MATH 728D: Machine Learning Homework #2: Probability

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This homework is intended as exercises for you to familiarize yourself with the course material. It will not be collected or graded. If you have questions about the exercises, these can be answered through email or at office hours.

Warning: Your should expect that more exercises will be added to this assignment as we move through the slides on probability!

1 Properties of Probability Measures

(From "Probability Basics" slides):
Assume that P is a probability measure defined on a sample space Ω, that A and B are events (measurable subsets of Ω), and that A^C represents the event that A *does not* occur. Verify the following statements:

1. $P(A^C) = 1 - P(A)$;
2. $P(B \cap A^C) = P(B) - P(A \cap B)$;
3. $P(A \cup B) = P(A) + P(B) - P(A \cap B)$;
4. $P(\cup_{i \in I} A_i) \leq \sum_{i \in I} P(A_i)$;
5. $P(A) = \sum_{i \in I} P(A \cap C_i)$, if $\{C_i\}$ is a partition of Ω;
6. $A \subseteq B \Rightarrow P(A) \leq P(B)$;

2 Roulette

An American roulette wheel has 36 numbered slots, half red and half black, as well as two green slots labeled "0" and "00". A player can wager an amount $B on a single number D between 1 and 36. The wheel is spun and the ball randomly lands in a slot S. The payoff table is:

Slot S	Payoff
D	player wins 35 * $B
anything else	player loses $B

Assume that the roulette ball is equally likely to land in any of the 38 slots. The *expected value* of the game is found by computing the sum of the payoff of every outcome times its probability.

1. What is the probability that S is red?
2. What is the probability that S lands in a slot numbered between 1 and 10?
3. What is the probability that S lands on slot #7?
4. If the player has bet B=$100, what is the expected value of the game?

3 Chuck-a-Luck

In the game of chuck-a-luck, a player chooses a particular "lucky number" D between 1 and 6. Then three fair dice are rolled, and the player wins or loses depending on how many dice show the value D. The payoff table is:

D's	Payoff
0	player loses $B
1	player wins $B
2	player wins $2*B
3	player wins $10*B

1. What is the probability of exactly 0, 1, 2 or 3 D's showing up? (Note that 1 or 2 D's can show up in 3 different ways!)
2. If the player bets B=$100, what is the expected value of the game?;
3. Suppose we can increase the payoff for the triple D result from $10*B to $N*B. What is the smallest (integer) value of N so that the expected value of the game is in the player's favor (making the expected value positive)?;

4 The Game of Life

In the game of life, you're born, you live a while, and then you die. Here are some simplified statistics for a population of 100,000 people, recording how many of those people survive to age 10, 20, ..., up to age 120 (surprise, none!).

Age	Population
0	100,000
10	99,184
20	98,771
30	97,393
40	95,603
50	92,632
60	85,802
70	73,100
80	50,564
90	17,915
100	932
110	2
120	0

We can use the same idea of expected value to estimate the expected lifetime of a person who has just been born into this population.

1. What is the probability that you will live to *at least* age 50?
2. What is the probability that you will die sometime between age 50 and age 60?
3. What is the expected lifetime of a person in this population? We want to estimate the area under a graph whose x axis is age, and whose y axis is the probability of reaching that age. The y value is found simply by dividing the number of people by 100,000. Simpson's rule for estimating the area under the function $y(x)$ given n data values at points with equal spacing h is:

$$\text{Area} \approx (\frac{1}{2}y_1 + y_2 + ... + y_{n-1} + \frac{1}{2}y_n) * h$$

5 Covariance Definition

(From "Probability Basics" slides):
Let $X = (X_1, ..., X_n)^T, Y = (Y_1, ..., Y_n)^T$ be vectors of random variables with joint distribution P. The covariance of X and Y is the rank-one matrix

$$\text{cov}[X, Y] := \mathbb{E}[(X - \mathbb{E}[X])\,(Y - \mathbb{E}(Y))^T] = \mathbb{E}[X\,Y^T] - \mathbb{E}[X]\,\mathbb{E}[Y]^T$$

Verify the second equality.

6 Tail Bounds

(From "Probability Basics" slides):

1. Show that, for any nonnegative random variable X:

$$\mathbb{E}[X] = \int_0^\infty \text{Prob}(X \geq t)\,dt$$

   and re-derive Markov's inequality:

2. Let $\phi(t)$ be any strictly monotonically increasing nonnegative function. Show that, for any random variable X and any $t \in \mathbb{R}$:

$$\text{Prob}(X \geq t) \leq \frac{\mathbb{E}[\phi(X)]}{\phi(t)}$$

3. From the previous result,

$$\text{Prob}(X \geq t) \leq \frac{\mathbb{E}[\phi(X)]}{\phi(t)}$$

   re-derive Chebychev's inequality that, for an arbitrary random variable X and $t > 0$, one has

$$\text{Prob}(|X - \mathbb{E}[X]| \geq t) \leq \frac{\text{var}[X]}{t^2}$$

7 Mean and Variance By Sampling

(From "Probability Basics" slides):
Suppose we make N draws from a Gaussian distribution whose true mean is μ and true variance is σ^2.

The maximum likelihood estimates for mean and variance, μ_{ML} and σ^2_{ML}, depend on the random draws X, and are therefore random variables. We can compute the expectation of these quantities. Show that

1. $$\mathbb{E}[\mu_{ML}] = \mu$$

2. $$\mathbb{E}[\sigma^2_{ML}] = \frac{(N-1)}{N}\sigma^2$$