# Introduction to the finite element method

M. M. Sussman

sussmanm@math.pitt.edu

Office Hours: 11:10AM-12:10PM, Thack 622

May 12 – June 19, 2014

# Resources

- Strang, G., Fix, G., *An Analysis of the Finite Element Method 2nd Edition*, Wellesley-Cambridge Press, 2008
- FEniCS Book: Volume 84 of Springer Lecture Notes in Computational Science and Engineering series:
  Anders Logg, Kent-Andre Merdal, Garth Wells, "Automated Solution of Differential Equations by the Finite Element Method"
  ISBN: 978-3-642-23098-1 (Print) 978-3-642-23099-8 (Online)
  **`http://launchpad.net/fenics-book/trunk/final/+download/fenics-book-2011-10-27-final.pdf`**
- FreeFem++ Book:
  **`http://www.freefem.org/ff++/ftp/freefem++doc.pdf`**
- Reference: Hecht, F. New development in freefem++. J. Numer. Math. 20 (2012), no. 3-4, 251-265. 65Y15
- Zienkiewicz, O. C., *The Finite Element Method in Engineering Science*, McGraw-Hill, 1971.

# Topics

# History

- Roots of method found in math literature: Rayleigh-Ritz
- Popularized in the 1950s and 1960s by engineers based on engineering insight with an eye toward computer implementation
- Winning idea: based on low-order piecewise polynomials with increased accuracy coming from smaller pieces, not increasing order.
- First use of the term “Finite element” in Clough, R. W., “The finite element in plane stress analysis,” *Proc. 2dn A.S.C.E. Conf. on Elecronic Computation,* Pittsburgh, PA, Sept. 1960.
- Strang and Fix, first 50 pages: introduction

# Topics

# Generalized derivatives

## Definition

The space of $C^\infty(\Omega)$ functions whose support is a compact subset of $\Omega$ is denoted $C_0^\infty(\Omega)$

- If a function $f$ is differentiable, then $\int_\Omega \frac{df}{dx}\phi dx = -\int_\Omega f\frac{d\phi}{dx}$ for all $\phi \in C_0^\infty$.

## Definition

If $f$ is a measurable function and if there is a measurable function $g$ satisfying $\int_\Omega g\phi dx = -\int_\Omega f\frac{d\phi}{dx}$ for all $\phi \in C_0^\infty$, then $g$ is said to be the "generalized derivative" of $f$.

# $L^2$

Suppose $\Omega$ is a “domain” in $\mathbb{R}^n$

- $u \in L^2(\Omega) \Leftrightarrow \|u\|_{L^2}^2 = \int_\Omega |u|^2 < \infty$
- $L^2(\Omega)$ is a Hilbert space with the inner product $(u, v) = \int_\Omega uv$
- $L^2(\Omega)$ is the completion of $C(\Omega)$ under the inner product $\| \cdot \|$.
- $L^2$ contains functions that are measurable but continuous nowhere.

# $H^k$

- A seminorm is given by $|u|_k^2 = \sum \int |D^k u|^2$, where $D^k$ is any derivative of total order $k$
- $H^k$ is the completion of $C^k$ under the norm $\|u\|_k^2 = \sum_{i=0}^k |u|_k^2$
- $H^0 = L^2$
- In 1D, functions in $H^1$ are continuous, but derivatives are only measurable.
- In dimensions higher than 1, functions in $H^1$ may not be continuous
- Functions in $H^1(\Omega)$ have well-defined "trace" on $\partial\Omega$.

# Topics

# A 2-point BVP

$$
\begin{aligned}
-\frac{d}{dx}\left(p(x)\frac{du}{dx}\right) + q(x)u &= f(x) \\
u(0) &= 0 \qquad \text{(BVP)} \\
\frac{du}{dx}(\pi) &= 0
\end{aligned}
$$

- Shape of a rotating string
- Temperature distribution along a rod

# Theory

- Suppose $f \in L^2$ (finite energy)
- Define linear operator $L : H_B^2 \to L^2$
  - $L : u \mapsto f$ from (BVP)

# Theory

- Suppose $f \in L^2$ (finite energy)
- Define linear operator $L : H^2_B \to L^2$
  - $L : u \mapsto f$ from (BVP)
- $L$ is SPD
- $L$ is 1-1
- For each $f \in L^2$, (BVP) has a unique solution $u \in H^2_B$
- $\|u\|_2 \leq C\|f\|_0$

# A solution

Assume $p > 0$ and $q \geq 0$ are constants

- Orthonormal set of eigenvalues, eigenfunctions

$$\lambda_n = p(n - \frac{1}{2})^2 + q \qquad u_n(x) = \sqrt{\frac{\pi}{2}} \sin(n - \frac{1}{2})x$$

- Expand

$$f(x) = \sum_{n=1}^{\infty} a_n \sqrt{\frac{\pi}{2}} \sin(n - \frac{1}{2})x$$

- Converges in $L^2$ since $\|f\|_0^2 = \sum_0^\infty a_n < \infty$
- $f$ need not satisfy b.c. pointwise!
- Solution

$$u = \sum \frac{a_n}{\lambda_n} u_n = \sqrt{\frac{pi}{2}} \sum_0^\infty \frac{a_n \sin(n - \frac{1}{2})x}{p(n - \frac{1}{2})^2 + q}$$

# Variational form: minimization

- Solving $Lu = f$ is equivalent to minimizing $I(v) = (Lv, v) - 2(f, v)$
- $(f, v) = \int_0^\pi f(x)v(x)\,dx$
- $(Lv, v) = \int_0^\pi [-(pv')' + qv]v\,dx$

# Variational form: minimization

- Solving $Lu = f$ is equivalent to minimizing $I(v) = (Lv, v) - 2(f, v)$
- $(f, v) = \int_0^\pi f(x)v(x)\, dx$
- $(Lv, v) = \int_0^\pi [-(pv')' + qv]v\, dx$
- Integrating by parts: $(Lv, v) = \int_0^\pi [p(v')^2 + qv^2]\, dx - [pv'v]_0^\pi$
- $v$ satisfies b.c.

# Variational form: minimization

- Solving $Lu = f$ is equivalent to minimizing $I(v) = (Lv, v) - 2(f, v)$
- $(f, v) = \int_0^\pi f(x)v(x)\,dx$
- $(Lv, v) = \int_0^\pi [-(pv')' + qv]v\,dx$
- Integrating by parts: $(Lv, v) = \int_0^\pi [p(v')^2 + qv^2]\,dx - [pv'v]_0^\pi$
- $v$ satisfies b.c.
- $I(v) = \int_0^\pi [p(x)(v'(x))^2 + q(x)v(x)^2 - 2f(x)v(x)]\,dx$

# Variational form: stationary point

- Solving $Lu = f$ is equivalent to finding $u$ so that $(Lu, v) = (f, v)$ for all $v$.
- Integrating by parts: $(Lu, v) = \int_0^\pi [p\,u'v' + q\,uv]\,dx - [puv']_0^\pi$
- Assume $u$ and $v$ satisfy b.c.
- $a(u, v) = \int_0^\pi [p\,u'v' + q\,uv]\,dx = (f, v)$
- Euler equation from minimization of $I(u)$

# Enlarge the search space

- Enlarge space to any function that is *limit* of functions in $H_B^2$ in the sense that $I(v - v_k) \to 0$
- Only need $H^1$
- Only the essential b.c. ($v(0) = 0$) survives!
- Admissible space is $H_E^1$
- Solution function *will* satisfy both b.c.

# Why only essential b.c.?

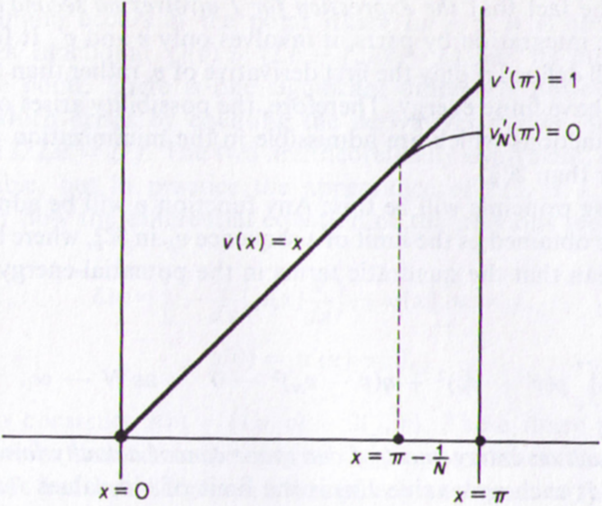



**Fig. 1.2** Convergence in $\mathcal{H}^1$, with $v_N'(\pi) = 0$ but $v'(\pi) \neq 0$.

From Strang and Fix.

# Relaxing space of $f$

- $f$ can now come from $H^{-1}$
- Functions whose derivatives are $L^2$ (also written $H^0$)
- $L : H^1_E \to H^{-1}$
- Dirac delta function!

# Topics

# Rayleigh-Ritz method

- Start from the (minimization) variational form
- Replace $H^1_E$ with sequence of finite-dimensional subspaces $S^h \subset H^1_E$
- Elements of $S^h$ are called “trial” functions
- Ritz approximation is minimizer $u^h$

$$I(u^h) \leq I(v^k) \quad \forall v^k \in S^h$$

# Example: Eigenvectors as trial functions

Assume $p, q > 0$ are constants

- $I(v) = \int_0^\pi [p\,(v')^2 + qv^2 - 2fv]\,dx$
- Choose eigenfunctions $j = 1, 2, \ldots, N = 1/h$
  $\phi_j(x) = \sqrt{\frac{\pi}{2}}\sin(j - \frac{1}{2})x$ with eigenvalues $\lambda_j = p(j - \frac{1}{2})^2 + q$
- Express $v^k = \sum_1^N v_j^h \phi_j(x)$

# Example: Eigenvectors as trial functions

Assume $p, q > 0$ are constants

- $I(v) = \int_0^\pi [p\,(v')^2 + qv^2 - 2fv]\,dx$
- Choose eigenfunctions $j = 1, 2, \ldots, N = 1/h$
  $\phi_j(x) = \sqrt{\frac{\pi}{2}}\sin(j - \frac{1}{2})x$ with eigenvalues $\lambda_j = p(j - \frac{1}{2})^2 + q$
- Express $v^k = \sum_1^N v_j^h \phi_j(x)$
- Plug in $I(v^k) = \sum_1^N [(v_j^k)^2 \lambda_j - 2\int_0^\pi f v_j^k \phi_j\,dx]$

# Example: Eigenvectors as trial functions

Assume $p, q > 0$ are constants

- $I(v) = \int_0^\pi [p\,(v')^2 + qv^2 - 2fv]\,dx$
- Choose eigenfunctions $j = 1, 2, \ldots, N = 1/h$
  $\phi_j(x) = \sqrt{\frac{\pi}{2}} \sin(j - \frac{1}{2})x$ with eigenvalues $\lambda_j = p(j - \frac{1}{2})^2 + q$
- Express $v^k = \sum_1^N v_j^h \phi_j(x)$
- Plug in $I(v^k) = \sum_1^N [(v_j^k)^2 \lambda_j - 2 \int_0^\pi f v_j^k \phi_j\,dx]$
- Minimize: $v_j^k = \int_0^\pi f \phi_j\,dx / \lambda_j$ for $j = 1, 2, \ldots, N$

# Example: Eigenvectors as trial functions

Assume $p, q > 0$ are constants

- $I(v) = \int_0^\pi [p\,(v')^2 + qv^2 - 2fv]\,dx$
- Choose eigenfunctions $j = 1, 2, \ldots, N = 1/h$
  $\phi_j(x) = \sqrt{\frac{\pi}{2}}\sin(j - \frac{1}{2})x$ with eigenvalues $\lambda_j = p(j - \frac{1}{2})^2 + q$
- Express $v^k = \sum_1^N v_j^h \phi_j(x)$
- Plug in $I(v^k) = \sum_1^N [(v_j^k)^2 \lambda_j - 2\int_0^\pi f v_j^k \phi_j\,dx]$
- Minimize: $v_j^k = \int_0^\pi f\phi_j\,dx/\lambda_j$ for $j = 1, 2, \ldots, N$
- Thus, $u^h = \sum_1^N (f, \phi_j)\phi_j/\lambda_j$

# Example: Eigenvectors as trial functions

Assume $p, q > 0$ are constants

- $I(v) = \int_0^\pi [p\,(v')^2 + qv^2 - 2fv]\,dx$
- Choose eigenfunctions $j = 1, 2, \ldots, N = 1/h$
  $\phi_j(x) = \sqrt{\frac{\pi}{2}}\sin(j - \frac{1}{2})x$ with eigenvalues $\lambda_j = p(j - \frac{1}{2})^2 + q$
- Express $v^k = \sum_1^N v_j^h \phi_j(x)$
- Plug in $I(v^k) = \sum_1^N [(v_j^k)^2 \lambda_j - 2\int_0^\pi f v_j^k \phi_j\,dx]$
- Minimize: $v_j^k = \int_0^\pi f\phi_j\,dx/\lambda_j$ for $j = 1, 2, \ldots, N$
- Thus, $u^h = \sum_1^N (f, \phi_j)\phi_j/\lambda_j$
- These are projections of true solution $u = \sum_1^\infty (f, \phi_j)\phi_j/\lambda_j$

# Example: Eigenvectors as trial functions

Assume $p, q > 0$ are constants

- $I(v) = \int_0^\pi [p\,(v')^2 + qv^2 - 2fv]\,dx$
- Choose eigenfunctions $j = 1, 2, \ldots, N = 1/h$
  $\phi_j(x) = \sqrt{\frac{\pi}{2}}\sin(j - \frac{1}{2})x$ with eigenvalues $\lambda_j = p(j - \frac{1}{2})^2 + q$
- Express $v^k = \sum_1^N v_j^h \phi_j(x)$
- Plug in $I(v^k) = \sum_1^N [(v_j^k)^2 \lambda_j - 2\int_0^\pi f v_j^k \phi_j\,dx]$
- Minimize: $v_j^k = \int_0^\pi f\phi_j\,dx/\lambda_j$ for $j = 1, 2, \ldots, N$
- Thus, $u^h = \sum_1^N (f, \phi_j)\phi_j/\lambda_j$
- These are projections of true solution $u = \sum_1^\infty (f, \phi_j)\phi_j/\lambda_j$
- Converges as $f_j/\lambda_j \approx f_j/j^2$.

# Example: Polynomials as trial functions

Assume $p, q > 0$ are constants

- Choose $v^k(x) = \sum_{j=1}^{N} v_j^k x^j$
- $v^k(0) = 0$
- $I(v^k) = \int_0^\pi [p(\sum v_j^k j x^{j-1})^2 + q(\sum v_j^k x^j)^2 - 2f \sum v_j^k x^j]\, dx$
- Differentiating I w.r.t. $v_j^k$ gives $N \times N$ system

$$KV = F$$

  where $K_{ij} = \frac{pij\pi^{i+j-1}}{i+j-1} + \frac{q\pi^{i+j+1}}{i+j+1}$ and $F_j = \int_0^\pi fx^j\, dx$
- $K$ is like the Hilbert matrix, very bad for n>12
- Can be partially fixed using orthogonal polynomials

# Topics

# FEM

- Strang and Fix discuss FEM in terms of minimization form
- Customary today to use stationary form
- $a(u, v) = \int_0^\pi [p\, u' v' + q\, uv]\, dx$
- $(f, v) = \int_0^\pi fv\, dx$
- Find $u \in H_E^1(0, \pi)$ so that $a(u, v) = (f, v)$ for all $v \in H^1(0, \pi)$.
- Choose a finite-dimensional subspace $S^h \subset H_E^1(0, \pi)$
- Find $u^h \in S^h$ so that $a(u^h, v^h) = (f, v^h)$ for all $v^h \in S^h$.
- Functions $v^h$ are called “test” functions.

# Stiffness matrix

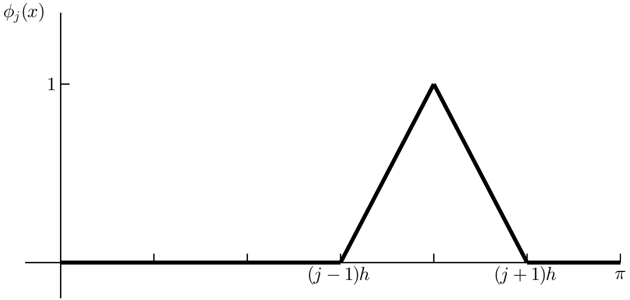
- Let $\{\phi_j\}_{j=1}^N$ be a basis of $S^h$
- $u^h(x) = \sum_{j=1}^n u_j^h \phi_j(x)$
- For each $\phi_i$,

$$
\begin{aligned}
a(u^h, \phi_i) &= \int_0^\pi \sum_j [p\, u_j^h \phi_j' \phi_i' + q\, u_j^h \phi_j \phi_i]\, dx \\
&= \sum_j \left( \int_0^\pi p \phi_i' \phi_j' + q \phi_i \phi_j \right) u_j^h\, dx \\
&= K_{ij} u_j^h = K U^h
\end{aligned}
$$

- $K$ is the “stiffness matrix”

# FEM: Piecewise linear functions

- Divide the interval $[0, \pi]$ into $N$ subintervals, each of length $h = \pi/N$ using $N + 1$ points $x_j = (j - 1)h$ for $j = 1, \ldots, N + 1$
- Construct $N$ "hat" functions $\phi_j$
  - $\phi_j(x_i) = \delta_{ij}$
  - Piecewise linear
  - $\phi_j(0) = 0$
- At most 2 $\phi_j$ are nonzero on any element.



# Assembling the system

Take $p = q = 1$

- Elementwise computation, for $e_\ell = [x_{\ell-1}, x_\ell]$

$$K^h_{ij} = \int_0^\pi \phi_i' \phi_j' + \phi_i \phi_j \, dx$$
$$= \sum_\ell \int_{e_\ell} \phi_i' \phi_j' + \phi_i \phi_j \, dx$$
$$(f, \phi_i) = b_i = \sum_\ell \int_{e_\ell} \phi_i f(x) \, dx$$

- System becomes $K^h U^h = b^h$

# First part: $\kappa_1 = \int_{e_\ell} \phi_i' \phi_j'$

- For each element, there is a Left endpoint and a Right endpoint

$$\int_e \phi_L' \phi_L' = \frac{1}{h} \qquad \int_e \phi_L' \phi_R' = -\frac{1}{h} \qquad \int_e \phi_R' \phi_L' = -\frac{1}{h} \qquad \int_e \phi_R' \phi_R' = \frac{1}{h}$$

- For the first element, there is only a right endpoint

$$(h)(\kappa_1) = \begin{bmatrix} 1 & & & \\ & & & \\ & & & \\ & & & \end{bmatrix} + \begin{bmatrix} 1 & -1 & & \\ -1 & 1 & & \\ & & & \\ & & & \end{bmatrix} + \cdots + \begin{bmatrix} & & & \\ & & & \\ & & 1 & -1 \\ & & -1 & 1 \end{bmatrix}$$

$$(\kappa_1) = \frac{1}{h} \begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -1 & 1 \end{bmatrix}$$

# More terms

- The second stiffness term is similar

$$(\kappa_2) = \frac{h}{6}\begin{bmatrix} 4 & 1 & & & \\ 1 & 4 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & 4 & 1 \\ & & & 1 & 2 \end{bmatrix}$$

- If $f$ is given by its nodal values $f_i = f(x_i)$ then $b = \kappa_2 f_i$.

# Integration

In more complicated situations, it is better to compute the integrals using Gauß integration. This involves a weighted sum over a few points inside the element.

# Topics

# Error and convergence

## Theorem

*Suppose that u minimizes $I(v)$ over the full admissible space $H^1_E$, and $S^h$ is any closed subspace of $H^1_E$. Then:*

1. *The minimun of $I(v^h)$ and the minimum of $a(u - v^h, u - v^h)$, as $v^h$ ranges over the subspace $S^h$, are achieved by the same function $u^h$. Therefore*

$$a(u - u^h, u - u^h) = min_{v^h \in S^h} a(u - v^h, u - v^h)$$

2. *With respect to the energy inner product, $u^h$ is the projection of u onto $S^h$. Equivalently, the error $u - u^h$ is orthogonal to $S^h$:*

$$a(u - u^h, v^h) = 0 \qquad \forall v^h \in S^h$$

3. *The minimizing function satisfies*

$$a(u^h, v^h) = (f, v^h) \qquad \forall v^h \in S^h$$

4. *In particular, if $S^h$ is the whole space $H^1_E$, then*

$$a(u, v) = (f, v) \qquad \forall v \in H^1_E$$

# First error estimate (Cea)

- $a(u - u^h, v) = (f, v) - (f, v) = 0$
- $a(u - u^h, u - u^h) = a(u - u^h, u - v) + a(u - u^h, v - u^h)$
- Since $u - u^h$ is not in $S^h$,

$$a(u - u^h, u - u^h) = a(u - u^h, u - v)$$

- Since $p(x) \geq p_0 > 0$,

$$p_0 \|u - u^h\|_1^2 \leq (\|p\|_\infty + \|q\|_\infty) \|u - u^h\|_1 \|u - v\|_1$$

- So that

$$\|u - u^h\|_1 \leq \frac{\|p\|_\infty + \|q\|_\infty}{p_0} \|u - v\|_1 \qquad \text{(Cea)}$$

- Choose $v$ to be the linear interpolant of $u$, so, from Taylor's theorem (integral form)

$$\|u - u^h\|_1 \leq Ch \|u\|_2$$

# Second error estimate (Nitsche)

- Let $w$ and $w^h$ be the true and approximate solutions of $a(w, v) = (u - u^h, v)$
- Clearly, $\|u - u^h\|^2 = a(w, u - u^h)$
- And $a(u - u^h, w^h) = 0$
- So $\|u - u^h\|^2 = a(u - u^h, w - w^h)$
- By Cauchy-Schwarz, $\|u - u^h\|^2 \leq \sqrt{a(u - u^h, u - u^h)}\sqrt{a(w - w^h, w - w^h)}$
- Applying first estimate, $\|u - u^h\|^2 \leq C^2 h^2 \|u\|_2 \|w\|_2$
- From definition of $w$, $\|w\|_2 \leq C\|u - u^h\|$
- Finally, $\|u - u^h\| \leq Ch^2 \|u\|_2$

# Error estimate steps

1. Cea tells us the error $\|u - u^h\|_1$ is smaller than the best approximation error
2. Choice of element tells us approximation error is $O(h)$ in $\|\cdot\|_1$
3. Nitsche tells us to error is $O(h^2)$ in $\|\cdot\|$

*These results are generally true!*