Implementation of *A Posteriori* Methods for Enforcing Conservation of Potential Enstrophy and Mass in Discretized Shallow-Water Equations Models

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ABSTRACT

A Sasaki variational approach is for the first time applied to enforce *a posteriori* conservation of potential enstrophy and total mass in long-term integrations of two ADI finite-difference approximations of the nonlinear shallow-water equations on a limited-area domain. The performance and accuracy of the variational approach is compared with that of a modified Bayliss-Isaacson *a posteriori* technique, also designed to enforce conservation of potential enstrophy and total mass, and with that of a periodic application of a two-dimensional high-order Shapiro filter.

While both the variational and the Bayliss-Isaacson *a posteriori* techniques yielded very satisfactory results after 20 days of numerical integration with regard to conservation of the integral constraints of the shallow-water equations and the accuracy of the solution, the high-order filtering approach performed in a less satisfactory way. This is attributed to the effects of the boundary conditions in the limited-area shallow-water equations models.

The Bayliss-Isaacson technique was found to be more robust and less demanding of CPU time, while the modified Sasaki variational technique is highly dependent on the updating procedure adopted for the Lagrange multiplier. The filtering technique is the most economical in terms of CPU time, but it is inadequate for limited-area domains with non-periodic boundary conditions and coarse meshes. In conclusion further research in this direction is suggested as these techniques provide viable alternatives to the rather complex conserving schemes proposed by other investigators.

1. Introduction

It is now well known (Burridge, 1980) that in discrete models of the shallow-water equations the most intensive numerical instabilities occur because the quadratic nonlinearity of the horizontal advection process generates aliasing errors. It has become evident through the work of Fjortoft (1953), Arakawa (1966), Lilly (1965) and Sadourny (1973, 1975) that the maintenance in the discrete representation of the integral constraints satisfied by invariants associated with the continuous equations can help to inhibit or prevent nonlinear instability.

Arakawa (1966), Sadourny (1975) and Kalnay-Rivas (1976) have all shown that the conservation of potential enstrophy in nondivergent barotropic flows (such as flows represented by the shallow-water equations) not only prevents nonlinear instability, but is essential if the dynamics and the energy exchanges between the different scales of motion are to be represented correctly.

If potential enstrophy is not conserved in a shallow-water equations model a spurious computational cascade of energy to smaller scales brings about a catastrophy (blow-up) after a finite time (Brissaud *et al.*, 1973; Sadourny, 1973, 1975; Fairweather and Navon, 1980).

Sadourny (1975) and Fairweather and Navon (1980) also have shown that formal conservation of potential enstrophy is more important than formal conservation of total energy, in that the models that conserve potential enstrophy are inherently more stable and maintain more realistic energy spectra. Moreover, it turns out that a shallow-water equations model conserving mass and potential enstrophy also conserves total energy very accurately if total energy is not conserved formally in its discretized form (Sadourny, 1973; Burridge, 1980).

Considerable effort has been devoted to devising and designing spatial finite-difference approximations of the shallow-water equations that retain the integral constraints of the continuous system. Following Arakawa (1966), a generalized Arakawa scheme for the shallow-water equations conserving energy and enstrophy was given by Grammeltvedt (1969), who also presented a detailed discussion of the conservation properties of several finite-difference schemes. Sadourny (1973, 1975) presented a potential-enstrophy conserving model for the shallow-water equations in a global cylindrical coordi-

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nate system. In a recent review on the numerical modeling of conservation laws Sadourny (1980) proposed a class of vorticity terms permitting numerical conservation of energy and potential enstrophy in isentropic coordinates. An optimum scheme combining both energy and potential enstrophy numerical conservations for triangular (Arakawa C-grid) meshes was reported by Kim (1978). Arakawa and Lamb (1979) have devised a second-order space difference scheme for the shallow-water equations that conserves both potential enstrophy and total energy in the presence of bottom topography.

All these numerical schemes modeling the conservation laws result, however, in rather complicated finite-difference forms that are difficult to generalize to fluid dynamics problem of interest. A different approach is to enforce the required conservation relationships explicitly by modifying the forecast fields values at each time step of the numerical integration.

Sasaki (1975, 1976, 1977) proposed such a variational approach and applied it to conserve total energy and total mass in one- and two-dimensional shallow-water equations models on a rotating plane, but he used an additional smoothing operation that in effect was responsible for stabilizing his scheme.

Bayliss and Isaacson (1975), Isaacson and Turkel (1976) and Isaacson (1977) presented a simple method of making any finite-difference scheme conservative with respect to any quantity. In their approach the conservative constraints were linearized about the predicted values by means of a gradient method for modifying the predicted values at each time-step of the numerical integration. Isaacson *et al.* (1979a) and Isaacson *et al.* (1979b) have implemented the same technique in terms of simultaneous conservation constraints for the shallow-water equations over a sphere, taking into account orography effects. Their approach has been tested by Kalnay-Rivas *et al.* (1977) with enstrophy as the conserved quantity. Kalnay-Rivas *et al.* (1977, 1979) found that the use of an enstrophy conserving scheme can be successfully replaced by using a fourth-order quadratically conserving scheme on a global domain, combined with the periodic application of 16th order Shapiro filter removing wave shorter than four times the grid size before they attain finite-amplitude. Kalnay-Rivas *et al.* (1979) extended this result to formally nonconservative schemes, also on a global domain. A similar result was obtained by Navon and Riphagen (1979) for a compact fourth-order scheme in conservation-law form (Haltiner and Williams, 1980, p. 137).

In the present paper we have made an attempt to show that it is possible to achieve stable long-term integrations of the shallow-water equations using the abovementioned techniques. To do so, we first propose a modified Sasaki variational approach to enforce conservation of total mass and potential enstrophy in long-term integrations of two ADI finite-difference approximations for a two-dimensional nonlinear shallow-water equations model. We then compare the performance of the variational approach with that of a modified Bayliss-Isaacson technique also designed to enforce conservation of potential enstrophy and total mass, and with the application of a two-dimensional high-order Shapiro filter. The two ADI finite-difference algorithms used are the nonlinear ADI Gustafsson (1971) method and the linear algorithm due to Fairweather and Navon (1980). Both conserve total energy and mass but do not conserve potential enstrophy, and both have a critical blow-up time for long-term integrations in the absence of dissipation (Fairweather and Navon, 1980). As such they are suitable for testing the two methods for enforcing potential enstrophy conservation (see the Appendix).

In Section 2 we detail the variational design for the two-dimensional nonlinear ADI shallow-water equations models, while the numerical algorithm for iteratively solving the resulting coupled nonlinear Euler-Lagrange equations is presented in Section 3, which also includes the discretized variational formulation for enforcing conservation of potential enstrophy and total mass.

In Section 4 the modified Bayliss-Isaacson conservative algorithm is described and its numerical implementation detailed, while in Section 5 we apply the two-dimensional high-order Shapiro filter.

In Section 6 we present the numerical results for long-term runs for both ADI shallow-water equations models, based on a commonly used test problem. Accuracy tests for the two *a posteriori* techniques and the two-dimensional high-order Shapiro filter are provided, also.

A quantitative assessment of the results of the proposed *a posteriori* numerical methods for enforcing potential enstrophy and total mass conservation on the discretized shallow-water equations is presented in Section 7.

In Appendix A the ADI shallow-water equations numerical models are briefly presented (see also Mitchell and Griffiths, 1980, pp. 253–254).

2. Formulation of the modified Sasaki variational method for enforcing conservation of potential enstrophy and mass

The potential enstrophy conservation law can be written as

$$Z = \frac{1}{2}\int_0^L \int_0^D \frac{Q^2}{h}\,dxdy$$
$$= \frac{1}{2}\int_0^L \int_0^D \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} + f\right)^2 h^{-1}dxdy. \quad (1)$$

The discrete finite-difference analog may be written as

$$Z_{dis} = \sum_{j=1}^{N_x} \sum_{k=1}^{N_y} \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} + f \right)_{jk}^{2} h_{jk}^{-1} \Delta x \Delta y = Z^0, \quad (2)$$

where Z^0 is the value of Z_{dis} at time $t = 0$, i.e., the initial potential enstrophy.

In the variational design of the numerical method we have adopted three basic principles due to Sasaki (1976) and at this point shall mention the first two:

1) Conservation laws valid for the true solution of the differential equations should also hold for the finite-difference solution.

2) The solution (u, v, h) at a given time step is also a stationary value that minimizes a weighted sum of the variances of $(u - \tilde{u})$, $(v - \tilde{v})$ and $(h - \tilde{h})$ integrated over the entire domain.

Here $(\tilde{u}, \tilde{v}, \tilde{h})$ are the predicted variables at the nth time level as determined by use of one of the two finite-difference ADI algorithms for solving the shallow-water equations, while (u, v, h) are the values adjusted by the variational method to enforce and satisfy the required conservation laws.

Based on these two hypotheses a variational formulation of the problem of enforcing potential enstrophy conservation yields the functional

$$J = \sum_{j=1}^{N_x} \sum_{k=1}^{N_y} [\tilde{\alpha}(u - \tilde{u})^2 + \tilde{\alpha}(v - \tilde{v})^2 + \tilde{\beta}(h - \tilde{h})^2]_{jk} + \lambda_E \left\{ \sum_{j=1}^{N_x} \sum_{k=1}^{N_y} \left[\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} + f \right)_{jk}^{2} h_{jk}^{-1} \right] \times \Delta x \Delta y - Z^0 \right\}, \quad (3)$$

where for simplicity of notation we have omitted the subscripts from u_{jk}, v_{jk}, h_{jk}. This functional will have a stationary value if its first variation equals zero, i.e.,

$$\delta J = 0, \quad (4)$$

where δ is the variational operator.

In (4) the summation is taken over all (j, k) mesh points. $\tilde{\alpha}$ and $\tilde{\beta}$ are weights determined following a technique proposed by Sasaki (1976). The weight $\tilde{\alpha}$ was taken to be unity.

We then applied Sasaki's third principle which states: The relative weight is so chosen as to make the fractional adjustment of variables proportional to the fractional magnitude of the truncation errors in the predicted variables.

Following Sasaki (1976), $\tilde{\beta}$ was chosen so that

$$\tilde{\beta} = g/H, \quad (5)$$

where H is the mean depth of the fluid (in our case $H = 2000$ m).

We now take λ_E as the Lagrange multiplier, assumed constant with respect to space but, in general, varying in time. Taking the first variation of (3) with respect to u, v, h and λ_E we obtain

$$\delta J = \sum_{j=1}^{N_x} \sum_{k=1}^{N_y} [2\tilde{\alpha}(u - \tilde{u})\delta u + 2\tilde{\alpha}(v - \tilde{v})\delta v + 2\tilde{\beta}(h - \tilde{h})\delta h] + \delta\lambda_E \left\{ \sum_{j=1}^{N_x} \sum_{k=1}^{N_y} \left[\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} + f \right)_{jk}^{2} h_{jk}^{-1} \Delta x \Delta y \right] - Z_0 \right\} + \lambda_E \sum_{j=1}^{N_x} \sum_{k=1}^{N_y} [(h\delta K - K\delta h)h^{-2}]_{jk} \Delta x \Delta y, \quad (6)$$

where we have denoted by K

$$K = \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} + f \right)^2. \quad (7)$$

The last term in (6) can be written in a simpler and more useful form using algebraic commutation formulas between finite-difference and variational operators as derived explicitly by Sasaki (1969, 1970a, 1970b). This procedure shifts the derivative operators from the variational quantities, i.e.,

$$\sum_i \psi \nabla_x \delta T_0 = - \sum_i \nabla_x \psi \delta T_0. \quad (8)$$

Eq. (8) is valid for the periodic variable x but not for an arbitrary prescribed boundary value y. However, in our case (see the Appendix) we have the y-direction boundary condition

$$v(x, 0, t) = v(x, D, t) = 0$$

(rigid-wall boundary condition) and for the u and ϕ variables it is possible to choose a finite-difference form of the boundary condition that satisfies Eq. (8). Then the last term in (6) can be written

$$\tfrac{1}{2}\lambda_E \sum_{j=1}^{N_x} \sum_{k=1}^{N_y} (h\delta K - K\delta h)_{jk} h^{-2} \Delta x \Delta y = \lambda_E \sum_{j=1}^{N_x} \sum_{k=1}^{N_y} [2h(\nabla_x v - \nabla_y u + f)h^{-2}(\nabla_x \delta v - \nabla_y \delta u) - (\nabla_x v - \nabla_y u + f)^2 h^{-2} \delta h]_{jk} \Delta x \Delta y, \quad (9)$$

and if the commutation formula in (8) is used, Eq. (9) takes the form

$$\tfrac{1}{2}\lambda_E \sum_{j=1}^{N_x} \sum_{k=1}^{N_y} [2hh^{-2}(-\nabla_x \nabla_x v \cdot \delta v + \nabla_x \nabla_y v \cdot \delta u + \nabla_y \nabla_x u \cdot \delta v - \nabla_y \nabla_y u \cdot \delta u) - (\nabla_x v - \nabla_y u + f)^2 h^{-2} \delta h]_{jk} \Delta x \Delta y, \quad (10)$$

where the discretized finite-difference operators ∇_x and ∇_y are defined as

$$\left.\begin{aligned}\nabla_x f = D_{0x} f = (f_{j+1,k} - f_{j-1,k})/2\Delta x \\ \nabla_y f = D_{0y} f = (f_{j,k+1} - f_{j,k-1})/2\Delta y\end{aligned}\right\} . \quad (11)$$

Substituting expression (10) into (6) rearranging terms yield

$$\begin{aligned}\delta J = \sum_{j=1}^{N_x} \sum_{k=1}^{N_y} \{[2\tilde{\alpha}(u - \tilde{u}) \\ + 2\lambda_E \Delta s^2 h^{-1}(\nabla_x \nabla_y v - \nabla_y \nabla_y u)]\delta u \\ + [2\tilde{\alpha}(v - \tilde{v}) + 2\lambda_E \Delta s^2 h^{-1}(\nabla_y \nabla_x u - \nabla_x \nabla_x v)]\delta v \\ + [2\tilde{\beta}(h - \tilde{h}) - \lambda_E \Delta s^2 h^{-2}(\nabla_x v - \nabla_y u + f)]^2 \delta h\}_{jk} \\ + \delta\lambda_E \{\sum_{j=1}^{N_x} \sum_{k=1}^{N_y} [(\nabla_x v - \nabla_y u + f)^2 \\ \times \Delta s^2 h^{-1}]_{jk} - Z_0\},\end{aligned} \quad (12)$$

where

$$\left.\begin{aligned}\Delta x = \Delta y \\ (\Delta s)^2 = \Delta x \Delta y\end{aligned}\right\} . \quad (13)$$

Since the variations of δu, δv, δh and $\delta\lambda_E$ are nonzero arbitrary values, the coefficients of each variation should vanish individually to satisfy the stationary conditions. We then obtain a set of nonlinear coupled partial differential equations which are the Euler-Lagrange equations for u, v, h and λ_E for each grid point (j, k):

$$2\tilde{\alpha}(u - \tilde{u}) + (\lambda_E/h)(\Delta s)^2(\nabla_x \nabla_y v - \nabla_y^2 u) = 0, \quad (14a)$$

$$2\tilde{\alpha}(v - \tilde{v}) + (\lambda_E/h)(\Delta s)^2(\nabla_y \nabla_x u - \nabla_x^2 u) = 0, \quad (14b)$$

$$2\beta(h - \tilde{h}) - (\lambda_E/2h^2)(\Delta s)^2(\nabla_x v - \nabla_y u + f)^2 = 0, \quad (14c)$$

$$\tfrac{1}{2} \sum_{j=1}^{N_x} \sum_{k=1}^{N_y} [(\nabla_x v - \nabla_y u + f)^2 h^{-1}(\Delta s)^2]_{jk} - Z_0 = 0. \quad (14d)$$

3. The numerical variational algorithm

a. Iterative solution of the discretized Euler-Lagrange equations

As we have a system of coupled nonlinear partial differential equations, the numerical solutions u, v, h and λ_E may be obtained by using an iterative technique.

We can write the discretized version of the Euler-Lagrange equations (14a)–(14d) as

$$\begin{aligned}2\tilde{\alpha} u_{jk}^{\nu+1} - \frac{2\lambda_E}{4h_{jk}^{(\nu)}}[u_{j,k+2}^{(\nu+1)} - 2u_{jk}^{(\nu+1)} + u_{j,k-2}^{(\nu+1)}] \\ = 2\tilde{\alpha}\tilde{u}_{jk} - \frac{2\lambda_E}{h_{jk}^{(\nu)}}[v_{j+1,k+1}^{(\nu)} - v_{j+1,k-1}^{(\nu)} \\ - v_{j-1,k+1}^{(\nu)} + v_{j-1,k-1}^{(\nu)}],\end{aligned} \quad (15a)$$

$$\begin{aligned}2\tilde{\alpha} v_{jk}^{(\nu+1)} - \frac{2\lambda_E}{h_{jk}^{(\nu)}}[v_{j+2,k}^{(\nu+1)} - 2v_{jk}^{(\nu+1)} + v_{j-2,k}^{(\nu+1)}] \\ = 2\tilde{\alpha}\tilde{v}_{jk} - \frac{2\lambda_E}{h_{jk}^{(\nu)}}[u_{j+1,k+1}^{(\nu+1)} - u_{j+1,k-1}^{(\nu+1)} \\ - u_{j-1,k+1}^{(\nu+1)} + u_{j-1,k+1}^{(\nu+1)}],\end{aligned} \quad (15b)$$

$$h_{jk}^{\nu+1} = \tilde{h}_{jk} - \frac{\lambda_E \Delta s^2}{h_{ij}^{(\nu)2} 2\tilde{\beta}}\left[\frac{v_{j+1,k}^{(\nu+1)} - v_{j-1,k}^{(\nu+1)}}{2\Delta x} - \frac{u_{j,k+1}^{(\nu+1)} - u_{j,k-1}^{(\nu+1)}}{2\Delta y} + f_k\right]^2, \quad (15c)$$

$$\frac{1}{2}\sum_{j=1}^{N_x}\sum_{k=1}^{N_y}\left\{\frac{\dfrac{[v_{j+1,k}^{(\nu+1)} - v_{j-1,k}^{(\nu+1)}]}{2\Delta x} - \dfrac{[u_{j,k+1}^{(\nu+1)} - u_{j,k-1}^{(\nu+1)}]}{2\Delta y}}{h_{jk}^{(\nu+1)}} + f_k\right\}^2 \times \Delta x \Delta y - Z_0 = 0. \quad (15d)$$

If the difference between Z_0, the initial total potential enstrophy, and the total potential enstrophy computed from the forecasted variables (using one of the two ADI solution algorithms) was less than a given limit (10^{-5} times Z_0 was used), the iterative process was stopped. The initial guess at each time step is the value calculated from either of the two ADI forecasting programs, i.e.,

$$u_{jk}^{(0)} = \tilde{u}_{jk}, \quad v_{jk}^{(0)} = \tilde{v}_{jk}, \quad h_{jk}^{(0)} = \tilde{h}_{jk}. \quad (16)$$

For typical differences the error was reduced below the limit in three to five iterations.

b. Determination of the Lagrange multiplier λ_E'

By substituting $u^{(\nu+1)}$, $v^{(\nu+1)}$ and $h_{jk}^{(\nu+1)}$ from Eqs. (15a, b, c) into (15d), we obtain

$$F(\lambda_E) \equiv \frac{1}{2}\sum_{j=1}^{N_x}\sum_{k=1}^{N_y} \times \left\{\frac{\left[\dfrac{v_{j+1,k}^{(\nu+1)} - v_{j-1,k}^{(\nu+1)}}{2\Delta x} - \dfrac{u_{j,k+1}^{(\nu+1)} - u_{j,k-1}^{(\nu+1)}}{2\Delta y} + f_k\right]^2 \Delta x \Delta y}{h_{jk}^{(\nu+1)}}\right\} - Z_0 = 0. \quad (17)$$

This gives a highly nonlinear equation for λ_E', which can be solved iteratively by a Newton iteration

$$\lambda_E^{(k+1)} = \lambda_E^{(k)} - \frac{F(\lambda_E^{(k)})}{F'[\lambda_E^{(k)}]}, \quad (18)$$

where F is given by Eq. (17). This way of determining λ_E turned out to be detrimental to the iterative solution of the Euler-Lagrange equations, whose stability proved to be critically dependent on the magnitude of λ_E.

Much better results were obtained when we first determined the Lagrange multiplier for total energy conservation λ_S using the Sasaki (1976) formula

$$\lambda_S = 2\tilde{\alpha}[(1/X) - 1]/H, \tag{19}$$

where

$$X = \{E^0[\sum_{j=1}^{N_x}\sum_{k=1}^{N_y} (\tilde{u}^2 + \tilde{v}^2 + \tilde{\phi}^2/4)\tilde{\phi}^2 \times (4g\Delta x\Delta y)^{-1}]^{-1}\}^{1/2}, \tag{20}$$

where E^0 is the initial energy [using Eq. (9)].

This λ_S, retaining its sign, was then scaled for the potential enstrophy conservation Eq. (15d) and modified, using trial and error, to improve the convergence of the nonlinear iterative solution of the Euler-Lagrange equations [(15a)–(15d)]. We started with

$$\lambda_E = 10^{12} \tag{21}$$

and then at time step $n + 1$

$$\lambda_E^{(n+1)} = \lambda_E^{(n)} \cdot \frac{\lambda_S^{(n+1)}}{\lambda_S^{(n)}}. \tag{22}$$

c. The final variational algorithm

Another important integral of the shallow-water equations is the total mass as given by Eq. (10). In order to enforce the constraint of mass conservation we treated it separately from the potential enstrophy constraint. To achieve mass conservation, the heights forecast by the finite-difference ADI models were first adjusted by using the formula

$$\tilde{h}_{jk}^{n*} = \tilde{h}_{jk}^{n} - (\sum_{j=1}^{N_x}\sum_{k=1}^{N_y} \tilde{h}_{jk}^{n} - \sum_{j=1}^{N_x}\sum_{k=1}^{N_y} \tilde{h}_{jk}^{0})(N_xN_y)^{-1}$$

$$= \tilde{h}_{jk}^{n} - (\sum_{j=1}^{N_x}\sum_{k=1}^{N_y} \tilde{h}_{jk}^{n} - H_0)(N_xN_y)^{-1},$$

for each

$$\left.\begin{array}{l} j = 1, 2, \ldots, N_x \\ k = 1, 2, \ldots, N_y \end{array}\right\}, \tag{23}$$

where H_0 is the value of

$$\sum_{j=1}^{N_x}\sum_{k=1}^{N_y} \tilde{h}_{jk} \quad \text{at time} \quad t = 0.$$

The variational algorithm applied at a given time step may then be summarized as follows:

1) Obtain $(\tilde{u}, \tilde{v}, \tilde{h})$ using either of the finite-difference ADI models of Section 2.

2) Adjust the height field $\tilde{h}$ to ensure total mass conservation.

3) Solve iteratively the Euler-Lagrange equations to enforce potential enstrophy conservation up to a predetermined accuracy limit.

4) Recheck total mass conservation and if the variation exceeds $10^{-4}H_0$ repeat mass adjustment (2) and then repeat the iterative solution (3). Not more than three repetitions of this procedure were necessary.

To test the success of this technique, long-term integrations should be conducted beyond the critical time T_c (see Sadourny, 1973, 1975), Fairweather and Navon (1980), when a blow up occurs due to the spurious computational cascade of energy to smaller scales when potential enstrophy is not conserved.

d. A generalized variational functional

A generalized variational functional including the simultaneous constraints of total mass, total energy and potential enstrophy conservation takes the form

$$J_1 = \sum_{j=1}^{N_x}\sum_{k=1}^{N_y} [\tilde{\alpha}(u - \tilde{u})^2 + \tilde{\alpha}(v - \tilde{v})^2 + \tilde{\beta}(h - \tilde{h})^2]_{jk}$$
$$+ \lambda_S\{\tfrac{1}{2} \sum_{j=1}^{N_x}\sum_{k=1}^{N_y} [h(u^2 + v^2) + gh^2]_{jk}\Delta x\Delta y - E^0\}$$
$$+ \lambda_E\left\{\frac{1}{2}\sum_{j=1}^{N_x}\sum_{k=1}^{N_y}\left[\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} + f\right]_{jk}^2 \times h_{jk}^{-1}\Delta x\Delta y - Z_0\right\}$$
$$+ \lambda h\}\sum_{j=1}^{N_x}\sum_{k=1}^{N_y} [h_{jk}]\Delta x\Delta y - H^0\}. \tag{25}$$

After some algebra (necessitated by taking the first variation of J with respect to u, v, h, λ_S, λ_E, and λ_h and requiring the coefficients of each variation to vanish individually in order to satisfy the stationary conditions), we obtain the following Euler Lagrange equations for u, v, h, λ_E, λ_S and λ_h:

$$2\tilde{\alpha}(u - \tilde{u}) + \lambda_E\Delta s^2h^{-1}(\nabla_x\nabla_y v - \nabla_y{}^2u) + \lambda_S\lambda s^2hu = 0, \tag{26a}$$

$$2\tilde{\alpha}(v - \tilde{v}) + \lambda_E\Delta s^2h^{-1}(\nabla_y\nabla_x u - \nabla_x{}^2v) + \lambda_S\Delta s^2hv = 0, \tag{26b}$$

$$2\tilde{\beta}(h - \tilde{h}) - \tfrac{1}{2}\lambda_E\Delta s^2h^{-2}(\nabla_x v - \nabla_y u + f)^2 + \tfrac{1}{2}\lambda_S\Delta s^2(u^2 + v^2) + \lambda_S\Delta s^2gh + \lambda_h\Delta s^2 = 0, \tag{26c}$$

$$\sum_{j=1}^{N_x}\sum_{k=1}^{N_y} h_{jk}\Delta s^2 - H_0 = 0, \tag{26d}$$

$$\tfrac{1}{2}\sum_{j=1}^{N_x}\sum_{k=1}^{N_y} [h(u^2 + v^2) + gh^2]_{jk}\Delta s^2 - E_0 = 0, \tag{26e}$$

$$\tfrac{1}{2}\sum_{j=1}^{N_x}\sum_{k=1}^{N_y} [(\nabla_x v - \nabla_y u + f)\alpha h^{-1}(\Delta s)^2]_{jk} - Z_0 = 0. \tag{26f}$$

The initial Lagrange multipliers λ_E, λ_S and λ_h should be obtained from a linearized model using scaling considerations similar to those of Sasaki (1970, 1976). However, this idea of simultaneously satisfying all the integer constraints of the shallow-water equations by the variational *a posteriori* method has not yet been tested.

4. The Bayliss-Isaacson algorithm

a. The theoretical framework

Alvin Bayliss and Eugene Isaacson (1975) arrived independently at the idea of modifying the predicted values by using the constraints of vorticity, enstrophy and energy (Isaacson, 1977). Their approach linearizes the constraints about the predicted values to make any finite-difference scheme conservative with respect to any quantity. This method was tested by Kalnay-Rivas *et al.* (1977) for a fourth-order approximation of the enstrophy constraint.

In what follows we shall first describe the theoretical framework of the Bayliss-Isaacson method, then our modified version of it, and finally, the numerical implementation of the latter.

Assume we have a mixed initial-boundary value problem

$$\mathbf{u}_t = B(\mathbf{u}) \tag{27}$$

for the vector $\mathbf{u}$, and that the solution $\mathbf{u}$ satisfies certain integral invariants (conservation laws)

$$g_k(\mathbf{u}) = 0, \quad k = 1, 2, \ldots, K. \tag{28}$$

By discretizing the integral invariants and representing the integrals as sums, we obtain the approximating integral invariants

$$G_k[U_{ij}^n] = 0, \quad k = 1, 2, \ldots, K, \tag{29}$$

where U_{ij}^n is a net function defined at the grid points (x_i, y_j, t_n) and $U(x_i, y_j, t_n)$ approximates $\mathbf{u}(x_i, y_j, t_n)$. At time t_{n+1}, the difference operator solving (for instance) the shallow-water equations (i.e., solving for the vector $\mathbf{u}$) has the form

$$W(n+1) = C[W(n), W(n-1), \ldots, W(n-s)] \equiv CW(n), \tag{30}$$

where $W(n)$ is a net function at time t_n. We now wish to modify the scheme (30) in such a way that it will produce a net function $U(n+1)$ that will satisfy the discretized integral constraints (29), i.e., a corrective net function $V(n+1)$ is to be found such that

$$\left.\begin{aligned} U(n+1) &= CU(n) + V(n+1) \\ G_k[U(n+1)] &= 0, \quad k = 1, 2, \ldots, K \end{aligned}\right\}, \tag{31}$$

and such that some norm of the perturbation $V(n+1)$ should be minimized:

$$\min\|V(n+1)\|. \tag{32}$$

To determine $V(n+1)$ a simpler method than the Sasaki variational approach is to solve (31)–(32) by linearizing the discretized invariants $G_k(U(n+1))$ about the predicted value $CU(n)$. This can be written as

$$\begin{aligned} G_k[U(n+1)] &= G_k[CU(n) + V(n+1)] \\ &\approx G_k[CU(n)] + \operatorname{grad}G_k \cdot V(n+1) \\ &= G_k[CU(n)] + \left.\frac{\partial G_k}{\partial U(n+1)}\right|_{U(n+1)=CU(n)} \\ &\times V(n+1) = 0, \quad k = 1, \ldots, K. \end{aligned} \tag{33}$$

Now, $\|V(n+1)\|$ is minimized subject to the K linear constraints (33).

Since any vector $\mathbf{V}$ has a unique representation in the form

$$\mathbf{V}(n+1) = \sum_{k=1}^{K} \alpha_k \operatorname{grad}G_k + \mathbf{P}, \tag{34}$$

where $\mathbf{P}$ is a vector orthogonal to the K gradients, it follows that any solution of the K simultaneous linear equations is also of that form. If the Gramian matrix $(\operatorname{grad}G_k \cdot \operatorname{grad}G_r)$ is nonsingular, by substituting the expression (34) for $V(n+1)$ into (33), the K scalar coefficients α_k are determined by solving the K linear equations (33), i.e.,

$$G_k[CU(n)] + \operatorname{grad}G_k \cdot \left(\sum_{r=1}^{K} \alpha_r \operatorname{grad}G_r\right) = 0, \quad k = 1, \ldots, K, \tag{35}$$

we have used the orthogonality conditions

$$\operatorname{grad}G_k \cdot \mathbf{P} = 0. \tag{36}$$

Note that the arbitrary vector $\mathbf{P}$ must be zero, if $V(n+1)$ is to have minimum norm.

If we deal with a single integral constraint—say the enstrophy constraint—and assume the correction

$$\mathbf{V}(n+1) = (U', V') \tag{37}$$

is added to the predicted values of the velocity field $\bar{U}, \bar{V}$, then using (31) we get

$$(U', V') = \alpha\left(\frac{\partial G}{\partial U}, \frac{\partial G}{\partial V}\right)_{\bar{U},\bar{V}}, \tag{38}$$

$$\left(\frac{\partial G}{\partial U}\right)_{\bar{U},\bar{V}} \cdot U' + \left(\frac{\partial G}{\partial V}\right)_{\bar{U},\bar{V}} \cdot V' = G(U_0, V_0) - G(\bar{U}, \bar{V}) \tag{39}$$

and α can now be determined from (39) as

$$\alpha = \frac{G(U_0, V_0) - G(\bar{U}, \bar{V})}{\left|\dfrac{\partial G}{\partial U}\right|^2 + \left|\dfrac{\partial G}{\partial V}\right|^2}, \tag{40}$$

where $G(U, V)$ denotes a consistent approximation of the mean-square vorticity, and $G(U_0, V_0)$ is the initial (time $t = 0$) mean-square vorticity.

b. Discretized Bayliss-Isaacson algorithm for enforcing conservation of potential enstrophy

The approximating functional for potential enstrophy conservation is

$$G' = \sum_i \sum_j \left(\frac{\partial V}{\partial x} - \frac{\partial U}{\partial y} + f\right)^2_{ij} (h_{ij}{}^1)^{-1} \Delta x \Delta y. \quad (41)$$

At a given grid location we get

$$G'^n_{ij} = \sum_i \sum_j \left(\frac{v^n_{i+1,j} - v^n_{i-1,j}}{2\Delta x} - \frac{u^n_{i,j+1} - u^n_{i,j-1}}{2\Delta y} + f_j\right)^2 (h^n_{ij})^{-1} \Delta x \Delta y. \quad (42)$$

The derivatives $\partial G'/\partial U|_{ij}$, $\partial G'/\partial V|_{ij}$ and $\partial G'/\partial h|_{ij}$ are

$$\left.\frac{\partial G'}{\partial U}\right|_{ij} = \frac{1}{h_{i,j+1}} \overset{\text{(a)}}{\left[\frac{v^n_{i+1,j+1} - v^n_{i-1,j+1}}{2\Delta x} - \frac{u^n_{i,j+2} - u^n_{i,j}}{2\Delta y} + f_{j+1}\right]} \Delta x - \frac{1}{h_{i,j-1}} \overset{\text{(b)}}{\left[\frac{v^n_{i+1,j-1} - v^n_{i-1,j-1}}{2\Delta x} - \frac{u^n_{i,j} - u^n_{i,j-2}}{2\Delta y} + f_{j-1}\right]} \Delta x, \quad (43)$$

$$\left.\frac{\partial G'}{\partial V}\right|_{ij} = \frac{1}{h_{i-1,j}} \overset{\text{(c)}}{\left[\frac{v^n_{i,j} - v^n_{i-2,j}}{2\Delta x} - \frac{u^n_{i-1,j+1} - u^n_{i-1,j-1}}{2\Delta y} + f_j\right]} \Delta y - \frac{1}{h_{i+1,j}} \overset{\text{(d)}}{\left[\frac{v^n_{i+2,j} - v^n_{i,j}}{2\Delta x} - \frac{u^n_{i+1,j+1} - u^n_{i+1,j-1}}{2\Delta y} + f_j\right]} \Delta y, \quad (44)$$

$$\left.\frac{\partial G'}{\partial h}\right|_{ij} = -\frac{\left(\dfrac{v^n_{i+1,j} - v^n_{i-1,j}}{2\Delta x} - \dfrac{u^n_{i,j+1} - u^n_{i,j-1}}{2\Delta y} + f_j\right)^2 \Delta x \Delta y}{h^2_{ij}}. \quad (45)$$

α is determined from

$$\alpha = \frac{G'(U_0, V_0) - G'(\bar U, \bar V)}{\left|\dfrac{\partial G}{\partial U}\right|^2 + \left|\dfrac{\partial G}{\partial V}\right|^2 + \left|\dfrac{\partial G}{\partial h}\right|^2}, \quad (46)$$

which in our case takes the form

$$\alpha = \left\{\sum_i \sum_j \frac{\left(\dfrac{v^0_{i+1,j} - v^0_{i-1,j}}{2\Delta x} - \dfrac{u^0_{i,j+1} - u^0_{i,j-1}}{2\Delta y} + f_j\right)^2 \Delta x \Delta y}{h^0_{ij}} - \sum_i \sum_j \frac{\left(\dfrac{v^n_{i+1,j} - v^n_{i-1,j}}{2\Delta x} - \dfrac{u^n_{i,j+1} - u^n_{i,j-1}}{2\Delta y} + f_j\right)^2 \Delta x \Delta y}{h^n_{ij}}\right\}$$
$$\div \left\{\sum_i \sum_j \left[\frac{1}{h_{i,j+1}} \cdot \text{(a)} - \frac{1}{h_{i,j-1}} \cdot \text{(b)}\right]^2 \Delta x^2 + \sum_i \sum_j \left[\frac{1}{h_{i-1,j}} \cdot \text{(c)} - \frac{1}{h_{i+1,j}} \cdot \text{(d)}\right]^2 \Delta y^2 + \sum_i \sum_j \frac{\left[\dfrac{v^n_{i+1,j} - v^n_{i-1,j}}{2\Delta x} - \dfrac{u^n_{i,j+1} - u^n_{i,j-1}}{2\Delta y} + f_j\right]^4 \Delta x^2 \Delta y^2}{(h^n_{ij})^4}\right\}. \quad (47)$$

The stages of the numerical algorithm were the following. To achieve mass conservation, adjust the heights h_{ij} forecast by the ADI models, i.e., use Eq. (23). Then

(i) calculate $(\partial G'/\partial U)_{ij}$, $(\partial G'/\partial V)_{ij}$, $(\partial G'/\partial h)_{ij}$;
(ii) calculate α following (46);
(iii) calculate the corrections u'_{ij}, v'_{ij}, h'_{ij} using

$$u'_{ij} = \alpha\left(\frac{\partial G'}{\partial U}\right)_{ij}, \quad v'_{ij} = \alpha\left(\frac{\partial G'}{\partial V}\right)_{ij},$$

$$h'_{ij} = \alpha\left(\frac{\partial G'}{\partial h}\right)_{ij}; \tag{48}$$

(iv) calculate the new potential enstrophy using the corrected fields, i.e.,

$$G'(\bar{U} + U', \quad \bar{V} + V', \quad \bar{h} + h'). \tag{49}$$

(v) Use the new corrected fields $U^* = U + U'$, $V^* = V + V'$, $h^* = h + h'$ as starting values for the new time-step calculation.

The Bayliss-Isaacson algorithm for enforcing enstrophy conservation follows closely the algorithm for enforcing conservation of potential enstrophy using

$$G = \sum_i \sum_j \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} + f\right)^2_{ij} \Delta x \Delta y \tag{50}$$

and Eq. (40). Finally, the corrected enstrophy is given by

$$G_{\text{correct}} = G(\bar{U} + U', \quad \bar{V} + V') \tag{51}$$

which should then be equal to the initial enstrophy $G(U_0, V_0)$.

5. The method of periodic high-order filtering

Kalnay-Rivas *et al.* (1977, 1979), using the GLAS fourth-order global atmospheric model, showed that a formally nonconservative scheme, combined with the application of a two-dimensional 16th order Shapiro applied at every time step of the numerical integration, will conserve potential enstrophy by eliminating waves shorter than four times the grid size before they attain significant amplitudes.

However, the case of a limited area as considered here (see the Appendix) proves to be totally different. Whereas we have periodic boundary conditions in the x direction, in the y direction we have zero boundary conditions which present a problem for the application of the Shapiro high-order filter as they can cause amplification of long-wave spurious components (Shapiro, 1970). Although this effect can be eliminated by a tedious procedure, another problem arises. As we approach the y boundary we have to reduce the order of the Shapiro filter as it requires points outside the limited area domain, about which we have no information (Shapiro, 1977, p. 35).

As a compromise we used a refined mesh ($\Delta x = \Delta y = 200$) by more than doubling the number of mesh points in each coordinate direction (31×23). Then we applied the full 16th order Shapiro filter in the x direction (17 points) while in the y direction, for points situated eight grid-points or less from either boundary, 9-, 7-, 5- and 3-point operators were used as appropriate (see Shapiro, 1977). This, of course, is bound to decrease the selectivity of the Shapiro filter and to affect longer wavelengths in the boundary regions of the domain, which are more strongly smoothed in as much as lower-order filter operators are used near the boundaries. In long-term integrations this effect contaminates the whole limited area domain.

6. Numerical results of long-term integrations

a. The test problem

To compare the modified Sasaki variational method with the modified Bayliss-Isaacson method for enforcing potential enstrophy conservation, the test problem of Gustafsson (1971) was used, *viz.*, the initial height field condition No. 1 used by Grammeltvedt (1969), i.e.,

$$h(x, y) = H_0 + H_1 \tanh\left[\frac{9(D/2 - y)}{2D}\right] + H_2 \operatorname{sech}^2\left[\frac{9(D/2 - y)}{D}\right] \sin\left[\frac{2\pi x}{L}\right]. \tag{52}$$

The initial velocity fields were derived from the initial height field via the geostrophic relationship

$$U = \left(\frac{-g}{f}\right)\frac{\partial h}{\partial y}, \quad V = \left(\frac{g}{f}\right)\frac{\partial h}{\partial x}. \tag{53}$$

The constants used were

$$\left.\begin{array}{ll} L = 4400 \text{ km} & g = 10 \text{ ms}^{-2} \\ D = 6000 \text{ km} & H_0 = 2000 \text{ m} \\ \hat{f} = 10^{-4} \text{ s}^{-1} & H_1 = 220 \text{ m} \\ \beta = 1.5 \times 10^{-11} \text{ s}^{-1} \text{ m}^{-1} & H_2 = 133 \text{ m} \end{array}\right\}. \tag{54}$$

For the long runs conducted here, the space and time increments used were

$$\Delta x = \Delta y = 500 \text{ km}, \quad \Delta t = 3600 \text{ s}. \tag{55}$$

The long-term runs were conducted for periods of over 20 days with no dissipation added to the two ADI models ADIF and GUSTAF. That both the modified variational technique and the Bayliss-Isaacson modified technique are successful, will be evidenced by the perfect conservation of the mass and potential enstrophy invariants and by the non-appearance of a blow up at the critical time T_c which was of the order of 12–14 days (Fairweather and Navon, 1980). For the high-order filtering method the space and time increments used were $\Delta x = \Delta y = 200$ km, $\Delta t = 3600$ s.

b. Results of the a posteriori techniques with the ADIF model

The starting values of the integral invariants of total mass, energy, enstrophy and potential enstrophy (in arbitrary units) were

$$\left.\begin{aligned} H_0 = 2000:\quad E_0 &= 6.10304E + 20 \\ Z_0 &= 1.37643E + 2 \\ Z_0' &= 4.13503E + 8 \end{aligned}\right\} . \quad (56)$$

After 20 days (using the ADIF model) we obtained the results given in Table 1. Similar results were obtained using the GUSTAF nonlinear ADI model (see Gustafsson, 1971; Navon, 1978).

c. *Discussion of the numerical results*

Up to five iterations were required to solve the nonlinear Euler-Lagrange equations [(15a)–(15b)] and up to three external mass adjustment corrections were allowed for at each time step of the numerical integration. The variational technique enforcing conservation of potential enstrophy and total mass prevented the numerical integration from blowing up and the numerically integrated results are quite satisfactory.

The modified Bayliss-Isaacson *a posteriori* method performed better than the variational method in conserving the potential enstrophy, while the results for energy and mass conservation were similar—and conserved up to an error of 1%.

The high-order two-dimensional Shapiro filter gave errors of up to 10% when applied at each time step and errors of about 5% when applied periodically (every three time steps). The reason for these deceptive results is due entirely to the limited-area domain and its boundary conditions as discussed in Section 5.

d. *Numerical stability*

Both the modified variational method and the modified Bayliss-Isaacson method were integrated up to 20 days using a 60 min time step. Both methods behaved stably in conjunction with the ADI models employed for solving the shallow-water equations and there was no sign of impending numerical instability. The same applies to the high-order two-dimensional Shapiro filter.

e. *Efficiency*

The measure of efficiency used was the additional CPU time per time step required by either the variational method or the modified Bayliss-Isaacson method for enforcing conservation of total mass and potential enstrophy. It was found that the modified Bayliss-Isaacson method required roughly a third of the CPU time per time step as compared with the modified Sasaki variational method. The reason for this was that in the variational method the corrected fields and the λ_E Lagrange multiplier required iterative procedures that were relatively costly in CPU time per time step.

The Shapiro two-dimensional high-order filter was the less demanding on CPU time, but because of the amplitude and phase errors due to limited-area boundary effects, this advantage cannot be exploited in our case.

f. *Accuracy tests*

In order to provide a basis for comparison between the different *a priori* methods, in the absence of an analytic solution to the full nonlinear shallow-water equations, we assume that the exact solution of this mixed initial boundary-value problem W_{EX} is the solution of ADIF (A16) computed with a fine-mesh discretization, *viz.*, $\Delta x = \Delta y = 50$ km and $\Delta t = 450$ s. (using the same method to enforce the discrete integral constraints).

As in Gustafsson (1971) the relative error in an approximate solution, W_{AP}, is measured in the norm $\|\cdot\|$, defined by the inner product

$$\left.\begin{aligned} (\alpha, \beta) &= \Delta x \Delta y \sum_{j=1}^{N_x} \{ \sum_{k=1}^{N_y} \alpha_{jk}^T \beta_{jk} \\ &\qquad + \tfrac{1}{2}(\alpha_{j0}^T \beta_{j0} + \alpha_{j,N_y}^T \beta_{j,N_y}) \} \\ \|\alpha\|^2 &= (\alpha, \alpha) \end{aligned}\right\} , \quad (57)$$

where α and β are grid functions, satisfying the boundary conditions given in (A6)–(A7). The relative errors in the approximation determined by the linear ADI method ADIF, using the three *a priori*

TABLE 1. Relative accuracy of different *a posteriori* techniques using the ADIF model.

Time	Method	Z_0 (potential enstrophy)	Z_0' (enstrophy)	E_0 (total energy)	H (mass)
$t = 20$ days		Ratio between final and initial values	Ratio between final and initial values	Ratio between final and initial values	Ratio between final and initial values
	Variational	0.983	0.995	1.007	0.999
	Modified Bayliss-Isaacson	0.994	0.993	1.007	0.999
	High-order Shapiro filtering (at every time-step)	1.125	1.117	0.905	0.997
	High-order Shapiro filtering (every three time-steps)	1.067	1.054	0.952	0.997

TABLE 2. Relative errors of the various *a posteriori* methods.

Method for enforcing integral constraints	Δt = 3600 s
Variational	3.1×10^{-3}
Modified Bayliss-Isaacson	2.9×10^{-3}
High-order Shapiro filter (applied at every time-step)	1.3×10^{-2}
High-order Shapiro filter (applied every three time-steps)	5.3×10^{-3}

methods namely the variational method, the Bayliss-Isaacson method and the two-dimensional Shapiro high-order periodic filtering method are shown in Table 2. Here

$$E_{\mathrm{AP}} = W_{\mathrm{AP}} - W_{\mathrm{EX}} \tag{58}$$

and the relative error between the approximate and the exact solution is

$$\text{relative error} = \frac{\|E_{\mathrm{AP}}\|}{\|W_{\mathrm{EX}}\|}, \quad \Delta t = 450 \text{ s}, \quad t = 20 \text{ days}. \tag{59}$$

The results point to a slightly better accuracy for the Bayliss-Isaacson method.

7. Summary and conclusions

Three methods were tested, all intended to enforce conservation of total mass and potential enstrophy in long-term integrations of two ADI finite-difference approximations of the nonlinear shallow-water equations model.

The first technique was a modified Sasaki variational method which resulted in a set of coupled nonlinear partial differential Euler-Lagrange equations, which were solved by an iterative method. The main issue proved to be the updating of the Lagrange multiplier λ_E. Although a working procedure was found to perform well enough, more sophisticated mathematical methods should be employed if it is intended to refine the method.

These methods relate mathematically to solving nonlinearly constrained optimization problems, and some very efficient FORTRAN programs have been described by Purcell (1977), based on techniques due to Powell (1969), Hestenes (1969) and Bertsekas (1975).

It was found that for the 20 days' integrations not only were potential enstrophy and mass conserved, but total energy was also conserved well enough. The modified Sasaki variational method required a number of iterations at each time step, both for solving the Euler-Lagrange equations and for imposing the constraint of conservation of total mass. The writer feels that this method could be greatly improved when coupled with a refined technique for updating λ_E between time steps.

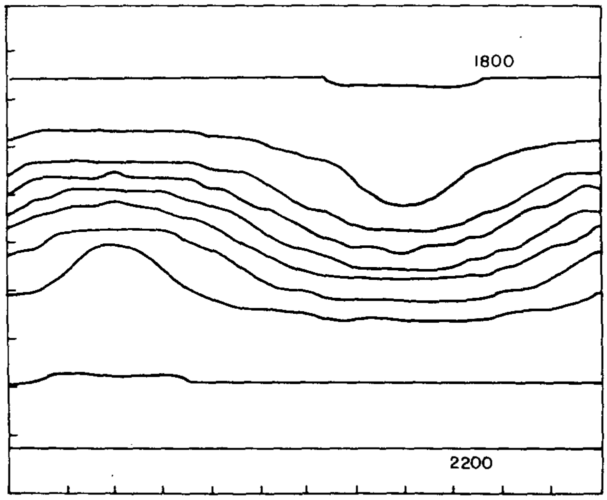

FIG. 1. The initial distribution of the height field depicted by isopleths drawn at 50 m intervals. The channel walls are shown as solid horizontal lines, the cyclic boundaries as vertical lines. The domain is covered by 13 grid points in horizontal (east–west) and 10 points in the vertical (north–south).

Use of the generalized variational functional of Section 4d would probably result in satisfactory results being obtained by applying the adjustments less frequently (every few time steps) and then the method could compete in efficiency with the Bayliss-Isaacson method.

The second technique, due to Bayliss and Isaacson, proved to be very robust and less demanding of CPU time. The results obtained after 20 days compared well with those obtained with the modified Sasaki variational technique, and in some instances were better. Moreover, the implementation of this method required less computer coding, and

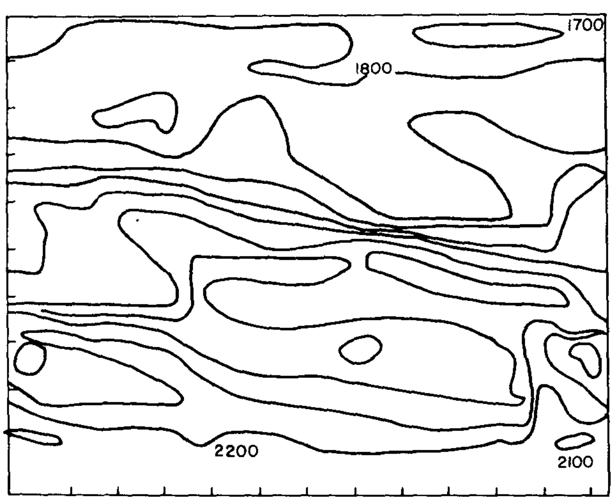

FIG. 2. The 20-day forecast by height field using the ADIF model with the variational enforcing of potential enstrophy and mass conservation, isoplethed at intervals of 50 m.

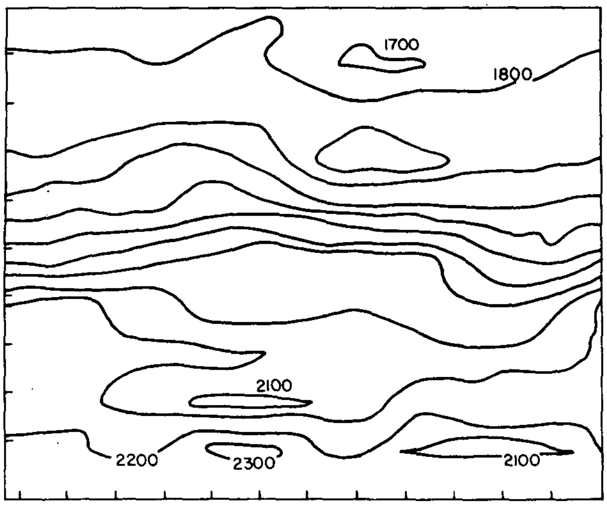

FIG. 3. The 20-day forecast of height field using the ADIF model with the Bayliss-Isaacson method of enforcing potential enstrophy and mass conservation isoplethed at intervals of 50 m.

in view of our experience with it, we would recommend its use for explicitly enforcing the required conservation relationships. The Bayliss-Isaacson technique produced smoother patterns after 20 days of integration than the modified Sasaki variational technique (see Figs. 1–3).

It should be noted that the Bayliss-Isaacson technique was originally formulated in terms of simultaneous conservation constraints, rather than successive as implemented in this paper. This could account for the fact that the conservation of integral constraints is not perfect.

The third technique, that is, the periodic application of the Shapiro (16th) high-order filter does not achieve the same degree of accuracy as the other two *a posteriori* techniques. This is due to the fact that we are applying the filter to a limited area domain. However, when applied periodically, on a refined mesh this method regains some accuracy. When applied to a global domain this method is more efficient than the other *a posteriori* methods.

More research should be worthwhile pursuing with the aim of evolving a refined updating method for the Lagrange multiplier λ_E in the Sasaki variational technique.

To sum up, the approach of enforcing conservation of total mass and potential enstrophy appears to be a valid alternative to the design of complex finite-difference conserving schemes, and the first two methods discussed above can be recommended for long-time integrations of the shallow-water equations on limited area domains, while the filtering method is the most efficient on global or doubly periodic domains.

Acknowledgments. Special thanks are extended to Mrs. Antoinette Venter, whose brilliant programming work was essential to the success of this research.

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APPENDIX

The Numerical Shallow-Water Equations Models

1. The Fairweather-Navon (1980) linear ADI model (ADIF)

The shallow-water equations can be written (Houghton, Kasahara and Washington, 1966) as

$$\left.\begin{aligned} &\frac{\partial \mathbf{w}}{\partial t} = \mathbf{A}(\mathbf{w})\frac{\partial \mathbf{w}}{\partial t} + \mathbf{B}(\mathbf{w})\frac{\partial \mathbf{w}}{\partial y} + \mathbf{C}(y)\mathbf{w} \\ &0 \le x \le L, \quad 0 \le y \le D, \quad t \ge 0 \end{aligned}\right\}, \tag{A1}$$

where L and D are the dimensions of a rectangular domain of area $\tilde{A} = L \cdot D$; $\mathbf{w}$ is a vector function

$$\mathbf{w} = (u, v, \phi)^{\mathrm{T}}; \tag{A2}$$

u and v are the velocity components in the x and y directions, respectively, while

$$\phi = 2\sqrt{gh}, \tag{A3}$$

where h is the depth of the fluid and g is the acceleration due to gravity.

In (A1) $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ are matrices given by

$$\mathbf{A} = \begin{bmatrix} u & 0 & \phi/2 \\ 0 & u & 0 \\ \phi/2 & 0 & u \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} v & 0 & 0 \\ 0 & v & \phi/2 \\ 0 & \phi/2 & v \end{bmatrix},$$

$$\mathbf{C} = \begin{bmatrix} 0 & f & 0 \\ -f & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{A4}$$

where f is the Coriolis term given by

$$f = \hat{f} + \beta(y - D/2), \tag{A5}$$

with $\hat{f}$ and β constants.

Periodic boundary conditions are assumed in the x direction

$$\mathbf{w}(x, y, t) = \mathbf{w}(x + L, y, t), \tag{A6}$$

whereas in the y direction the boundary condition is

$$v(x, 0, t) = v(x, D, t) = 0. \tag{A7}$$

With these boundary conditions and with the initial condition

$$\mathbf{w}(x, y, 0) = \varphi(x, y), \tag{A8}$$

the total energy

$$E = \frac{1}{2}\int_0^L \int_0^D (u^2 + v^2 + \phi^2/4)\phi^2(4g)^{-1}dxdy \tag{A9}$$

is independent of time. Also, the average value of the height of the free surface, proportional to the total mass, is conserved, i.e.,

$$\bar{h} = \bar{A}^{-1}\int_0^L\int_0^D hdxdy \tag{A10}$$

is independent of time.

Then the Fairweather-Navon (1980) linear ADI algorithm (ADIF) is set as follows: Let N_x and N_y be positive integers and set

$$\Delta x = L/N_x, \quad \Delta y = D/N_y. \tag{A11}$$

We shall denote by $\mathbf{w}_{jk}^n$ an approximation to $\mathbf{w}(j\Delta x, k\Delta y, n\Delta t)$, where Δt is the time step. The basic difference operations used in the algorithm are

$$\left.\begin{aligned} D_{0x}\mathbf{w}_{jk}^n &= (\mathbf{w}_{j+1,k}^n - \mathbf{w}_{j-1,k}^n)/2\Delta x \\ D_{+x}\mathbf{w}_{jk}^n &= (\mathbf{w}_{j+1,k}^n - \mathbf{w}_{j,k}^n)/\Delta x \\ D_{-x}\mathbf{w}_{jk}^n &= (\mathbf{w}_{jk}^n - \mathbf{w}_{j-1,k}^n)/\Delta x \end{aligned}\right\}, \tag{A12}$$

with similar definitions for D_{0y}, D_{+y} and D_{-y}, respectively. We also define the operators P_{jk}^n and Q_{jk}^n by

$$\left.\begin{aligned} \mathbf{P}_{jk}^n &= \tfrac{1}{2}\Delta t[A(\mathbf{w}_{jk}^n)D_{0x} + \mathbf{C}_k^{(1)}] \\ \mathbf{Q}_{jk}^n &= \tfrac{1}{2}\Delta t[B(\mathbf{w}_{jk}^n)D_k + \mathbf{C}_k^{(2)}] \end{aligned}\right\}, \tag{A13}$$

where

$$\mathbf{C}_k^{(1)} = \begin{bmatrix} 0 & 0 & 0 \\ -f_k & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$\mathbf{C}_k^{(2)} = \begin{bmatrix} 0 & f_k & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{A14}$$

and where, due to the boundary conditions imposed in the y direction,

$$D_k = \begin{cases} D_{0y}, & \text{for } k = 1, 2, \ldots, N_y - 1 \\ D_{+y}, & \text{for } k = 0 \\ D_{-y}, & \text{for } k = N_y. \end{cases} \tag{A15}$$

The Fairweather-Navon (1980) linear ADI algorithm for the shallow-water equations is then given by

$$\left.\begin{aligned} (\mathbf{I} - \mathbf{P}_{jk}^{n*})\mathbf{w}_{jk}^{(n+1)*} &= \mathbf{V}_{jk}^n \\ (\mathbf{I} - \mathbf{Q}_{jk}^{n*})\mathbf{w}_{jk}^{(n+1)} &= 2\mathbf{w}_{jk}^{(n+1)*} - \mathbf{V}_{jk}^n \end{aligned}\right\}, \tag{A16}$$

with

$$\mathbf{V}_{jk}^n = (\mathbf{I} + \mathbf{Q}_{jk}^{n*})\mathbf{w}_{jk}^n, \tag{A17}$$

and where

$$\left.\begin{aligned} P_{jk}^{n*} &= \tfrac{1}{2}\Delta t[A(\hat{\mathbf{w}}_{jk}^n)D_{0x} + C_k^{(1)}] \\ Q_{jk}^{n*} &= \tfrac{1}{2}\Delta t[B(\hat{\mathbf{w}}_{jk}^n)D_k + C_k^{(2)}] \end{aligned}\right\}, \tag{A18}$$

and

$$\left.\begin{aligned} \hat{\mathbf{w}}_{jk}^n &= \tfrac{1}{2}(3\mathbf{w}_{jk}^n - \mathbf{w}_{jk}^{n-1}), \quad \text{for } n \geq 1 \\ \hat{\mathbf{w}}_{jk}^{(0)} &= \mathbf{w}_{jk}^{(0)} + [P_{jk}^{(0)} + Q_{jk}^{(0)}]\mathbf{w}_{jk}^{(0)} \end{aligned}\right\}, \tag{A19}$$

and where $\mathbf{w}_{jk}^{(n+1)*}$ is an auxiliary solution. Note that owing to (19) we have to solve only sequences of systems of linear equations.

Also, owing to the assumption of periodic boundary conditions in the x direction, the resulting coefficient matrices arising from the application of the Fairweather-Navon linear algorithm along horizontal rows, are either block or scalar cyclic tridiagonal (see Navon, 1979; Fairweather and Navon, 1980). If we denote by Q the absolute vorticity, i.e.,

$$Q = \xi + f, \tag{A20}$$

where

$$\xi = \frac{\partial v}{\partial X} - \frac{\partial u}{\partial y}, \tag{A21}$$

then a third invariant of the shallow-water equations, *viz.*, the potential enstrophy, given by

$$Z = \int_0^D\int_0^L\left(\frac{Q^2}{h}\right)dxdy \tag{A22}$$

was found not to be conserved by the finite-difference version of the linear ADI algorithm (Fairweather and Navon, 1980).

2. The nonlinear Gustafsson (1971) ADI algorithm (GUSTAF)

Using the same notation, the Gustafsson (1971) nonlinear ADI difference scheme is defined by

$$[\mathbf{I} - \mathbf{P}_{jk}^{(n+1/2)}]\mathbf{w}_{jk}^{(n+1/2)} = [\mathbf{I} + \mathbf{Q}_{jk}^n]\mathbf{w}_{jk}^{(n)}, \tag{A23a}$$

$$[\mathbf{I} - \mathbf{Q}_{jk}^{(n+1)}]\mathbf{w}_{jk}^{n+1} = [\mathbf{I} + \mathbf{P}_{jk}^{(n+1/2)}]\mathbf{w}_{jk}^{n+1/2}, \tag{A23b}$$

where $\mathbf{w}_{jk}^{(n+1/2)}$ is an intermediary variable. These equations do not apply to the v component for $k = 0$ and $k = N_y$; for those values we use the conditions

$$v_{j,0}^n = v_{j,N_y}^n = 0, \quad n = 0, 1, \ldots. \tag{A24}$$

From Eqs. (23a) and (23b) it is evident that for each time step of the scheme, a sequence of systems of nonlinear equations have to be solved. Gustafsson (1971) used both a simple iteration technique and a quasi-Newton method for solving the resulting nonlinear equations. For a detailed computer program implementing the Gustafsson algorithm see Navon (1978).

As was found by Fairweather and Navon (1980), the discretized finite-difference version of Gustafsson's algorithm conserves total mass and total energy but again does not conserve potential enstrophy. As such it is subject to a blow up after a finite time T_c if no dissipation is included in the model (Fairweather and Navon, 1980); Gustafsson, 1971).

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