let $\partial\Omega$ denotes its boundary. At this point in our discussion of the finite element method, we only have the background to use finite element spaces which are tensor products of the one dimensional finite element spaces discussed in the last chapter. Consequently, when we move to the discretization stage we require that Ω be a rectangular domain. However, the weak formulations that we present hold for more general domains. In the next chapters we address the problem of discretizing using other elements suitable for more general domains. We let $\overline{\Omega}$ denote the closure of Ω; i.e., $\overline{\Omega} = \Omega \cup \partial\Omega$. Let $f = f(\vec{x})$ be a given function that is continuous on the closure of Ω; i.e., $f \in C(\overline{\Omega})$. We say that a function $u(\vec{x})$ defined on $\overline{\Omega}$ is a classical solution of the Poisson equation with homogeneous Dirichlet boundary conditions if $u \in C^2(\Omega)$, $u \in C(\overline{\Omega})$ and u satisfies

$$-\Delta u(\vec{x}) = f(\vec{x}) \quad \text{for } \vec{x} \in \Omega \tag{5.1a}$$

$$u(\vec{x}) = 0 \quad \text{for } \vec{x} \in \partial\Omega\,, \tag{5.1b}$$

where $\Delta u = u_{xx} + u_{yy}$ in $\mathbb{R}^2$ or analogously $\Delta u = u_{xx} + u_{yy} + u_{zz}$ in $\mathbb{R}^3$. It is well known that for sufficiently smooth $\partial\Omega$ there exists a unique classical solution of (6.1).

In the sequel, we assume enough smoothness of the boundary so that the domain admits the application of the divergence theorem. Every polygonal domain or a domain with a piecewise smooth boundary has sufficient smoothness for our purposes.

We make extensive use of Green's formula which is the analog of the integration by parts formula in higher dimensions and is derived from the divergence theorem of vector calculus. Let $\vec{n}$ denote the unit outer normal to $\partial\Omega$ and let dS denote the measure defined on the boundary and dV the measure of volume. We have that for $v \in C^1(\overline{\Omega})$, $w \in C^2(\overline{\Omega})$

$$\int_\Omega v\Delta w\, dV = \int_{\partial\Omega} v(\vec{n}\cdot\nabla w)\, dS - \int_\Omega \nabla w \cdot \nabla v\, dV$$

or equivalently

$$\int_\Omega v\Delta w\, dV = \int_{\partial\Omega} v\frac{\partial w}{\partial \vec{n}}\, dS - \int_\Omega \nabla w \cdot \nabla v\, dV\,. \tag{5.2}$$

5.1.1 Weak formulation

To define the weak formulation we first determine the underly As before, we impose the homogeneous Dirichlet boundary conditions by constraining our space $H^1(\Omega)$; in particular we have the space

$$H^1_0(\Omega) = \{v \in H^1(\Omega) \ : \ v = 0 \text{ on } \partial\Omega\}\,.$$

The weak formulation which we consider is

$$\begin{cases} \text{seek } u \in H^1_0(\Omega) \text{ such that} \\ A(u,v) = \displaystyle\int_\Omega \nabla v \cdot \nabla w\, dV = (f,v) \quad \forall\, v \in H^1_0(\Omega)\,. \end{cases} \tag{5.3}$$

The solution $u \in H^1_0(\Omega)$ of (6.3) is called the generalized or weak solution of (6.1).

If u satisfies the classical problem (6.1) then u satisfies the weak formulation (6.1) because

$$\begin{aligned} \int_\Omega fv\, dV &= -\int_\Omega \Delta uv\, dV \quad \forall\, v \in H^1_0(\Omega) \\ &= \int_\Omega \nabla u \cdot \nabla v\, dV - \int_{\partial\Omega} \frac{\partial u}{\partial \vec{n}} v\, dS \end{aligned}$$

$$= \int_\Omega \nabla u \cdot \nabla v \, dV$$
$$= A(u, v)\,,$$

where we have used Green's formula (6.2) and imposed the fact that $v = 0$ on $\partial\Omega$.

The existence and uniqueness of a weak solution to (6.3) can be verified by satisfying the hypotheses of the Lax-Milgram theorem. Recall that the norm on $H^1(\Omega)$ is defined by

$$\|u\|_1^2 = \int_\Omega \big(u^2 + \nabla u \cdot \nabla u\big)\, dV = \|u\|_0^2 + \|\nabla u\|_0^2 = \|u\|_0^2 + |u|_1^2\,.$$

The bilinear form is bounded on all of $H^1(\Omega)$ since

$$|A(u,v)| = \Big|\int_\Omega \nabla u \cdot \nabla v \, dV\Big| = |\,(\nabla u, \nabla v)\,|$$
$$\leq \|\nabla u\|_0 \|\nabla v\|_0 \leq \|u\|_1 \,\|v\|_1\;,$$

where we used the Cauchy-Schwarz inequality and the definition of the H^1 and L^2 norms.

We must now show coercivity of the bilinear form, i.e., there exists a constant $m > 0$ such that

$$A(u,u) = \int_\Omega (\nabla u \cdot \nabla u)\, dV \geq m\, \|u\|_1^2 \quad \forall\, u \in H_0^1(\Omega)\,.$$

Note that the bilinear form $A(u,u)$ can also be written as

$$A(u,u) = |u|_1^2 = \frac{1}{2}\big(|u|_1^2 + |u|_1^2\big)\,.$$

Our underlying Hilbert space is $H_0^1(\Omega)$ so we can use the Poincaré inequality to demonstrate that the standard H^1-norm is norm equivalent to this semi-norm and thus coercivity is guaranteed in an analogous manner to the homogeneous Dirichlet problem for the two-point boundary value problem of the last chapter. Specifically, we have

$$A(u,u) = \frac{1}{2}\big(|u|_1^2 + |u|_1^2\big) \geq \frac{1}{2}\min\{1, \frac{1}{C_p^2}\}\Big(\|u\|_0^2 + |u|_1^2\Big) = m\, \|u\|_1^2\;,$$

where C_p is the constant in the Poincaré inequality. We have demonstrated the boundedness and coercivity of the bilinear form defined in (**??**) and thus the Lax-Milgram theorem guarantees the existence and uniqueness of a solution to the weak problem (6.3) because the right-hand side is obviously a bounded linear functional on $H^1(\Omega)$. The bilinear form is symmetric and so we know that approximating the solution of the weak problem is equivalent to the minimization problem

$$\min_{v \in H_0^1(\Omega)} \int_\Omega \Big(\frac{1}{2}\nabla v \cdot \nabla v - f v\Big)\, dV\,.$$

5.1.2 Approximation using bilinear functions

In later chapters we consider finite element spaces over polygonal or curved domains. At present, we restrict the domain so that we can use rectangular elements; therefore, the finite element spaces can be constructed from the spaces used in the previous chapter. As in the one-dimensional case, we must now choose a finite dimensional subspace of $S_0^h(\Omega) \subset H_0^1(\Omega)$ in which to seek the approximate solution. For the discrete problem we have

$$\begin{cases} \text{seek } u^h \in S_0^h(\Omega) \text{ satisfying} \\ A(u^h, v^h) = \int_\Omega \left(\nabla u^h \cdot \nabla v^h\right) dV = \left(f, v^h\right) \quad \forall\, v^h \in S_0^h\,. \end{cases} \tag{5.4}$$

Existence and uniqueness of the solution to this problem is guaranteed by the Lax-Milgram theorem.

To approximate our finite element solution we consider the concrete case where Ω is the unit square or unit cube. We choose the space $S_0^h(\Omega)$ to be continuous, piecewise bilinear functions defined on $\Omega \subset \mathbb{R}^2$ or continuous, piecewise trilinear functions[1] for $\Omega \subset \mathbb{R}^3$. We formally construct the bilinear basis functions; the trilinear basis functions are defined analogously. Let N, M be positive integers and let $h_x = 1/(N+1)$, $h_y = 1/(M+1)$ and consider the subdivision of Ω into rectangles of size $h_x \times h_y$ where

$$x_i = ih_x,\;\; 0 \le i \le N+1, \quad y_j = jh_y,\;\; 0 \le j \le M+1\,.$$

See Figure 6.1 for a sample grid on a unit square with $h_y = 2h_x$. Let $\phi_i(x)$, $1 \le i \le N$ represent the standard "hat" piecewise linear basis functions in x and let $\phi_j(y)$, $1 \le j \le M$, be similarly defined, i.e.,

$$\phi_i(x) = \begin{cases} \dfrac{x - x_{i-1}}{h_x} & \text{for } x_{i-1} \le x \le x_i \\ \dfrac{x_{i+1} - x}{h_x} & \text{for } x_i \le x \le x_{i+1} \\ 0 & \text{elsewhere} \end{cases} \quad \phi_j(y) = \begin{cases} \dfrac{y - y_{j-1}}{h_y} & \text{for } y_{j-1} \le y \le y_j \\ \dfrac{y_{j+1} - y}{h_y} & \text{for } y_j \le y \le y_{j+1} \\ 0 & \text{elsewhere.} \end{cases}$$

On $\Omega = (0,1) \times (0,1)$ we now define the NM *bilinear functions*

$$\phi_{ij}(x,y) = \phi_i(x)\phi_j(y) \quad \text{for } 1 \le i \le N,\, 1 \le j \le M\,. \tag{5.5}$$

We easily see that $\phi_{ij}(x_i, y_j) = 1$ and $\phi_{ij}(x_k, y_l) = 0$ for $k \ne i$ **or** $l \ne j$. Also $\phi_{ij}(x,y)$ is zero outside of $[(i-1)h_x, (i+1)h_x] \times [(j-1)h_y, (j+1)h_y]$. The support of $\phi_j(x,y)$ is illustrated in Figure 6.1 and the shape of a specific bilinear function $\phi_{2,3}$ which is one at node (x_2, y_3) is given in Figure 6.2.

For Ω the unit square, we choose $S_0^h(\Omega) \equiv S_0^h(0,1) \otimes S_0^h(0,1)$ to be the tensor product of the subspaces $S_0^h(0,1)$ (one each in the $x-$ and $y-$ directions) of one-dimensional piecewise linear, continuous functions which vanish at zero and one.

[1] A bilinear or trilinear function is a function which is linear with respect to its variables because if we hold one variable fixed, it is linear in the other; for example $f(x,y) = xy$ is a bilinear function but $f(x,y) = x^2y$ is not.

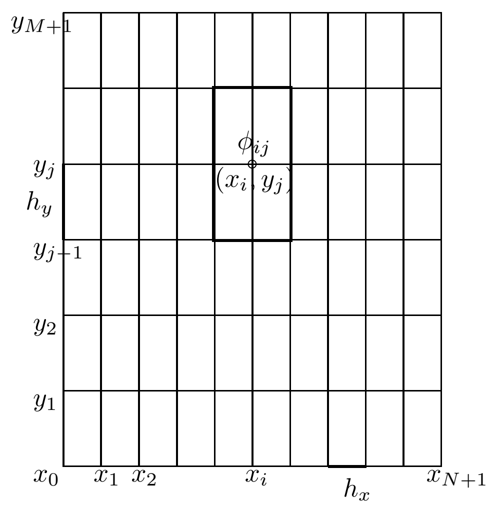


Figure 5.1. *Grid on a unit square with support of basis function $\phi_{ij}(x,y)$ indicated.*

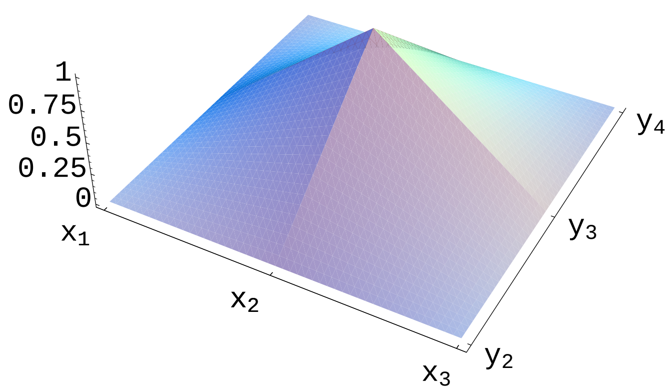


Figure 5.2. *Support of bilinear basis function $\phi_{2,3}$.*

$S_0^h(\Omega)$ consists of all functions $v(x,y)$ on $(0,1)\times(0,1)$ of the form

$$v(x,y) = \sum_{i=1}^{N}\sum_{j=1}^{M} c_{ij}\phi_i(x)\phi_j(y) = \sum_{i=1}^{N}\sum_{j=1}^{M} c_{ij}\phi_{ij}(x,y)\,. \qquad (5.6)$$

Note that the general form of a bilinear function in $\mathbb{R}^2$ is $a_0 + a_1x + a_2y + a_3xy$ compared with a linear function in two variables which has the general form $a_0 + a_1x + a_2y$. Clearly $S_0^h(\Omega)$ is the space of all continuous, piecewise bilinear functions (with respect to the given subdivision) which vanish on the sides of the unit square. Also, every piecewise bilinear function $w(x, y)$ can be written in the form (6.6) with $c_{ij} = f(x_i, y_j)$; i.e., it is a linear combination of the $P = NM$ linearly independent functions $\phi_{ij}(x, y)$. $S_0^h(\Omega)$ is an P-dimensional subspace of $H_0^1(\Omega)$; note that for $M = N$, S_0^h is an N^2 dimensional subspace whereas in one dimension, it was an N dimensional subspace. Of course this affects the size of our resulting matrix problem.

From previous discussions we know that once a basis is chosen for the approximating subspace, the discrete problem can be written as a linear system of equations. To investigate the structure of the coefficient matrix for our choice of bilinear basis functions, we let the basis functions $\phi_{ij}(x, y)$ for $S_0^h(\Omega)$ be rewritten in single index notation; for simplicity of exposition we choose $M = N$. We have

$$\{\psi_k(x, y)\}_{k=1}^{N^2} = \{\phi_{ij}(x, y)\}_{i,j=1}^{N} \ .$$

For example, $\psi_k = \phi_{k1}$ for $1 \leq k \leq N$, $\psi_{N+k} = \phi_{k2}$ for $1 \leq k \leq N$, etc. Our discrete weak formulation (6.4) is equivalent to seeking $u^h \in S_0^h$ satisfying

$$A(u^h, \psi_i) = (f, \psi_i) \quad \text{for } 1 \leq i \leq N^2 \, .$$

We now let $u^h = \sum_{j=1}^{N^2} c_j \psi_j$ and substitute into the above expression. The result is a linear system of N^2 equations in the N^2 unknowns $\{c_j\}_{j=1}^{N^2}$; i.e., $\mathcal{A}\vec{c} = \vec{\mathcal{F}}$ where $\vec{c} = (c_1, \ldots, c_{N^2})^T$, $\mathcal{F}_i = (f, \psi_i)$ and $\mathcal{A}_{ij} = A(\psi_i, \psi_j)$. Note that with the numbering scheme we are using for the basis functions, we are numbering our unknowns which correspond to the coefficients c_j across rows. Because we have assumed the same number of points in the x and y directions we could have easily numbered them along columns of the grid.

To determine the structure of the resulting matrix we consider the ith row of the matrix and decide how many nonzero entries are in the row. Because we know the matrix is symmetric, we only consider terms above the diagonal. Clearly there can be nonzero entries in columns i and $i+1$. The next nonzero entries occur for unknowns corresponding to basis functions in the next row of the grid. Specifically we can have nonzero entries in columns $i + N - 1$, $i + N$ and $i + N + 1$ where N is the number of unknowns across the row. The coefficient matrix $\mathcal{A}$ is an $N^2 \times N^2$ symmetric, positive definite matrix which has a block tridiagonal structure of the form

$$\mathcal{A} = \begin{pmatrix} A_0 & A_1 & 0 & & \cdots & 0 \\ A_1 & A_0 & A_1 & 0 & \cdots & 0 \\ & & \ddots & \ddots & \ddots & \\ 0 & \cdots & 0 & A_1 & A_0 & A_1 \\ 0 & \cdots & & 0 & A_1 & A_0 \end{pmatrix} , \tag{5.7}$$

where A_0 and A_1 are $N \times N$ tridiagonal matrices. (See exercises.) A matrix of this form can be solved efficiently by a banded Cholesky algorithm, a block tridiagonal solver or an iterative solver.

Error estimates

Galerkin's theorem provides us with the estimate

$$\left\|u - u^h\right\|_1 \leq \inf_{\chi^h \in S_0^h} \left\|u - \chi^h\right\|_1 . \tag{5.8}$$

As before, we turn to the interpolant of u in our finite dimensional space $S_0^h(\Omega)$ to obtain an estimate in terms of powers of h. Specifically for $\mathbb{R}^2$, we denote $I^h v$ as the unique function in $S_0^h(\Omega)$ which satisifes $(I^h v)(x_i, y_j) = v(x_i, y_j)$ for $0 \leq i, j \leq N+1$. We can write $I^h v$ as a linear combination of our basis functions; i.e., $(I^h v)(x, y) = \sum_{i,j=1}^N v(x_i, y_j)\phi_{ij}(x, y)$. For v defined on $\Omega \subset \mathbb{R}^2$, we let $I_x^h v$ and $I_y^h v$ denote the interpolation operators in the x- and y-directions; i.e.,

$$(I_x^h v)(x, y) = \sum_{i=1}^N v(x_i, y)\phi_i(x) \quad \text{and} \quad (I_y^h v)(x, y) = \sum_{j=1}^N v(x, y_j)\phi_j(y) .$$

Then we have that

$$\begin{aligned}(I_y^h I_x^h v)(x, y) = I_y^h \left(\sum_{i=1}^N v(x_i, y)\phi_i(x) \right) &= \sum_{j=1}^N \left(\sum_{i=1}^N v(x_i, y_j)\phi_i(x) \right) \phi_j(y) \\ &= (I^h v)(x, y) .\end{aligned}$$

Similarly, $I^h v = I_x^h I_y^h v$. For $\Omega \subset \mathbb{R}^3$ clearly $I^h v = I_x^h I_y^h v I_z^h$. This result can be used to prove the following theorem which gives us an estimate of the error in $v - I^h v$ when v is sufficiently smooth.

Lemma 5.1. *Let $v \in H^2(\Omega)$. Then if $I^h v$ is the interpolant of v in $S^h(\Omega)$, the space of continuous, piecewise bilinear functions, then there exist constants C_i, $i = 1, 2$ independent of v and h such that*

$$\left\|v - I^h v\right\|_0 \leq C_1 h^2 \left\|v\right\|_2 \tag{5.9}$$

and

$$\left\|v - I^h v\right\|_1 \leq C_2 h \left\|v\right\|_2 . \tag{5.10}$$

As in the case in one-dimension, we can now make use of the interpolation result to prove an optimal error estimate in the H^1-norm. To obtain a result for the L^2-norm we again use "Nitsche's trick" in a manner completely analogous to that in the one-dimensional case where now we make use of elliptic regularity. The details of the proof are left to the exercises.

Theorem 5.2. *Let $u \in H^2(\Omega) \cap H_0^1(\Omega)$ be the solution of (6.3) where $\Omega = (0,1) \times (0,1)$. Let $S_0^h(\Omega)$ be the space of piecewise bilinear functions which vanish on $\partial\Omega$ and let u^h be the Galerkin approximation to u in $S_0^h(\Omega)$ defined by (6.4). Then*

$$\left\| u - u^h \right\|_1 \leq Ch \left\| u \right\|_2 \tag{5.11}$$

and

$$\left\| u - u^h \right\|_0 \leq Ch^2 \left\| u \right\|_2 \tag{5.12}$$

for some constants C independent of h and u.

5.1.3 Higher order elements

Our discussion of approximating the problem (6.1) posed on $\Omega = (0,1) \times (0,1)$ has so far included only piecewise bilinear function spaces. Of course, we can also use tensor products of higher order spaces such as the quadratic or cubic functions in one space dimension. Note that a general biquadratic function has the form $a_0 + a_1 x + a_2 y + a_3 xy + a_4 x^2 + a_5 y^2 + a_6 x^2 y + a_7 x y^2 + a_8 x^2 y^2$ compared with a general quadratic function in two dimensions which has the form $a_0 + a_1 x + a_2 y + a_3 xy + a_4 x^2 + a_5 y^2$. As in the one-dimensional case, for a smooth enough solution, these spaces yield higher rates of convergence then that achieved with piecewise bilinear approximations. The construction of the basis functions in two or three dimensions is done analogous to the piecewise bilinear case; the details are left to the exercises.

5.1.4 Numerical quadrature

Once again, the entries in the matrix and right-hand side of our linear system are calculated using a numerical quadrature rule which has the form

$$\int_\Omega f(\vec{x})\, d\Omega \approx \sum_i f(\vec{q}_i)\omega_i \,,$$

where the points $\vec{q}_i$ are the quadrature points and ω_i are the quadrature weights. Because we are using rectangular elements with basis functions obtained by taking the tensor product of one-dimensional basis functions, the most straightforward approach is to use tensor products of the quadrature rules in one spatial dimension. Typically, we use the same quadrature rule in each spatial dimension. For example, if we have the rule

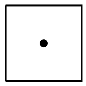
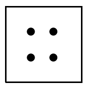
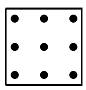
$$\int_a^b f(x)\; dx = \sum_i f(q_i) w_i$$

then we can write

$$\int_a^b \int_c^d f(x,y)\; dydx \approx \int_a^b \Big(\sum_j f(x, q_j) w_j \Big) \approx \sum_i \sum_j f(q_i, q_j) w_j w_i \,.$$

In one dimension we employed the Gauss-Legendre quadrature rules on $[-1,1]$. If we take the tensor products of a p-point Gauss rule in each direction then we would

Table 5.1. *Tensor product of Gauss quadrature rules in two dimensions*

	1-D rule	# points in $\mathbb{R}^2$	points q_i & weights w_i
	1 point Gauss	1	$q_1 = (0,0) \quad w_1 = 4$
	2 point Gauss	4	$q_i = \frac{1}{\sqrt{3}}\{(-1,-1),(1,-1),(-1,1),(1,1)$ $w_i = 1$
	3 point Gauss	9	$q_i = \sqrt{\frac{3}{5}}\Big\{(-1,-1),(0,-1),(1,-1),(-1,0),$ $((0,0),(1,0),(-1,1),(0,1),(1,1)\Big\}$ $w_i = \frac{1}{81}\{25,40,25,40,64,40,25,40,25\}$

have one point for the tensor product of the one-point rule, four points for the tensor product of the two-point rule, etc. The quadrature points in two dimensions formed by the tensor product of one-point through three-point Gauss quadrature rules are described in Table 6.1. Note that in three dimensions we have 1, 8, and 27 quadrature points for tensor products of these three quadrature rules. To apply these rules to an integral over an arbitrary rectangular domain, we must perform a change of variables in both the x and y directions analogous to the one-dimensional case. For our example, if we are using bilinear or trilinear elements, then the tensor product of the one-point Gauss rule is adequate; for biquadratics or triquadratics we need to use the tensor product of the two-point Gauss rule.

5.2 The Poisson equation with Neumann boundary data

In this section we consider solving Poisson's equation on an open, bounded domain in $\mathbb{R}^2$ or $\mathbb{R}^3$ where we specify Neumann data on a portion of the boundary and Dirichlet data on the remainder of the boundary. In particular, we seek a function u satisfying

$$\begin{aligned} -\Delta u(\vec{x}) &= f(\vec{x}) \quad \text{for } \vec{x} \in \Omega \\ u(\vec{x}) &= 0 \quad \text{for } \vec{x} \in \Gamma_1 \\ \frac{\partial u}{\partial \vec{n}}(\vec{x}) &= g(\vec{x}) \quad \text{for } \vec{x} \in \Gamma_2\,, \end{aligned} \tag{5.13}$$

where $\Gamma_1 \cup \Gamma_2 = \partial\Omega$, $\Gamma_1 \cap \Gamma_2$ is a set of measure zero, and $\partial u/\partial\vec{n}$ denotes the directional derivative of u in the direction of the unit outward normal to the boundary of the domain. We note that if $\Gamma_1 = \partial\Omega$ then we have the purely Dirichlet problem discussed in Section 6.1; in the case $\Gamma_2 = \partial\Omega$ we have a purely Neumann problem. As expected, in the latter case the problem does not have a unique solution. It is well known that for sufficiently smooth $\partial\Omega$ there exists a unique classical solution of (6.13) provided, of course, that Γ_1 is measurable.

5.2.1 Weak Formulation

For this problem we define $H^1_B(\Omega)$ as

$$H^1_B(\Omega) = \{u \in H^1(\Omega) \ : \ u = 0 \text{ on } \Gamma_1\}\,. \tag{5.14}$$

Our weak formulation is

$$\left\{ \begin{array}{l} \text{seek } u \in H^1_B(\Omega) \text{ satisfying} \\ A(u,v) \equiv \displaystyle\int_\Omega \nabla u \cdot \nabla v \, d\Omega = (f,v) + \int_{\Gamma_2} gv \quad \forall\, v \in H^1_B(\Omega)\,. \end{array} \right. \tag{5.15}$$

If u is a solution of the classical problem (6.13) then by Green's theorem u satisfies

$$\begin{aligned} (f,v) = -\int_\Omega \Delta u v \, d\Omega &= \int_\Omega \nabla u \cdot \nabla v \, d\Omega - \int_\Gamma \frac{\partial u}{\partial \vec{n}} v \; ds \\ &= A(u,v) - \int_{\Gamma_1} \frac{\partial u}{\partial \vec{n}} v \; ds - \int_{\Gamma_2} \frac{\partial u}{\partial \vec{n}} v \; ds \\ &= A(u,v) - \int_{\Gamma_2} g(\vec{x}) v \; ds \quad \forall\, v \in H^1_B(\Omega)\,, \end{aligned}$$

where we have used the fact that the boundary integral over Γ_1 is zero since $v \in H^1_B(\Omega)$ and for the boundary integral over Γ_2 we have used $\partial u/\partial\vec{n} = g(\vec{x})$. In this problem the Dirichlet boundary condition on Γ_1 is essential whereas the Neumann boundary condition on Γ_2 is natural. It's interesting to compare the weak formulation (6.15) with the analogous weak formulation (**??**) for the two-point boundary value problem. In the one-dimensional case, we simply have the value of the derivative at a point times the test function at the same point. In two spatial dimensions with inhomogeneous Neumann boundary conditions we have a line integral on the right-hand side of the weak form and in three spatial dimensions we have a surface integral. This complicates the implementation of the method but it is straightforward; for example, for $\Omega \subset \mathbb{R}^2$ we have a line integral on the boundary which can be approximated using a Gauss quadrature rule. The existence and uniqueness of a solution to (6.15) is demonstrated in an analogous manner to the purely Dirichlet problem discussed in Section 6.1. The only complication is demonstrating that the right-hand side, which now contains a boundary integral, is a bounded linear functional on $H^1(\Omega)$.

When the classical problem is a purely Neumann problem, i.e., when $\Gamma_2 = \partial\Omega$, it is clear that there is not a unique solution. Thus, we can not expect the hypotheses of the Lax-Milgram theorem to be satisfied. In particular, we are unable to demonstrate coercivity of the bilinear form.

5.2.2 Approximation using bilinear functions

As a concrete example we once again take $\Omega = (0,1) \times (0,1)$; we choose Γ_1 to be the top and bottom portions of the boundary, i.e., when $y = 0$ and $y = 1$; Γ_2 is the remainder of the boundary. We subdivide our domain into rectangles of size $h \times h$ where $h = 1/(N+1)$, $x_i = ih$, $y_j = jh$, $i, j = 0, \ldots, N+1$. If we approximate using continuous, piecewise bilinear functions as in Section 6.1, then we seek our solution in the space $\hat{S}^h(\Omega$ which is the space of all continuous, piecewise bilinear functions on Ω which are zero at $y = 0$ and $y = 1$. In the x-direction we have the $N+2$ basis functions $\phi_i(x)$, $i = 0, 1, \ldots, N+1$ and N basis functions in the y-direction $\phi_j(y)$, $j = 1, \ldots, N$. In this case we have the $N(N+2)$ basis functions $\phi_{ij}(x, y)$ which are the tensor products of the one-dimensional basis functions. The basic structure of the matrix is the same as in the previous example. Optimal error estimates are derived in a completely analogous manner to the previous section when $u \in H^2(\Omega) \cap H^1_B(\Omega)$.

We note that if we attempt to discretize the purely Neumann problem, i.e., when $\Gamma_2 = \partial\Omega$, then the resulting $(N+2)^2$ matrix would be singular. This is to be expected because we could not prove uniqueness of the solution to the weak problem. A unique solution to the system can be found by imposing an additional condition on u^h such as specifying u^h at *one* point or requiring the solution to have zero mean, i.e., $\int_\Omega u\, dV = 0$.

5.3 Other examples

In this section we make a few brief remarks concerning some additional examples. In particular, we consider Poisson's equation with inhomogeneous Dirichlet boundary data and with a mixed boundary condition, a purely Neumann problem for the Helmholtz equation and a fourth order equation.

5.3.1 Other boundary conditions

As in the one-dimensional case, we can consider problems with inhomogeneous Dirichlet boundary conditions such as

$$\begin{aligned} -\Delta u &= f \quad \vec{x} \in \Omega \\ u(\vec{x}) &= q(\vec{x}) \quad \text{on } \Gamma\,. \end{aligned} \tag{5.16}$$

To treat the inhomogeneous Dirichlet condition we proceed formally as before and define a function $g(\vec{x}) \in H^1(\Omega)$ such that $g(\vec{x}) = q(\vec{x})$ on Γ. Then we convert the problem into one which has homogeneous Dirichlet boundary conditions. Then our solution is $u(\vec{x}) = w(\vec{x}) + g(\vec{x})$ where w is the unique solution in $H^1_0(0,1)$ of

$$A(w, v) = (f, v) - A(q, v) \quad \forall\, v \in H^1_0(0,1)\,,$$

where, as before,

$$A(w, v) = \int_\Omega \nabla w \cdot \nabla v \, dV\,.$$

The most serious difficulty arises when we try to approximate. In the one-dimensional case, it was easy to determine a function g^h in our approximating subspace; however, for higher dimensions this is not the case. When we choose a finite dimensional approximating subspace, in general, we are not able to find a function g^h in this subspace to use for the function g above. If g^h is not in the approximating subspace then $u^h = w^h + g^h$ is not in the approximating space. We postpone discussion of this problem until a later chapter.

A problem with a mixed boundary condition such as

$$\begin{aligned} -\Delta u &= f \quad \vec{x} \in \Omega \\ \frac{\partial u}{\partial \vec{n}} + \alpha(\vec{x})u(\vec{x}) &= q(\vec{x}) \quad \text{on } \Gamma \end{aligned} \tag{5.17}$$

can be handled analogous to the one-dimensional case; i.e., we merely include a term $\int_\Gamma \alpha uv\, ds$ in the bilinear form and add the boundary integral $\int_\Gamma qv\, ds$ to the right-hand side. Recall that in the one-dimensional case, we had to add point values of the solution and/or test function to the bilinear form or right-hand side whereas in the two-dimensional case we are modifying the bilinear form and the right-hand side by a boundary integral.

5.3.2 A Neumann problem for the Helmholtz equation

We have seen that the purely Neumann problem for Poisson's equation; i.e., when $\Gamma_2 = \partial\Omega$, does not have a unique solution and if we attempt to discretize then we are lead to a singular matrix. If, however, we consider the Neumann problem for the Helmholtz equation

$$\begin{aligned} -\Delta u + \sigma^2 u &= f \quad \text{in } \Omega \\ \frac{\partial u}{\partial \vec{n}} &= 0 \quad \text{on } \partial\Omega \end{aligned}$$

then the problem possesses a unique solution for $u \in C^2$ and sufficiently smooth $\partial\Omega$. In this case the weak formulation is to find $u \in H^1(\Omega)$ such that

$$A(u,v) \equiv \int_\Omega \big(\nabla u \cdot \nabla v + uv\big)\, dV = (f,v) \quad \forall\, v \in H^1(\Omega)\,.$$

This bilinear form is coercive on $H^1(\Omega)$ as well as bounded. In fact for $k^2 = 1$

$$A(u,u) = \int_\Omega \nabla u \cdot \nabla u + u^2\, dV = \|u\|_1^2$$

and in general

$$A(u,u) = \int_\Omega \nabla u \cdot \nabla u + u^2\, dV A(u,u) = \int_\Omega \nabla u \cdot \nabla u + u^2\, dV = \|u\|_1^2\, gemin1, k^2 \, \|u\|_1^2 \,.$$

5.3.3 A fourth order problem

The biharmonic equation is the fourth order partial differential equation

$$\Delta\Delta u = \Delta^2 u = f \quad \text{in } \Omega \ .$$

We may impose boundary conditions such as

$$u = \frac{\partial u}{\partial \vec{n}} = 0 \quad \text{on } \partial\Omega\,.$$

We define $H_0^2(\Omega)$ to be the space

$$H_0^2(\Omega) = \{v \in H^2(\Omega) \mid v = \frac{\partial v}{\partial n} = 0 \ \text{ on } \partial\Omega\}.$$

A weak formulation is to seek $u \in H_0^2(\Omega)$ such that

$$A(u,v) = F(v) \quad \forall\, v \in H_0^2(\Omega)$$

where

$$A(u,v) = \int_\Omega \Delta u \Delta v \, d\Omega \quad \forall\, u, v \in H_0^2(\Omega)$$

and

$$F(v) = \int_\Omega f v \, d\Omega \quad \forall\, v \in H_0^2(\Omega)$$

It can be shown that if u is the classical solution of the biharmonic problem then u satisfies this weak problem; moreover, the weak problem has a unique solution u in $H_0^2(\Omega)$. (See exercises.) To discretize this problem we must use a subspace of $H_0^2(\Omega)$ such as bicubic splines or bicubic Hermites in $\Omega \subset \mathbb{R}^2$. However, in the next chapter we see that when we use a triangular element, it is not so easy to obtain a subspace of H^2.

5.4 Computational examples

Before looking at a specific example, we first compare the number of nodes, the number of unknowns, and the number of quadrature points required to approximate the solution of the problem $-\Delta u + u = f$ with homogeneous, Neumann boundary conditions in one, two and three dimensions. Note that in this purely Neumann problem the number of unknowns is the same as the number of nodes. Specifically we compare the number of unknowns for various values of h for linear, bilinear and trilinear elements as well as for tensor products of quadratic and cubic spaces. We also provide the minimum number of quadrature points that are used in each case. Recall that the number of unknowns corresponds to the size of the matrix and the number of quadrature points influences the amount of work required to compute the entries in the matrix and right-hand sides. In all cases we assume a uniform grid with spacing h in each dimension. The "curse of dimensionality" can clearly be seen from Table 6.2.

Table 5.2. *Comparison of number of unknowns for solving a problem on a domain* $(0,1)^n$, $n = 1,2,3$ *using tensor products of one-dimensional elements.*

	Number of unknowns			Number of
	$h = 0.1$	$h = 0.01$	$h = 0.001$	quadrature pts.
linear	11	101	1001	1
bilinear	121	10,201	1.030×10^6	1
trilinear	1331	1.030×10^6	1.003×10^9	1
quadratic	21	201	2001	2
biquadratic	441	40,401	4.004×10^6	4
triquadratic	9261	8.121×10^6	8.012×10^9	8
cubic	31	301	3001	3
bicubic	961	90,601	9.006×10^6	9
tricubic	29,791	2.727×10^7	2.703×10^{10}	27

We now turn to providing some numerical results for the specific problem

$$\begin{aligned} -u''(x) &= (x^2+y^2)\sin(x,y) \qquad \forall\,(x,y)\in\Omega \\ u &= \sin(xy) \quad \text{on } \partial\Omega \end{aligned} \tag{5.18}$$

where $\Omega = \{(x,y)\ :\ 0 \le x \le 3,\quad 0 \le y \le 3\}$. The exact solution to this problem is $u(x,y) = \sin(xy)$ whose solution is plotted in Figure 6.3 along with a contour plot of the solution. Note that we are imposing inhomogeneous Dirichlet boundary conditions in this example. The results presented here use bilinear and biquadratic elements on a uniform grid of size h in each dimension; for the quadrature rule we use the tensor product of the one point Gauss rule for bilinears and the tensor product of the two point Gauss rule for biquadratics. As usual, a higher order quadrature rule is used to calculate the error. The numerical rates of convergence are obtained using (**??**). The results are presented in Table 6.3 and some results are plotted for the bilinear case in Figure **??**. Note that as expected, the optimal rates of convergence are obtained.

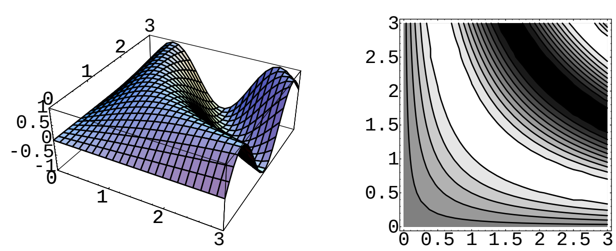


Figure 5.3.

Table 5.3. *Numerical results for (6.18) using bilinear and biquadratic elements.*

element	h	No. of unknowns	$\|u-u^h\|_1$	rate	$\|u-u^h\|_0$	rate
bilinear	1/4	144	0.87717		0.76184×10^{-1}	
bilinear	1/8	529	0.0.43836	1.0007	0.19185×10^{-1}	1.9895
bilinear	1/16	2209	0.21916	1.0001	0.48051×10^{-2}	1.9973
bilinear	1/32	9216	0.0.10958	1.0000	0.12018×10^{-3}	1.9994
biquadratic	1/4	529	0.70737×10^{-1}		0.22488×10^{-2}	
biquadratic	1/8	2209	0.17673×10^{-1}	1.9758	0.28399×10^{-3}	2.9853
biquadratic	1/16	9025	0.44175×10^{-2}	1.9940	0.35604×10^{-4}	2.9957
biquadratic	1/32	36,491	0.11043×10^{-2}	1.9986	0.44539×10^{-5}	2.9990