Application of a New Partly-Implicit Time-Differencing Scheme for Solving the Shallow-Water Equations

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Abstract

A new partly-implicit time-differencing scheme is applied to the non-linear shallow-water equations on a limited-area domain, employing a semi-momentum space-differencing scheme. Numerical integration is performed with time steps 2,5 times as large as the greatest allowed for by the CFL (COURANT, FRIEDRICHS and LEVY)- criterion for the leapfrog explicit scheme. It is shown that an unconditionally stable scheme can be obtained by coupling implicitly the continuity equation and one of the momentum equations. Numerical experiments with the barotropic equations are performed both in a channel and in combination with coarse-mesh data, using the partly implicit scheme with well-posed boundary conditions. Integral invariants of the shallow-water equations are conserved well during the numerical integrations and it is found that the simulation of the process of geostrophic adjustment is more realistic when this partly-implicit scheme is used.

Zusammenfassung. Anwendung eines neuen teil-impliziten zeitlichen Differenzen-Verfahrens zur Lösung der „Flachwasser-Gleichungen"

Ein neues teilweise implizites zeitliches Differenzenverfahren wird auf die nichtlinearen Gleichungen für eine barotrope Flüssigkeit (Flachwasser-Gleichungen) für ein begrenztes Gebiet angewandt. Dabei wird als räumliches Differenzschema das Semi-Momentum-Verfahren benutzt. Die numerische Integration erfolgt mit Zeitschritten, die 2,5 mal so groß sind wie die größten Schritte, die das CFL (COURANT, FRIEDRICHS und LEVY) – Kriterium für das explizite "Leap Frog" – Verfahren erlaubt. Die Arbeit zeigt, daß sich ein unbedingt stabiles Schema ergibt, wenn man die Kontinuitätsgleichung und eine der Bewegungsgleichungen implizit koppelt. Numerische Experimente mit den barotropen Gleichungen werden sowohl in einem Kanal als auch in Kombination mit einem grobmaschigen Gitter durchgeführt, wobei das teilweise implizite Verfahren mit konsistenten Randbedingungen benutzt wird. Die integralen Invarianten der Flachwasser-Gleichungen bleiben während der numerischen Integration mit guter Genauigkeit erhalten, und es stellt sich heraus, daß die Simulation des Prozesses der geostrophischen Anpassung realistischer verläuft, wenn dieses teilweise implizite Verfahren verwendet wird.

Résumé: Application d'un nouveau schéma de différentiation temporelle partiellement implicite pour résoudre les équations en eau peu profonde.

Un nouveau schéma de différentiation temporelle partiellement implicite est appliqué aux équations non linéaires en eau peu profonde sur un domaine d'aire limitée, en utilisant un schéma de différentiation spatiale appliqué partiellement aux quantités de mouvement. L'intégration numérique est effectuée avec des pas temporels 2,5 fois plus grands que le maximum acceptable d'après le critère de COURANT-FRIEDRICHS-LEVY dans le schéma explicite „leap-frog". On montre qu'un schéma inconditionnellement stable peut être obtenu en couplant implicitement l'équation de continuité et l'une des équations du mouvement. On exécute les expériences numériques avec les équations barotropes, dans un canal et en combinaison avec un réseau à grandes mailles, en employant le schéma partiellement implicite avec des conditions aux limites bien posées. Les invariants intégraux des équations en eau peu profonde se conservent avec une bonne précision durant l'intégration numérique et on trouve que la simulation du processus d'ajustement géostrophique est plus réaliste quand ce schéma partiellement implicite est utilisé.

1 Introduction

During the last few years a number of investigators have devoted considerable attention to implicit-time integration schemes. In explicit-time difference approximations applied to the primitive equations the time step is restricted by a maximum value depending on the inverse velocity of the most rapid wave solutions, i.e. the fast gravity-wave solutions. In contrast the use of implicit-time schemes make stable integrations of the primitive equations possible with only little restriction on the time step. Further-more, implicit-time schemes can be formulated with built-in selective damping, thus alleviating the non-linear instability problem.

KURIHARA (1965) and MARCHUK (1965, 1974) have extensively investigated several implicit-time differencing schemes. GUSTAFSSON (1975) used an alternating-direction implicit (ADI) method for the shallow-water equations, solving the resulting non-linear algebraic equations by a quasi-Newton method, whereas FAIRWEATHER and NAVON (1977) propose a new linear economical ADI method based on a perturbation of a CRANK-NICOLSON-type discretization.

KWITZAK (1970) and KWITZAK and ROBERT (1971) used a semi-implicit time-integration scheme, i.e. only the terms representing the gravitational modes were treated implicitly, while the advective non-linear term was treated explicitly.

Economical explicit-integration schemes have been proposed by MAGAZENKOV et al. (1971), SHUMAN (1971) and SHUMAN et al. (1972). These schemes permit the use of a time step having twice the value of that used in the leap-frog scheme. GADD (1974) and MESINGER (1974, 1976) proposed similar economical explicit-integration schemes. SCHOENSTADT and WILLIAMS (1976) showed that the presence of a mean flow reduces the increase in the time step achieved by the use of the SHUMAN-technique.

In the present paper a new partly-implicit time-integration scheme is proposed. The equations used are given in Section 2, where the numerical procedure is also described. Section 3 provides a linear stability analysis, both of the partly implicit scheme and an unconditionally stable scheme. The numerical results of two test calculations are given in Section 4, demonstrating the stability of the method and its ability to control non-linear instabilities and to conserve linearized integral invariants of the shallow-water equations. A linearized frequency analysis shows that this scheme alleviates to a certain extent the process of geostrophic adjustment time scale retardation inherent in implicit schemes.

2 Equations and Numerical Procedure

a) The shallow-water equations

We consider the equations of a barotropic fluid in two space dimensions, using a local coordinate system with the positive x-axis pointing east and the positive y-axis north

$$\frac{\partial u}{\partial t} = -u\frac{\partial u}{\partial x} - v\frac{\partial u}{\partial y} + fv - \frac{\partial \phi}{\partial x}$$

$$\frac{\partial u}{\partial t} = -u\frac{\partial v}{\partial x} - v\frac{\partial v}{\partial y} - fu - \frac{\partial \phi}{\partial y} \qquad (1)$$

$$\frac{\partial \phi}{\partial t} = -\frac{\partial (\phi u)}{\partial x} - \frac{\partial (\phi v)}{\partial y}$$

where

ϕ = gh the geopotential
g = the acceleration of gravity
f = the Coriolis parameter
u = the eastward component of the wind
v = the northward component of the wind
h = the height of the free surface of the fluid above sea-level.

b) The partly-implicit scheme

Let A_u, A_v and A_ϕ denote the discretized right-hand sides of equation (1). The partly-implicit scheme is then written as

$$
\begin{aligned}
\phi_{ij}^{n+1} - \phi_{ij}^{n} &= \Delta t\, A_\phi(u_{ij}^{n}, v_{ij}^{n}, \phi_{ij}^{n+1}) \\
u_{ij}^{n+1} - u_{ij}^{n} &= \Delta t\, A_u(u_{ij}^{n+1}, v_{ij}^{n}, \phi_{ij}^{n+1}) \qquad (2) \\
v_{ij}^{n+1} - v_{ij}^{n} &= \Delta t\, A_v(u_{ij}^{n+1}, v_{ij}^{n+1}, \phi_{ij}^{n+1})
\end{aligned}
$$

where w_{ij}^{n} corresponds to the values assumed by w on a rectangular grid at the point $(i\Delta x, j\Delta y, n\Delta t)$ where i, j, n are integers and w stands for u, v or ϕ.

More explicitly, using the semi-momentum finite-difference scheme of SHUMAN and STACKPOLE (1968) we can specify A_ϕ, A_u and A_v as

$$
\begin{aligned}
A_\phi &= -\overline{\left[(\overline{u^{(n)}}^{y}\, \overline{\phi^{(n+1)}}^{y})_x + (\overline{v^{(n)}}^{x}\, \overline{\phi^{(n+1)}}^{x})_y \right]}^{xy} \\
A_u &= -\overline{\left[\overline{u^{(n+1)}}^{xy}\, \overline{u_x^{(n+1)}}^{y} + \overline{v^{(n)}}^{xy}\, \overline{u_y^{(n+1)}}^{x} + \overline{\phi_x^{(n+1)}}^{y} - \overline{f}^{xy}\, \overline{v^{(n)}}^{xy} \right]}^{xy} \qquad (3) \\
A_v &= -\overline{\left[\overline{u^{(n+1)}}^{xy}\, \overline{v_x^{(n+1)}}^{y} + \overline{v^{n+1}}^{xy}\, \overline{v_y^{(n+1)}}^{x} + \overline{\phi_y^{(n+1)}}^{x} + \overline{f}^{xy}\, \overline{u^{(n+1)}}^{xy} \right]}^{xy}
\end{aligned}
$$

where the following basic finite-difference operators were used:

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\begin{aligned}
\overline{F}^{\alpha} &= (F(\alpha + \tfrac{1}{2}\Delta\alpha) + F(\alpha - \tfrac{1}{2}\Delta\alpha))/2 \\
F_\alpha &= (F(\alpha + \Delta\alpha/2) - F(\alpha - \Delta\alpha/2))/\Delta\alpha \qquad (4)
\end{aligned}
$$

while multiple superscripts have the following meaning:

$$\overline{F}^{\alpha\alpha} = \overline{(\overline{F}^{\alpha})}^{\alpha} \qquad (5)$$

where α stands for x and y.

The following spacial distribution of the dependent variables ϕ, u, v was adopted (see Figure 1):

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$$

where w_{ij}^{n} corresponds to the values assumed by w on a rectangular grid at the point $(i\Delta x, j\Delta y, n\Delta t)$ where i, j, n are integers and w stands for u, v or ϕ.

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\end{aligned}
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where the following basic finite-difference operators were used:

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\begin{aligned}
&\overline{F}^{\alpha} = (F(\alpha + \tfrac{1}{2}\Delta\alpha) + F(\alpha - \tfrac{1}{2}\Delta\alpha))/2 \\
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\end{aligned}
\tag{4}
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while multiple superscripts have the following meaning:

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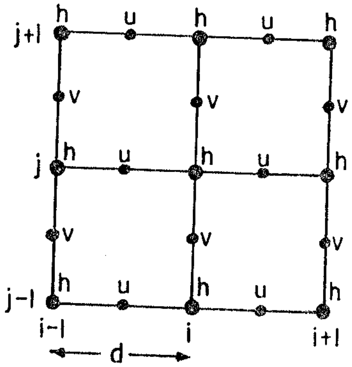
where α stands for x and y.

The following spacial distribution of the dependent variables ϕ, u, v was adopted (see Figure 1):



● **Figure 1** Spatial distribution of the dependent variables on a square grid.

● **Bild 1** Räumliche Verteilung der abhängigen Variablen auf dem benutzten quadratischen Gitter

corresponding to scheme C of ARAKAWA (1977), WINNINGHOFF (1968). The rather unconventional cho of the implicitly-treated terms, as well as the choice of the spatial distribution of the variables, was made with a view to a realistic simulation of the process of geostrophic adjustment. That is an advantage was confirmed by a subsequent frequency analysis.

For the sake of simplicity of notation and to avoid a number of tehnicalities the simplest space differenci scheme of secondorder accuracy will be used for specifying space derivatives in A_ϕ, A_u and A_v in Sections and d, i.e.

$$\frac{\partial u}{\partial u} = D_{ox} u_{ij} = (u^n_{i+1,j} - u^n_{i-1,j})(2\Delta x)^{-1} + 0(\Delta x^2)$$

$$\frac{\partial u}{\partial y} = D_{oy} u_{ij} = (u^n_{i,j+1} - u^n_{i,j-1})(2\Delta y)^{-1} + 0(\Delta y^2). \qquad (6)$$

c) Linearization procedure

The advective non-linear term in the momentum equations imposes upon discretization the solution of non-linear systems of algebraic equations. To avoid solving such systems, a linear analogue to th nonlinear advective term is obtained via a backward Taylor-series projection (see RICHTMYER and MORTON (1967, p. 203).

The space finite-difference approximation to the implicit non-linear advective term is expressed as

$$u^{n+1}_{ij} \left(\frac{\partial u}{\partial x}\right)^{n+1}_{ij} = u^{n+1}_{ij} D_{ox} u^{n+1}_{ij} = u^{n+1}_{ij} (u^{n+1}_{i+1,j} - u^{n+1}_{i-1,j})(2\Delta x)^{-1}. \qquad (7)$$

Taking the first term of the product and Taylor-projecting it about the time level $n\Delta t$, we have

$$\frac{u^{n+1}_{ij} u^{n+1}_{i+1,j}}{2\Delta x} = \frac{u^n_{ij} u^n_{i+1,j}}{2\Delta x} + \frac{\Delta t}{2\Delta x} u^n_{ij} \left(\frac{\partial u}{\partial t}\right)^n_{i+1,j} + \frac{\Delta t}{2\Delta x} u^n_{i+1,j} \left(\frac{\partial u}{\partial t}\right)_{ij} + 0(\Delta t^2) \qquad (8)$$

or using

$$\left(\frac{\partial u}{\partial t}\right)_{ij} = (u^{n+1}_{ij} - u^n_{ij})/\Delta t \qquad (9)$$

and inserting the corresponding values in (8) we obtain

$$\frac{u^n_{ij}u^{n+1}_{i+1,j}}{2\Delta x} \approx (u^n_{ij}u^{n+1}_{i+1,j} + u^{n+1}_{ij}u^n_{i+1,j} - u^n_{ij}u^n_{i+1,j})/2\Delta x. \tag{10}$$

The same procedure is applied to the second term of (7).

d) Generalized linearization procedure

If we have a system of equations implicitly coupled as in Section 3b a general linearization procedure was proposed by BRILEY and MCDONALD (1975, 1977), BEAM and WARNING (1976) and by STEGER and KUTLER (1977).

Writing the shallow-water equations in the form

$$\frac{\partial w}{\partial t} = A(w)\frac{\partial w}{\partial y} + C(y)w \tag{11}$$

where w is the vector

$$w = (u, v, \Phi)^T \tag{12}$$

and

$$\Phi = 2\sqrt{\phi} \tag{13}$$

while A, B and C are the matrices given by

$$A = \begin{bmatrix} w & 0 & \Phi/2 \\ 0 & u & 0 \\ \Phi/2 & 0 & u \end{bmatrix} \qquad B = \begin{bmatrix} v & 0 & 0 \\ 0 & v & \Phi/2 \\ 0 & \Phi/2 & v \end{bmatrix} \tag{14}$$

$$C = \begin{bmatrix} 0 & f & 0 \\ -f & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Taylor series linearizations are introduced by

$$A^{n+1} = A^n + \left(\frac{\partial A}{\partial w}\right)^n (w^{n+1} - w^n) + 0(\Delta t^2)$$

$$B^{n+1} = B^n + \left(\frac{\partial B}{\partial w}\right)^n (w^{n+1} - w^n) + 0(\Delta t^2) \tag{15}$$

where A and B stand for $A\frac{\partial w}{\partial x}$ and $B\frac{\partial w}{\partial y}$ and the matrices $\partial A/\partial w$ and $\partial B/\partial w$ are standard Jacobians and w is the vector defined by Equation (12).

e) Iterative improvement of the linearization procedure

The values obtained using the linearization procedure can be improved using the following iterative algorithm due to VON ROSENBERG (1969):

$$\frac{\phi^{n+1}_{ij,\,m+1} - \phi^{n+1}_{ij,\,0}}{\Delta t} = - u^{n+1}_{i,j,m} D_{ox}\phi^{n+1}_{i,j,m+1} - v^{n+1}_{i,j,m} D_{oy}\phi^{n+1}_{i,j,m+1}$$

$$- \phi^{n+1}_{i,j,m+1} D_{ox} u^{n+1}_{i,j,m} - \phi^{n+1}_{i,j,m+1} D_{oy} v^{n+1}_{i,j,m}$$

$$\frac{u^{n+1}_{i,j,m+1} - u^{n+1}_{i,j,0}}{\Delta t} = - u^{n+1}_{i,j,m+1} D_{ox}(\hat{u}^{n+1}_{i,j,m}) - v^{n+1}_{i,j,m} D_{oy} u^{n+1}_{i,j,m+1}$$

$$+ f v^{n+1}_{i,j,m} - D_{ox}\phi^{n+1}_{i,j,m+1}$$

$$\frac{v^{n+1}_{i,j,n+1} - v^{n+1}_{i,j,0}}{\Delta t} = - u^{n+1}_{i,j,m+1} D_{ox} v^{n+1}_{i,j,m+1} - v^{n+1}_{i,j,m+1} D_{oy}(\hat{v}^{n+1}_{i,j,m})$$

$$- f u^{n+1}_{i,j,m+1} - D_{oy}\phi^{n+1}_{i,j,m+1}$$

where the m-subscript denotes the iteration count and (ˆ) denotes a linearized value.

Also

$$u^{n+1}_{i,j,0} = u^{n}_{i,j};\; v^{n+1}_{i,j,0} = v^{n}_{i,j};\; \phi^{n+1}_{i,j,0} = \phi^{n}_{ij}. \qquad (16)$$

The iteration is continued until

$$|u^{n+1}_{i,j,m+1} - u^{n+1}_{i,j,m}| < \epsilon_w$$

$$|v^{n+1}_{i,j,m+1} - v^{n+ß}_{i,j,m}| < \epsilon_v \qquad (17)$$

$$|\phi^{n+1}_{i,j,m+1} - \phi^{n+1}_{i,j,m}| < \epsilon_\phi$$

where $\epsilon_u = \epsilon_v = 10^{-1}$ m/sec and $\epsilon_\phi = 1.\mathrm{m}^2\,\mathrm{sec}^{-2}$.

It was found that generally one iteration was sufficient in our case to satisfy (17).

f) Accuracy

The accuracy of the scheme is $0(\Delta t, \Delta x^2)$ i.e. only first-order in time. This is no inconvenience, since as shown by KWITZAK (1970) the errors due to the time discretization are an order of magnitude less than those due to space finite-difference discretization. To obtain second-order accuracy in time the scheme can be modified by staggering the velocity and the geopotential half a time step apart in time, while the linearized advective terms must be treated as proposed by STEPPELER (1975) by retaining an additional term in the Taylor backward series linearization procedure.

g) **Method of solution**

Upon performing the time and space discretizations, linear systems of finite-difference equations are obtained for ϕ_{ij}^{n+1}, u_{ij}^{n+1} and v_{ij}^{n+1}. We shall detail the solution procedure for ϕ_{ij}^{n+1}, the procedures for u_{ij}^{n+1} and v_{ij}^{n+1} being similar.
After the time and space discretization of the continuity equation in (3) one obtains

$$\begin{aligned}&\phi_{ij}^{n+1} - \phi_{ij}^{n} + a_1\phi_{i+1,j+1}^{n+1} + a_2\phi_{i,j+1}^{n+1} + a_3\phi_{i-1,j+1}^{n+1} + b_1\phi_{i+1,j}^{n+1} + \\ &b_2\phi_{ij}^{n+1} + b_3\phi_{i-1,j}^{n+1} + c_1\phi_{i+1,j-1}^{n+1} + c_2\phi_{i,j-1}^{n+1} + c_3\phi_{i-1,j-1} = 0\end{aligned} \tag{18}$$

where

$$\begin{aligned}
a_1 &= \frac{\gamma}{16}\left(2\bar{u}^{xxyy} - 2\bar{v}^{xxyy} + \bar{u}_x^{xxy} + \bar{v}_y^{xxy}\right)^{(n)}\\
a_2 &= \frac{\gamma}{8}\left(2\bar{v}^{xxyy} + \bar{u}_x^{xyy} + \bar{v}_y^{xxy}\right)^{(n)}\\
a_3 &= \frac{\gamma}{16}\left(-2\bar{u}^{xxyy} + 2\bar{v}^{xxyy} + \bar{u}_x^{xyy} + \bar{v}_y^{xxy}\right)^{(n)}\\
b_1 &= \frac{\gamma}{8}\left(2\bar{u}^{xxyy} + \bar{u}_x^{xyy} + \bar{v}_y^{xxy}\right)^{(n)}\\
b_2 &= \frac{\gamma}{4}\left(\bar{u}_x^{xyy} + \bar{v}_y^{xxy}\right)^{(n)}\\
b_3 &= \frac{\gamma}{8}\left(-2\bar{u}^{xxyy} + \bar{u}_x^{xyy} + \bar{v}_y^{xxy}\right)^{(n)}\\
c_1 &= \frac{\gamma}{16}\left(2\bar{u}^{xxyy} + 2\bar{v}^{xxyy} + \bar{u}_x^{xyy} + \bar{v}_y^{xxy}\right)^{(n)}\\
c_2 &= \frac{\gamma}{8}\left(-2\bar{v}^{xxyy} + \bar{u}_x^{xyy} + \bar{v}_y^{xxy}\right)^{(n)}\\
c_3 &= \frac{\gamma}{16}\left(-2\bar{u}^{xxyy} - 2\bar{v}^{xxyy} + \bar{u}_x^{xyy} + \bar{v}_y^{xxy}\right)^{(n)}
\end{aligned} \tag{19}$$

and

$$\gamma = \frac{\Delta t}{\Delta s} \tag{20}$$

with

$$\Delta s = \Delta x = \Delta y. \tag{21}$$

Assuming a rectangular domain of sides $M\Delta x$ and $N\Delta y$ a block-tridiagonal coefficient matrix is obtained when the unknown vector

$$x_{ij},\ i = 1, \ldots, M_x, \quad j = 1, \ldots, N_y$$

is ordered as

$$\{x_{1,1}, \ldots, x_{M_x,1}, x_{1,2}, \ldots, x_{M_x,2}, \ldots, x_{1,N_y}, \ldots, x_{M_xN_y}\}^T. \tag{22}$$

Each block matrix is square of order M and tridiagonal. Since the matrix has a block-tridiagonal structure and is block-diagonally dominant, it is permissible to use the successive block-overrelaxation scheme (CUTHILL and VARGA (1959), BENSON and EVANS (1972)).

The optimal block-overrelaxation factor w_b was derived from the minmax expression by CUTHILL and VARGA (1959). Furthermore, since each block submatrix has a simple tridiagonal structure, the solution of each block of unknowns is obtained via the Thomas tridiagonal algorithm (MITCHELL 1969).

An optimal block-overrelaxation factor of $w_b = 1.63$ was used and 10–15 iterations were required before relative accuracy test of 1.10^{-4} was valid on the solution vector between successive iterations.

3 Stability Analysis

a) Stability analysis of the partly-implicit scheme

A necessary condition for the stability of the non-linear dicretized shallow-water equations is that the linearized version of the disturbance equations should be stable.

From Equation (1) a system of linearized perturbation equations is derived, using the following assumptions:

$$\phi = \bar{\phi} + \phi' \quad u = U + u' \quad V = v' \quad \frac{\partial \phi}{\partial y} = - FU \tag{23}$$

where $\bar{\phi}$ is the mean geopotential of the atmosphere as a function of y, f is a constant, U is a basic constant zonal wind and u', v' and ϕ' are perturbation quantities assumed independent of y. The products of perturbation quantities and their derivaties are neglected. We also have that

$$\frac{\partial U}{\partial t} = \frac{\partial U}{\partial x} = \frac{\partial U}{\partial y} = \frac{\partial \phi}{\partial x} = \frac{\partial \phi}{\partial t} = 0. \tag{24}$$

Upon dropping the primes, the linearized equations are

$$\frac{\partial u}{\partial t} + U\frac{\partial u}{\partial x} - fv = - \frac{\partial \phi}{\partial X}$$

$$\frac{\partial v}{\partial t} + U\frac{\partial v}{\partial x} + fu = 0 \tag{25}$$

$$\frac{\partial \phi}{\partial t} + U\frac{\partial \phi}{\partial x} - fUv = - \bar{\phi}\frac{\partial u}{\partial x}.$$

Harmonic perturbations for u, v, ϕ are assumed, of the form

$$F = F_0 e^{ik(x - ct)} \tag{26}$$

where c is the phase speed and k the wave number, and a predictor operator X is defined such that it transforms the function F(t) into F(t + Δt), i.e.

$$F(t + \Delta t) = XF(t) \tag{27}$$

and also

$$F(t - \Delta t) = X^{-1} F(t). \tag{28}$$

X is identical with the amplification factor.

Note also that the application to (26) of a centered space-difference operator results in

$$\frac{\partial F}{\partial X} \simeq D_{ox}F = (F(x + \Delta x) + F(x - \Delta x))\,(2\Delta x)^{-1} = \frac{i\sin k\Delta x}{\Delta x}\,F. \tag{29}$$

Inserting these expressions into Equation (25) and applying the partly-implicit scheme, we have

$$\begin{aligned}
&\frac{X-1}{\Delta t}\,\phi_{ij}^{n} + UX\,\frac{i\sin k\Delta x}{\Delta x}\,\phi_{ij}^{n} - fUv_{ij}^{n} + \bar{\phi}\,\frac{i\sin k\Delta x}{\Delta x}\,u_{ij}^{n} = 0\\
&\frac{X-1}{\Delta t}\,u_{ij}^{n} + UX\,\frac{i\sin k\Delta x}{\Delta x}\,u_{ij}^{n} - fv_{ij}^{n} + iX\,\frac{\sin k\Delta x}{\Delta x}\,\phi_{ij}^{n} = 0\\
&\frac{X-1}{\Delta t}\,v_{ij}^{n} + UX\,\frac{i\sin k\Delta x}{\Delta x} + Xfu_{ij}^{n} = 0.
\end{aligned} \tag{30}$$

Upon requiring the determinant of Equation (30) to vanish for non-trivial solutions of ϕ, u and v, we obtain

$$\begin{vmatrix}
\left(\frac{X-1}{\Delta t} + UX\,\frac{i\sin k\Delta x}{\Delta x}\right) & \bar{\phi}\,\frac{i\sin k\Delta x}{\Delta x} & -fu\\
X\,\frac{i\sin k\Delta x}{\Delta x} & \left(\frac{X-1}{\Delta t} + UX\,\frac{i\sin k\Delta x}{\Delta x}\right) & -f\\
0 & Xf & \left(\frac{X-1}{\Delta t} + UX\,\frac{i\sin k\Delta x}{\Delta x}\right)
\end{vmatrix} = 0 \tag{31}$$

Expansion of the determinant (31) yields the following cubic equation in the amplification factor X:

$$\begin{aligned}
&\left(\frac{X-1}{\Delta t} + \frac{i\sin k\Delta x}{\Delta x}\,UX\right)^{3} - f^{2}UX^{2}\,\frac{i\sin k\Delta x}{\Delta x} + Xf^{2}\left(\frac{X-1}{\Delta t} + \frac{i\sin k\Delta x}{\Delta x}\,UX\right)\\
&+ X\bar{\phi}\,\frac{\sin^{2}k\Delta x}{\Delta x^{2}}\left(\frac{X-1}{\Delta t} + \frac{i\sin k\Delta x}{\Delta x}\,UX\right) = 0.
\end{aligned} \tag{32}$$

For U = 0 and f = 0, (32) reduces to

$$\frac{1}{\Delta t^{3}}\,(X-1)\left[(X-1)^{2} + \bar{\phi}\,\frac{\Delta t^{2}}{\Delta x^{2}}\,\sin^{2}k\Delta x\;X\right] = 0$$

with the roots $X_1 = 1$ and

$$X_{2,3} = 1 - \frac{\bar{\phi}\,\Delta t^{2}}{2\Delta t^{2}}\,\sin^{2}k\Delta x \pm \sqrt{\bar{\phi}\,\frac{\Delta t^{2}}{\Delta x^{2}}\,\sin^{2}k\Delta x\left(\frac{\bar{\phi}\,\Delta t^{2}}{4\Delta x^{2}}\,\sin^{2}k\Delta x - 1\right)} \tag{34}$$

yielding the stability condition (i.e. $|X| \leqslant 1$):

$$\sqrt{\bar{\phi}}\,\frac{\Delta t}{\Delta x} < 2. \tag{35}$$

For other small values of U and f the stability condition remains about the same.
Note that the improvement over the two-dimensional explicit leap-frog scheme is $2\sqrt{2}$.

b) A proposed absolutely stable time-differencing scheme

Taking into account the structure of the shallow-water equations, we show here that in order to obtain an absolutely stable time-differencing scheme it is sufficient to couple implicitly two out of three equations constituting the system of shallow-water equations: viz the continuity equation and one of the momentum equations.

Let us couple implicitly u and ϕ.

The proposed scheme is then written as

$$\begin{aligned}
\phi_{ij}^{n+1} - \phi_{ij}^{n} &= \Delta t A_{\phi}(u_{ij}^{n+1}, v_{ij}^{n}, \phi_{ij}^{n+1}) \\
u_{ij}^{n+1} - u_{ij}^{n} &= \Delta t A_{u}(u_{ij}^{n+1}, v_{ij}^{n}, \phi_{ij}^{n+1}) \\
v_{ij}^{n+1} - v_{ij}^{n} &= \Delta t A_{v}(u_{ij}^{n+1}, v_{ij}^{n+1}, \phi_{ij}^{n+1}).
\end{aligned} \tag{36}$$

The same semi-momentum space discretization as in Equation (3) applies here.

Let us now perform a stability analysis of this scheme by considering again the perturbed linearized Equation (15) and implicitly coupling u and ϕ.

By using the same procedure as in Section 3(a) we obtain

$$\begin{aligned}
&\frac{X-1}{\Delta t}\,\phi_{ij}^{n} + UX\,\frac{i\sin k\Delta x}{\Delta x}\,\phi_{ij}^{n} - fUv_{ij}^{n} + \bar{\phi} X\,\frac{i\sin k\Delta x}{\Delta x} = 0 \\
&\frac{X-1}{\Delta t}\,U_{ij}^{n} + UX\,\frac{i\sin k\Delta x}{\Delta x}\,u_{ij}^{n} - fv_{ij}^{n} + iX\,\frac{\sin k\Delta x}{\Delta x}\,\phi_{ij}^{n} = 0 \\
&\frac{X-1}{\Delta t}\,V_{ij}^{n} + UX\,\frac{i\sin k\Delta x}{\Delta x}\,v_{ij}^{n} + Xfu_{ij}^{n} = 0.
\end{aligned} \tag{37}$$

Again requiring the determinant of Equation (37) to vanish for non-trivial solutions of ϕ, u and v we obtain

$$\begin{vmatrix}
\left(\frac{X-1}{\Delta t} + UX\frac{i\sin k\Delta x}{\Delta x}\right) & \bar{\phi} X\frac{i\sin k\Delta x}{\Delta x} & -fU \\
X\frac{i\sin k\Delta x}{\Delta x} & \left(\frac{X-1}{\Delta t} + UX\frac{i\sin k\Delta x}{\Delta x}\right) & -f \\
0 & Xf & \left(\frac{X-1}{\Delta t} + UX\frac{i\sin k\Delta x}{\Delta x}\right)
\end{vmatrix} = 0 \tag{38}$$

Expansion of the determinant (38) yields the following cubic equation in the amplification factor X:

$$\begin{aligned}
&\left(\frac{X-1}{\Delta t} + UX\frac{i\sin k\Delta x}{\Delta x}\right) - f\,UX\,\frac{\sin^2 k\Delta x}{\Delta x} + f\,X\left(\frac{X-1}{\Delta t} + UX\frac{i\sin k\Delta x}{\Delta x}\right) \\
&+ X^2 \bar{\phi}\,\frac{\sin^2 k\Delta x}{\Delta x^2}\left(\frac{X-1}{\Delta t} + \frac{i\sin k\Delta x}{\Delta x}UX\right) = 0.
\end{aligned} \tag{39}$$

For $f = 0$ and $U = 0$, (39) simplifies to

$$\frac{1}{\Delta t^3}(X-1)\left[(X-1)^2 + X^2 \phi \frac{\Delta t^2}{\Delta x^2} \sin^2 k\Delta x\right] \tag{40}$$

the roots of which are $X_1 = 1$ and

$$X_{2,3} = \frac{1 \pm i\sqrt{\bar{\phi}\dfrac{\Delta t^2}{\Delta x^2}\sin^2 k\Delta x}}{1 + \bar{\phi}\dfrac{\Delta t^2}{\Delta x^2}\sin^2 k\Delta x} \tag{41}$$

$$\text{i.e. } |X| = \frac{1}{\sqrt{1 + \bar{\phi}\dfrac{\Delta t^2}{\Delta x^2}\sin^2 k\Delta x}} . \tag{42}$$

As $\bar{\phi}\dfrac{\Delta t^2}{\Delta x^2}\sin^2 k\Delta x$ is always positive, we obtain $|X| < 1$, i.e. unconditional stability.

For small values of U and f the unconditional stability is preserved. The algebraic manipulation is cumbersome and will not be reproduced here.

The result obtained is due to a property of the shallow-water equations shown most clearly when they are written in the form given by Equation (11).

Note that only (u, Φ) are coupled in the x-direction, whereas only (v, Φ) are coupled in the y-direction.

Preparations are in progress for numerical experiments with this scheme.

The stability calculations derived in this Section serve only as indicators for the full non-linear shallow-water equations. A similar stability analysis can be performed for a (v, Φ) implicit coupling using a modified linearized version of the shallow-water equations.

c) Frequency analysis of the partly implicit scheme

Considering again the system of linearized shallow-water Equations (25) and applying to it the partly implicit scheme, and assuming ϕ, u and v to be represented by

$$F = F_0 e^{ik(x - ct)} \tag{43}$$

one finds

$$\frac{F(t+\Delta t) - F(t)}{\Delta t} = S' \frac{\partial F}{\partial t} . \tag{44}$$

Using Equation (43) we obtain

$$\frac{\partial F}{\partial t} = -ikcF \tag{45}$$

but

$$\frac{F(t+\Delta t) - F(t)}{\Delta t} = \frac{e^{-(ikc\Delta t)} - 1}{\Delta t} F. \tag{46}$$

Combining the results we have

$$-\,ikcFS' = \left(\frac{e^{-ikc\Delta t}-1}{\Delta t}\right)F \tag{47}$$

i.e.

$$S' = \frac{1-e^{-ikc\Delta t}}{ik\Delta t} \tag{48}$$

$$|S'| = \sqrt{2}\,\frac{\sqrt{1-\cos kc\Delta t}}{kc\Delta t} = \frac{2\sin\frac{kc\Delta t}{2}}{kc\Delta t} < 1. \tag{49}$$

Also

$$F(t+\Delta t) = c'F(t)$$

$$c' = e^{-ikc\Delta t} \qquad |c'| = 1. \tag{50}$$

Applying those operators to the linearized Equations (25) one obtains

$$S'\frac{\partial\phi}{\partial t} + Uc'\frac{\partial\phi}{\partial y} - fUv + \overline{\phi}\frac{\partial u}{\partial x} = 0$$

$$S'\frac{\partial u}{\partial t} + Uc'\frac{\partial u}{\partial x} - fv + c'\frac{\partial\phi}{\partial x} = 0 \tag{51}$$

$$S'\frac{\partial v}{\partial t} + Uc'\frac{\partial v}{\partial x} + fc'u = 0.$$

Performing analytical differentiation with respect to the space coordinate x, using Equation (43) one obt

$$\frac{\partial F}{\partial x} = ikF \tag{52}$$

so that by denoting by C_c the phase speed of the computed physical modes we obtain

$$ik(Uc' - S'C_c)\,\phi + ik\overline{\phi}\,u - fUv = 0$$

$$ikc'\phi - fv + ik(Uc' - S'C_c)u = 0$$

$$fc'u + ik(Uc' - S'C_c)v = 0 \tag{53}$$

Expanding the determinant of the coefficients of the above equations (which must vanish for a solution) we have a cubic frequency equation for the computed phase velocities. Assumung $\overline{\phi} >> (Uc' - S'C_c)^2$ we obtain the approximate root for the meteorological mode given by

$$S'C_{c_1} = \frac{U}{1+\frac{f^2c'}{k^2c'\overline{\phi}}} = \frac{U}{1+\frac{f^2}{k^2\overline{\phi}}} \tag{54}$$

Next, by assuming $C_c >> U$ (and neglecting the term $ikf^2c'^2U$) we obtain the two approximate roots for the gravitational mode which are

$$S'C_{c_{2,3}} = Uc' \pm \sqrt{c'\overline{\phi} + \frac{f^2c'}{k^2}} = Uc' \pm \sqrt{c'\left(\overline{\phi}+\frac{f^2}{k^2}\right)}\,. \tag{55}$$

We compared our results with the true phase velocities given by

$$c_1 = U + 2\sqrt{\frac{-a}{3}}\cos\left(\frac{t}{3} + \frac{4}{3}\pi\right) \approx \frac{U}{1 + \frac{f^2}{k^2\bar{\phi}}}$$

$$c_2 = U + 2\sqrt{\frac{-a}{3}}\cos\frac{t}{3} \approx U + \sqrt{\bar{\phi} + \frac{f^2}{k^2}} \qquad (56)$$

$$c_3 = U + 2\sqrt{\frac{-a}{3}}\cos\left(\frac{t}{3} + \frac{2}{3}\pi\right) \approx U - \sqrt{\bar{\phi} + \frac{f^2}{k^2}}$$

where

$$a = -\frac{f^2}{k^2} - \bar{\phi},\ b = -\frac{f^2 U}{k^2},\ t = \tan^{-1}\left[\frac{-4a^3}{27b^2} - 1\right]^{\frac{1}{2}} \qquad (57)$$

(see THOMPSON (1961), KURIHARA (1965)), and with the semi-implicit computed phase velocities which are

$$S'C_{c3} = \frac{U - (f^2 U/k^2 c'\bar{\phi})}{(1 + f^2/k^2 x'^2 \bar{\phi})} \quad \text{and} \quad S'C_{c2,3} = U \pm \sqrt{c'^2\bar{\phi} + \frac{f^2}{k^2}} \qquad (58)$$

(see KWITZAK (1970)). $c'_{(\text{semi-implicit})} = \cos k\, C_c \Delta t$.

It was found that for the meteorological mode the ratio of the partly-implicit computed mode to the true mode is 1.

The semi-implicit scheme on the contrary accelerates the meteorological mode.

For the gravitational waves there is a certain deceleration over the true phase speeds but this is considerably less than the corresponding decleration caused by using the semi-implicit time differencing scheme.

This leads to the conclusion that the proposed partly-implicit time-differencing scheme can be useful in simulating the geostrophic adjustment process.

A similar conclusion was reached by JANJIC and WIIN-NIELSEN (1977) who state that unless very short time-steps are employed, the semi-implicit scheme is not adequate for simulating the process of geostrophic adjustment. Experiments by MCPHERSON and KISTLER (1973) also verified the delayed damping of the gravity waves by the semi-implicit scheme.

4 Numerical Results

a) Computations with coarse-mesh data

A limited-area integration was performed using time-dependent boundary values derived from coarse-mesh model data defined on a rectangular of grid 22 X 30 points. The space increment of the coarse-mesh model was $\Delta x = \Delta y = 400$ km and the initial conditions of Section 4(b) were used.

The limited-area integration region was a rectangular domain with dimensions 4400 km X 6000 km, with $\Delta x = \Delta y = 200$ km, contained within the coarse-mesh domain.

Noting that the shallow-water equations constitute a hyperbolic system, the number of boundary conditions to be prescribed at a given point of the limited-area boundary should equal the number of characteristics passing into the region at that boundary point. In this experiment the boundary conditions adopted were those applied by ELVIUS and SUNDSTRÖM (1972) and which have been shown by OLIGER and SUNDSTRÖM (1977) to be well-posed.

The boundary conditions are that both the value of the combination

$$v_n - 2\sqrt{\phi}$$

and the value of v_{tang} are specified at inflow points of the boundary ($v_n < 0$), while only the value of $v_n - 2\sqrt{\phi}$ is specified at the outflow points of the boundary ($v_n > 0$). A time-step of $\Delta t = 2700$ s was used. The initial height field and the height fields at different times are shown in Figures 2 to 5. (59)

The results for 72-hour integrations show good conservation of the total energy and the mean height integral invariants (Figures 6–7), and are an indication of the well-posedness of the boundary conditions.

b) Two space-dimensional channel computations

We use the initial condition employed by GRAMMELTVEDT (1969) describing a westerly jet flow with north-south perturbations of different wavelenghts and amplitudes along the zonal axis of the jet. This initial condition was also employed by GUSTAFSSON (1971) and FAIRWEATHER and NAVON (1977) and thus provides a basis for comparison.

The initial height field is

$$h(x, y) = H_0 + H_1 \tanh\left(\frac{9(D/2 - y)}{2D}\right) + H_2 \operatorname{sech}^2\left(\frac{9(D/2 - y)}{D}\right) \sin\frac{2\pi x}{L}. \tag{60}$$

The initial velocity components u and v are calculated from the geostrophic approximation, i.e.

$$u = -\left(\frac{g}{f}\right)\frac{\partial h}{\partial y}$$

$$v = \left(\frac{g}{f}\right)\frac{\partial h}{\partial x}. \tag{61}$$

The dimensions of the channel were

$$L = 4400 \text{ km} \quad \text{and} \quad D = 6000 \text{ km}$$

and the following constant values were adopted:

$$H_0 = 2000 \text{ m},\ H_1 = 220 \text{ m},\ H_2 = 133 \text{ m},$$

$$g = 10 \text{ m sec}^{-2},\ \hat{f} = 10^{-4} \text{ sec}^{-1},\ \beta = 1.5 \times 10^{-11} \text{ sec}^{-1}\text{ m}^{-1} \tag{62}$$

where

$$f = \hat{f} + \beta\left(y - \frac{D}{2}\right) \tag{63}$$

is the Coriolis parameter.

The scheme was run with the resolution

$$\Delta x = \Delta y = 200 \text{ km}$$

and a time step of $\Delta t = 3600$ sec.

Periodic boundary conditions were assumed in the x direction

$$u(x, y, t) = u(x + L, y, t)$$

$$v(x, y, t) = v(x + L, y, t)$$

$$\phi(x, y, t) = \phi(x + L, y, t) \tag{64}$$

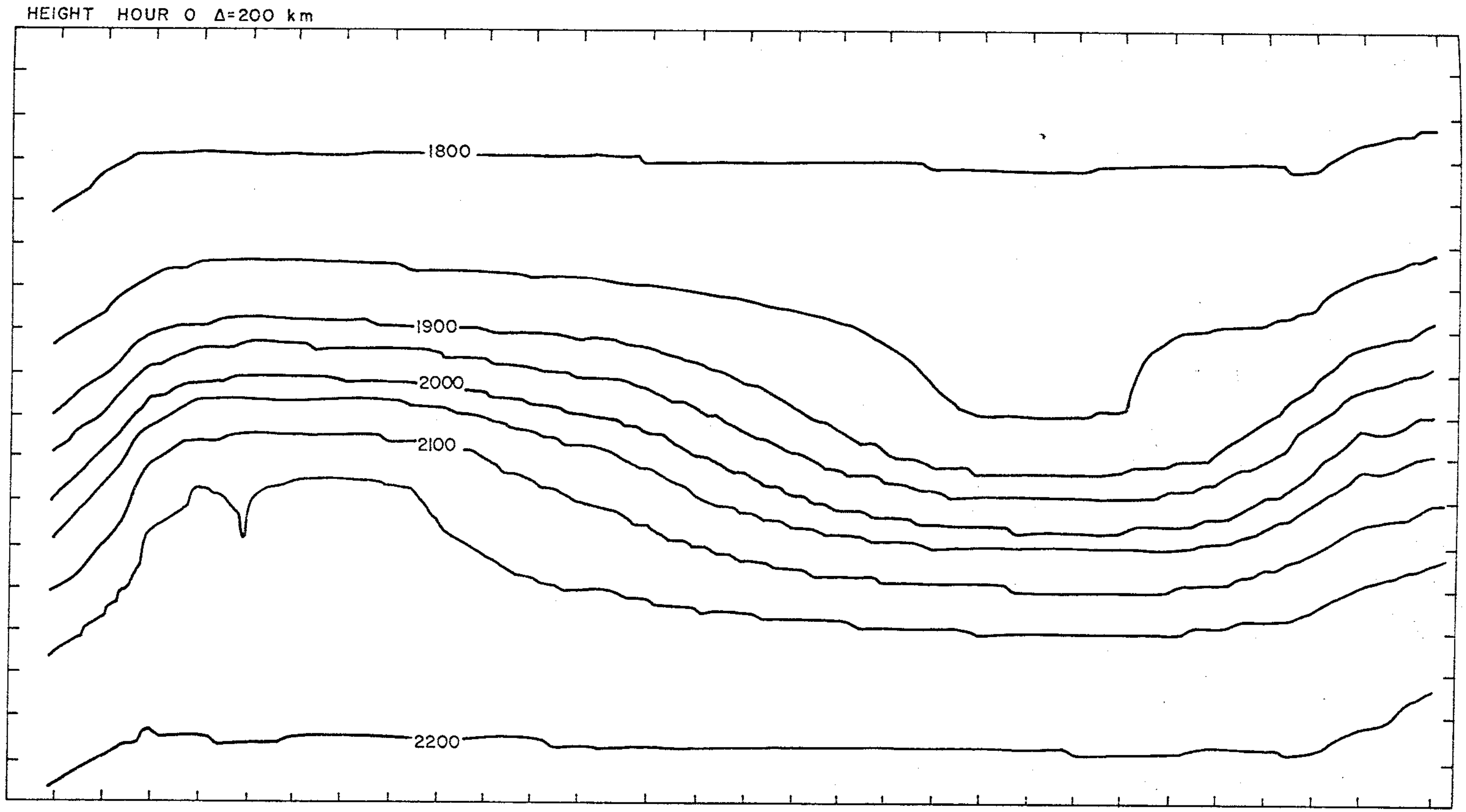
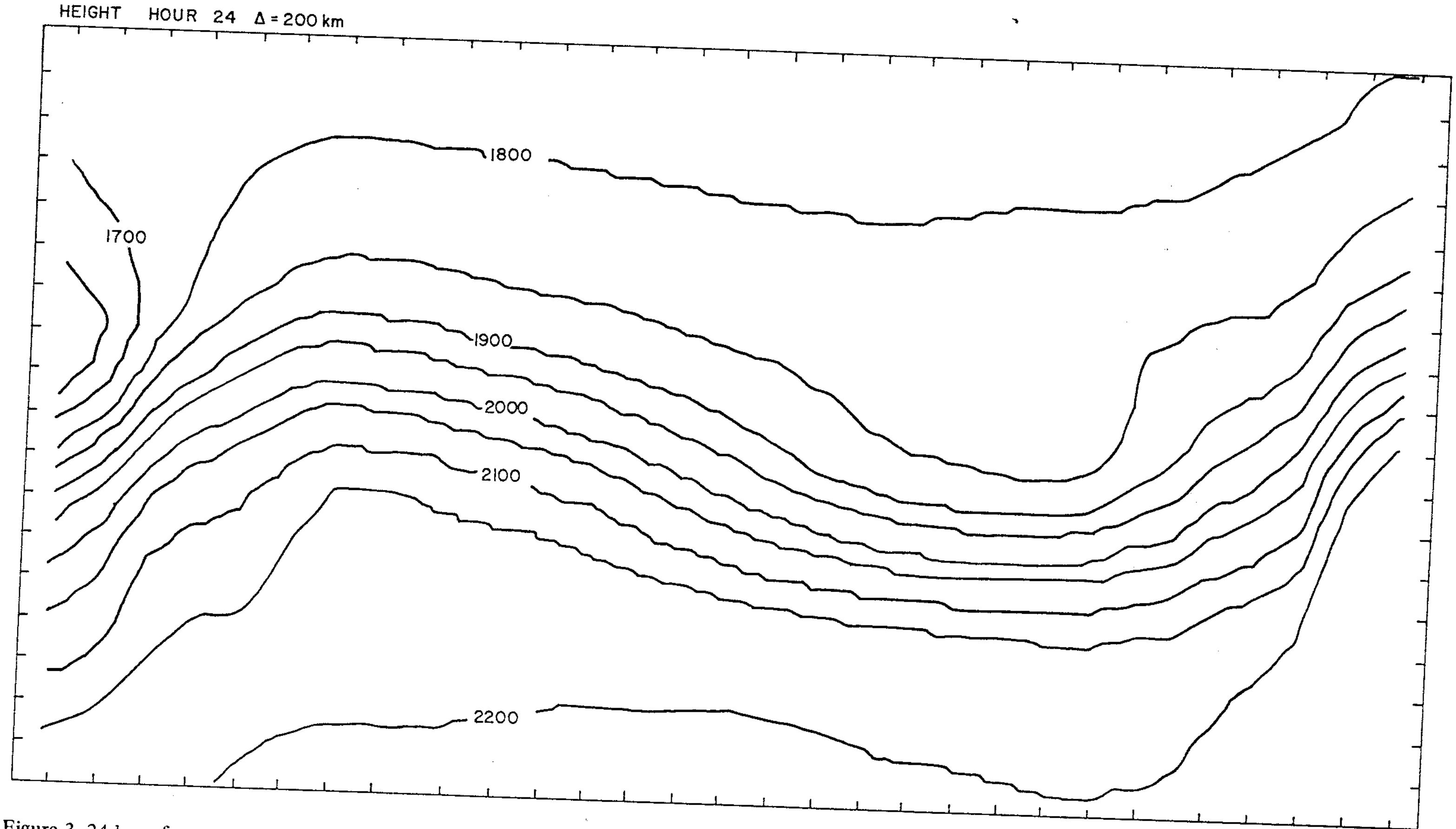
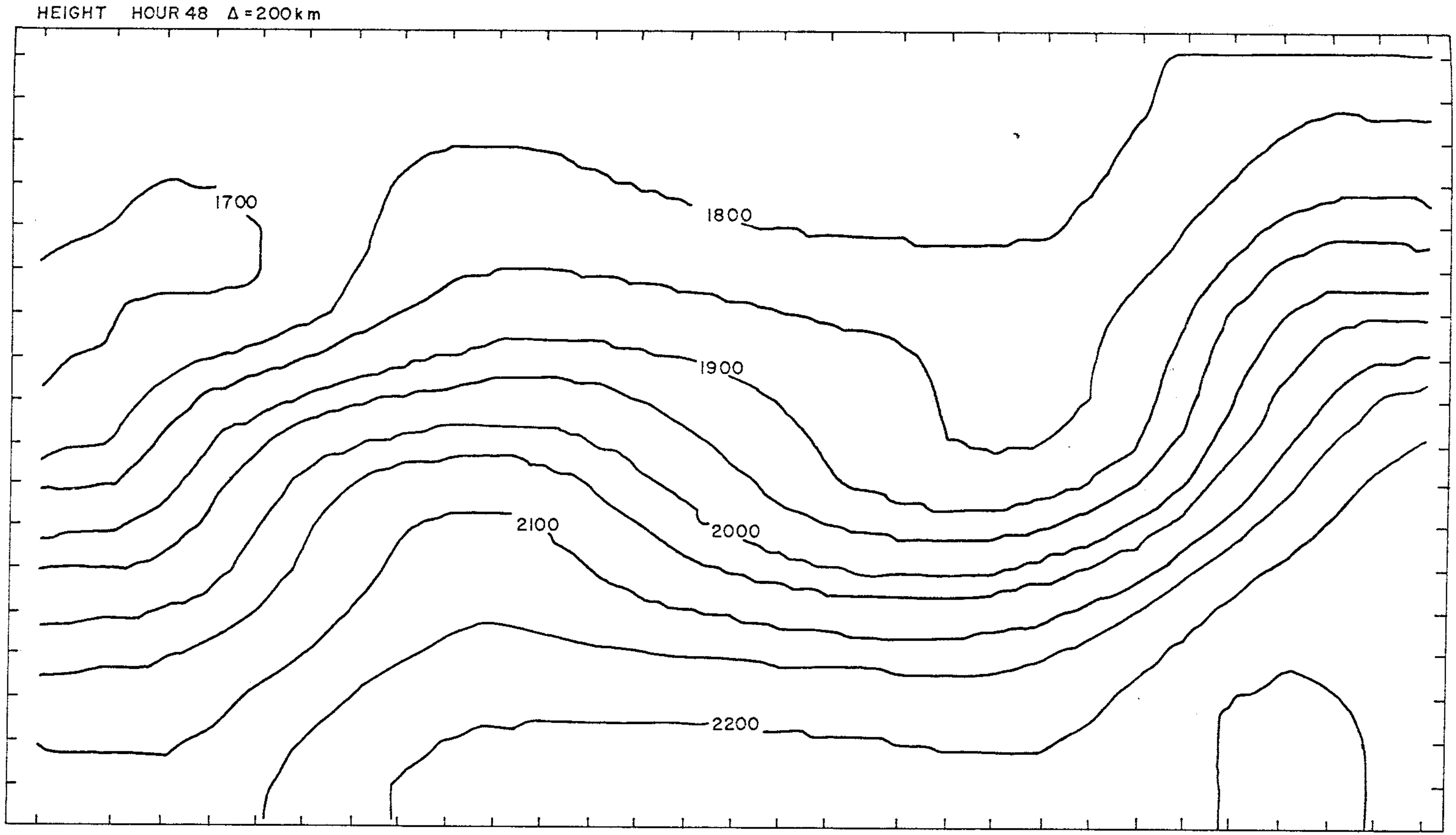


Figure 2 Initial height field for limited area domain. Contours are drawn every 50 m.

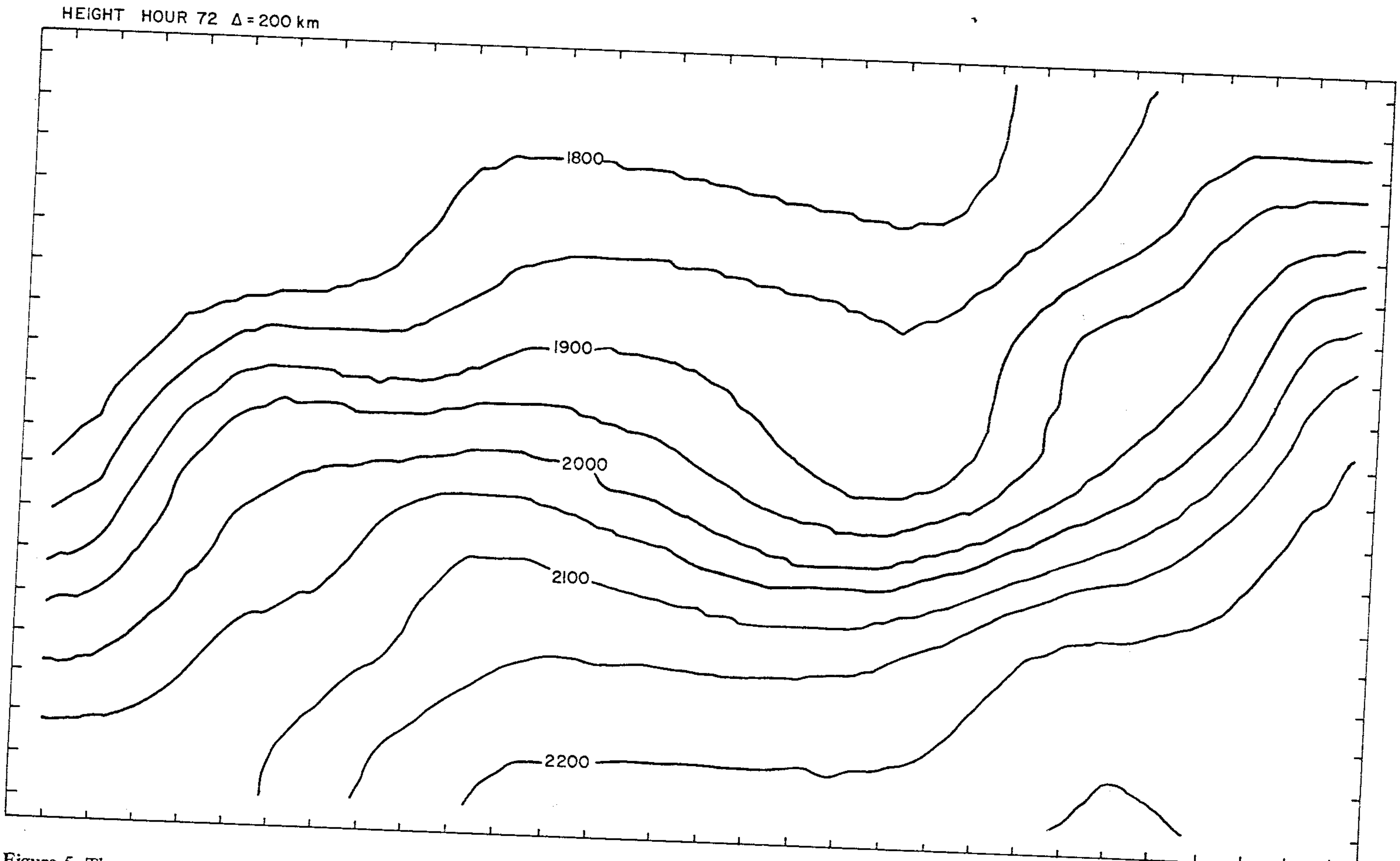
Bild 2 Anfangsfeld der Höhe für das begrenzte Gebiet. Das Konturenintervall ist 50 m.



- Figure 3 24-hour forecast by the partly implicit scheme and using time dependent boundary values from a coarse mesh model.
- Bild 3 24 Stunden-Vorhersage mit Hilfe des teilweise impliziten Verfahrens unter Benutzung von zeitabhängigen Randwerten des grobmaschigen Modells.



- Figure 4 The same as in Figure 3, but for 48 h
- **Bild 4** Wie in Bild 3, jedoch für 48 h



- **Figure 5** The same as in Figure 3, but for 72 h
- **Bild 5** Wie in Bild 3, jedoch für 72 h

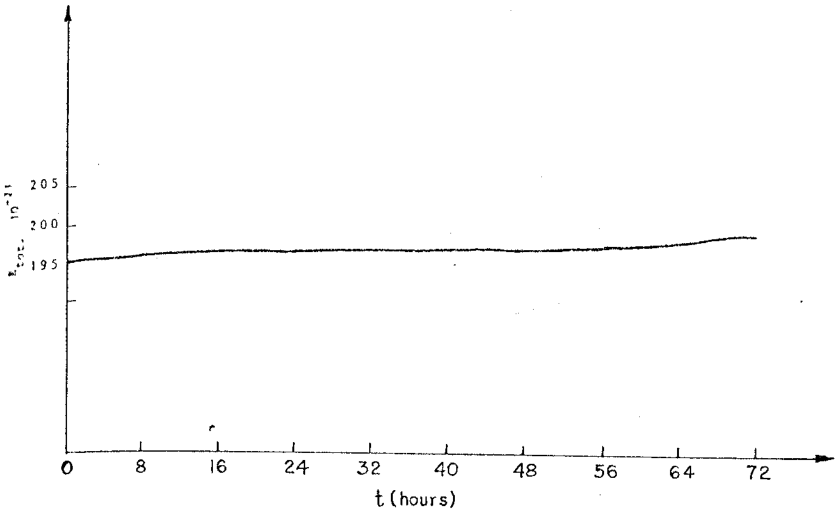


Figure 6 Total energy as a function of time for the limited-area domain (relative units).

Bild 6 Gesamte Energie in Abhängigkeit von der Zeit für das begrenzte Gebiet in relativen Einheiten.

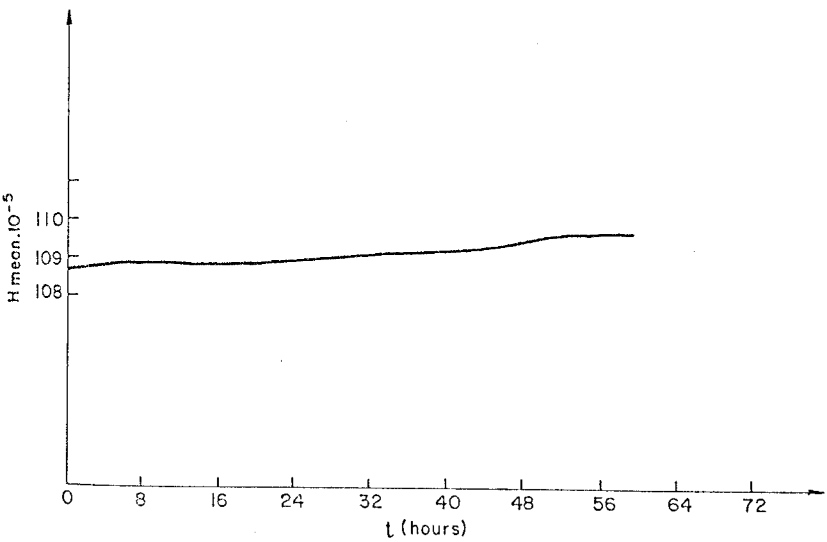


Figure 7 Mean height as a function of time for the limited-area domain (relative units).

Bild 7 Mittlere Höhe in Abhängigkeit von der Zeit für das begrenzte Gebiet in relativen Einheiten.

and the single boundary condition

$$v(x, 0, t) = v(x, D, t) = 0$$

in the y direction. Note that no boundary conditions are necessary for u and ϕ at y = 0, D. (65)

Using those boundary conditions, the energy

$$\frac{1}{2g}\int_0^L\int_0^D (u^2 + v^2 + \phi)\phi\, dx dy \qquad (66)$$

is independent of time (GUSTAFSSON (1971)).

As may be gathered from Table I, an almost perfect conservation of the total energy and mean height was obtained after a 48-hour integration.

Table I Mean height and total energy using the partly-implicit scheme for 48-hour forecast

Tabelle I Mittlere Höhe und gesamte Energie wie sie sich bei Benutzung des teilweise impliziten Verfahrens für eine 48 Stunden-Vorhersage ergeben.

Mean height at different integration times (given by subscript in hours)	Total energy at different integration times (given by subscript in hours)
h_0 = 2000 m	E_0 = 5.616926 E + 20
h_{24} = 1999.98 m	E_{24} = 5.617429 E + 20
h_{48} = 1999.99 m	E_{48} = 5.617067 E + 20

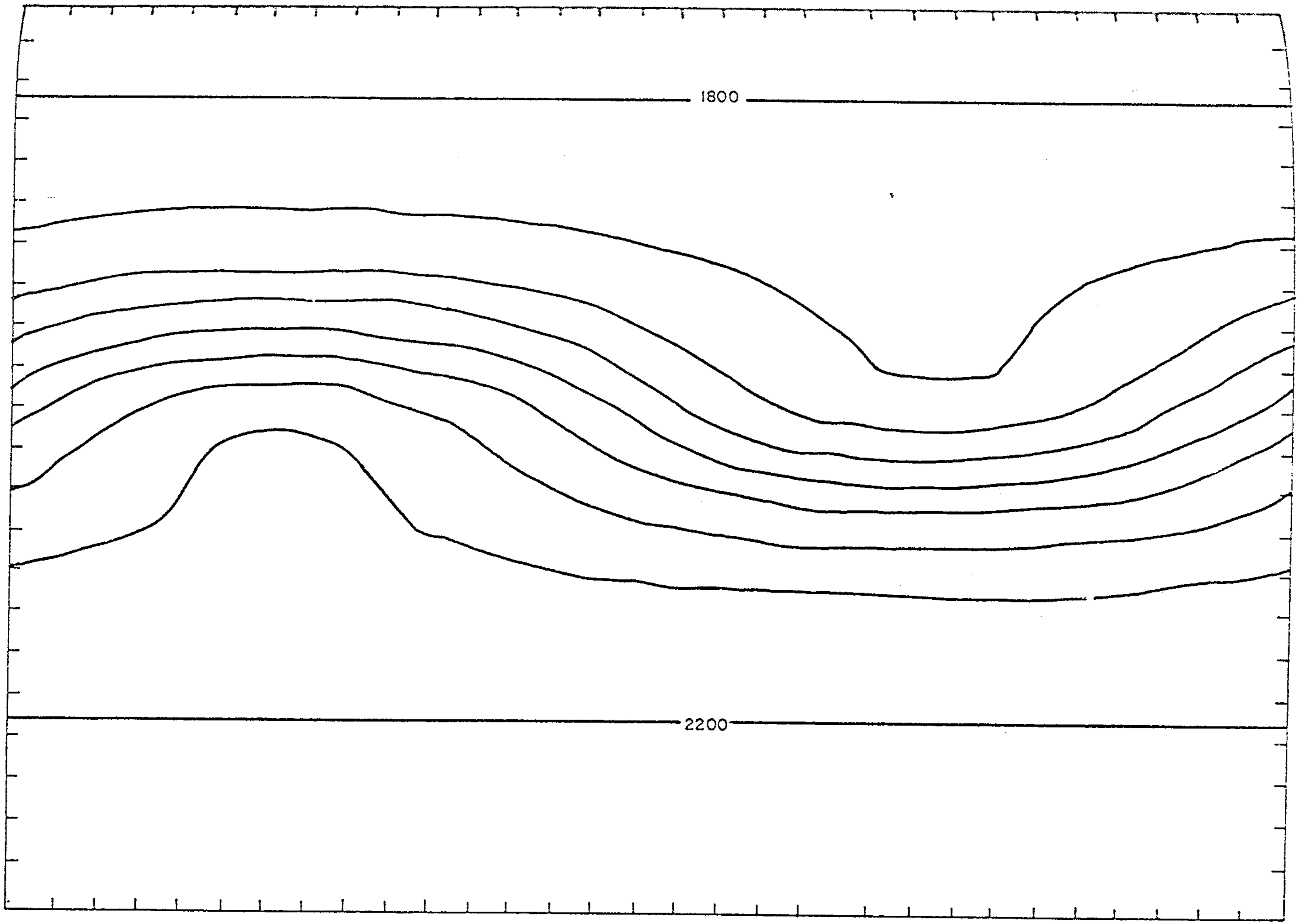
Figures 8 and 9 show the initial height field and the height field after 48 hours for the two-space dimensional channel computation using the partly-implicit scheme with Δt = 3600 sec, while Figure 10 shows the height field after 48 hours using the GUSTAFSSON (1971) QN3 (M = 6) method with Δt = 3600 sec. and with the same initial and boundary conditions. The results are very similar and show that for short-range forecasts the partly-implicit scheme performs satisfactorily.

5 Conclusions

The partly-implicit scheme was found to perform satisfactorily for all the test computations. The correct formulation of the boundary conditions was essential in the limited-area model, particularly in order to control the propagation of short gravity waves.

The physical reason for the proposed partly-implicit scheme is that in implicit schemes one exceeds the CFL condition by slowing down the phase speeds of the inertia-gravity waves. On the other hand, inertia-gravity waves constitute the mechanism for bringing the flow and pressure fields into geostrophic balance and errors in their estimation affect the adjustment time scales of numerical models. It is suggested that the partly-implicit scheme should be used in the initialization stages of a numerical weather prediction model to simulate the geostrophic adjustment process and that then a switch-over should be made to the semi-implicit scheme with its higher computational efficiency.

Numerical experimentation is required to test the proposed absolutely stable time differencing scheme with the full non-linear barotropic model equations.



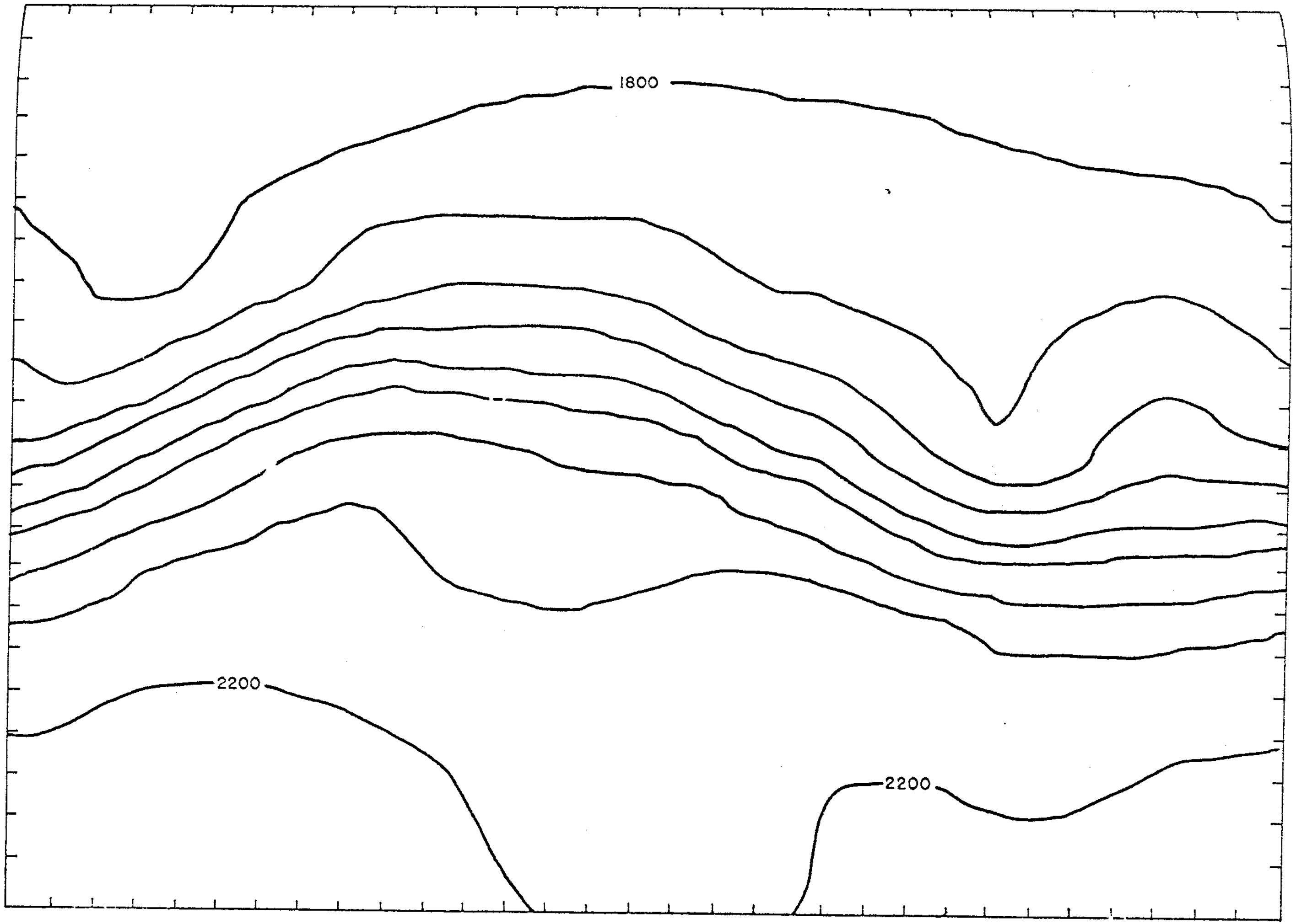
• **Figure 8** Initial height field for the two space dimensional channel.

• **Bild 8** Anfangsfeld der Höhe für den räumlich zweidimensionalen Kanal

Figure 9 48-hour forecast by the partly implicit method for the two-space dimensional channel.

Bild 9 48 Stunden-Vorhersage mit Hilfe des [illegible]

- **Figure 9** 48-hour forecast by the partly implicit method for the two-space dimensional channel.
- **Bild 9** 48 Stunden-Vorhersage mit Hilfe des teilweise impliziten Verfahrens für den räumlich zweidimensionalen Kanal.



- **Figure 10** 48-hour forecast by the GUSTAFSSON QN3 (M = 6) method for the two-space dimensional channel.
- **Bild 10** 48 Stunden-Vorhersage mit Hilfe der GUSTAFSSONschen QN3 (M = 6) Methode für den räumlich zweidimensionalen Kanal.

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