A dual-weighted POD approach for 4D-Var adaptive mesh ocean modelling

F. Fang[1], C.C. Pain[1], I.M. Navon[2], M.D. Piggott[1], G.J. Gorman[1], P.A. Allison[1] and A.J.H. Goddard[1]

[1] Applied Modelling and Computation Group
Department of Earth Science and Engineering
Imperial College London, UK

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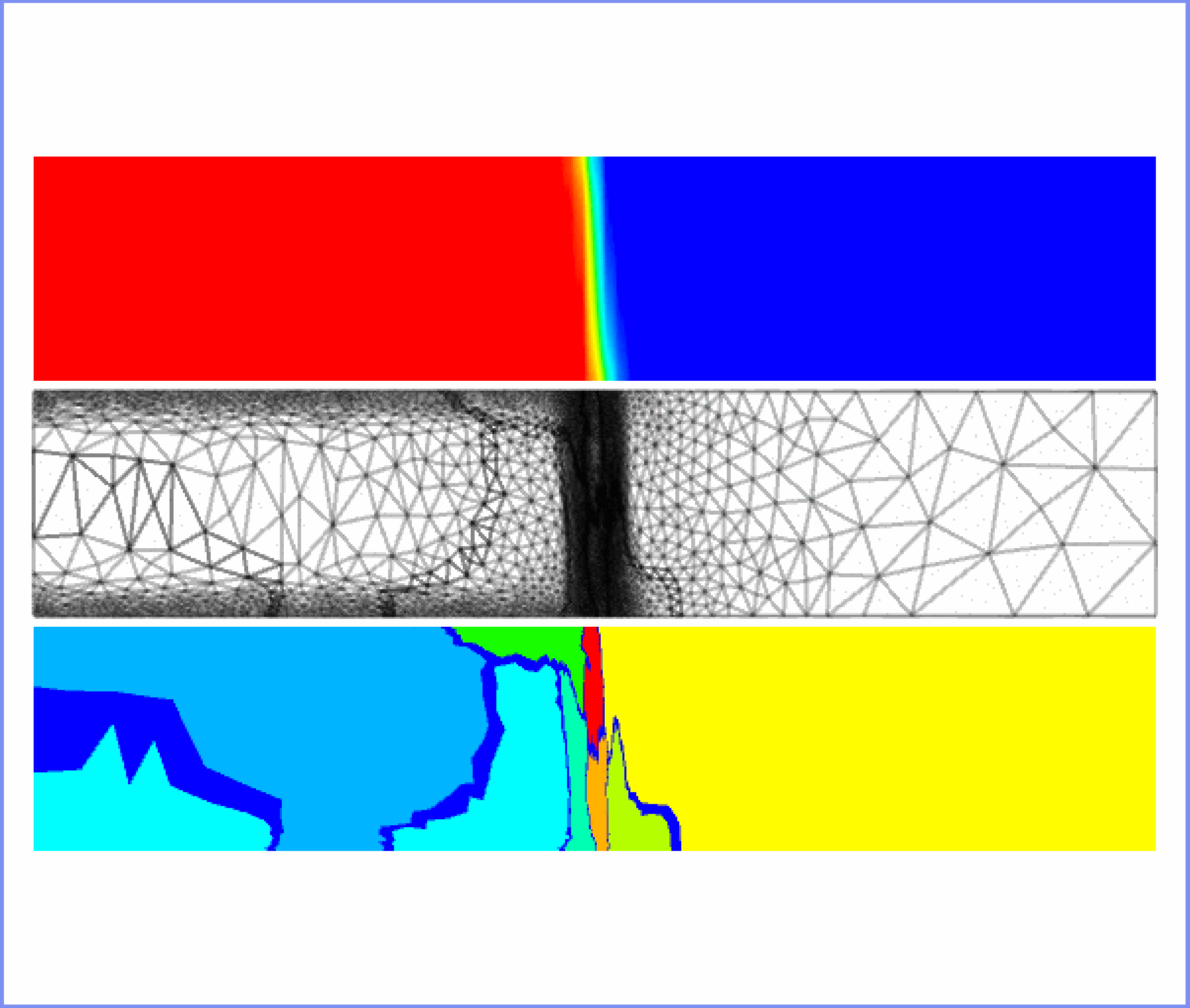
[2] School of Computational Science and Department of Mathematics
Florida State University

Outline of talk

- Aims and Objectives
- Proper Orthogonal Decomposition (POD) in ICOM
- Mesh adaptivity with POD
- Goal-based error measurement
- Test cases
- Conclusion

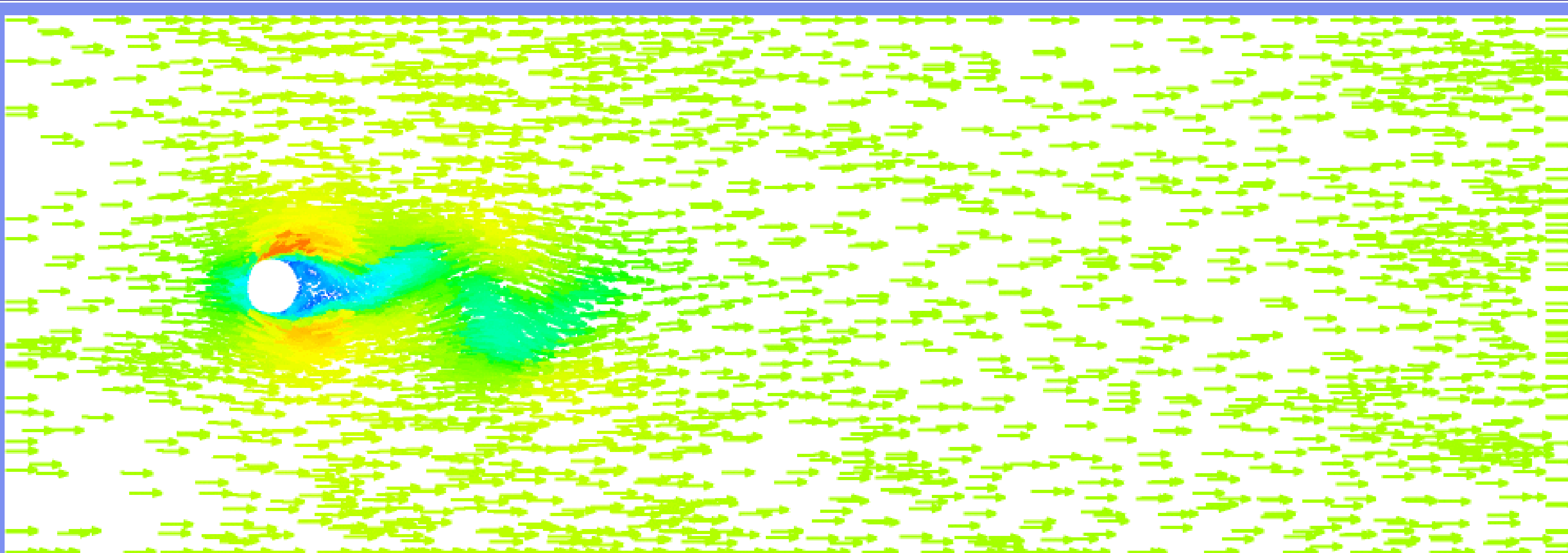
Intro. to ICOM e.g. Lock exchange problem

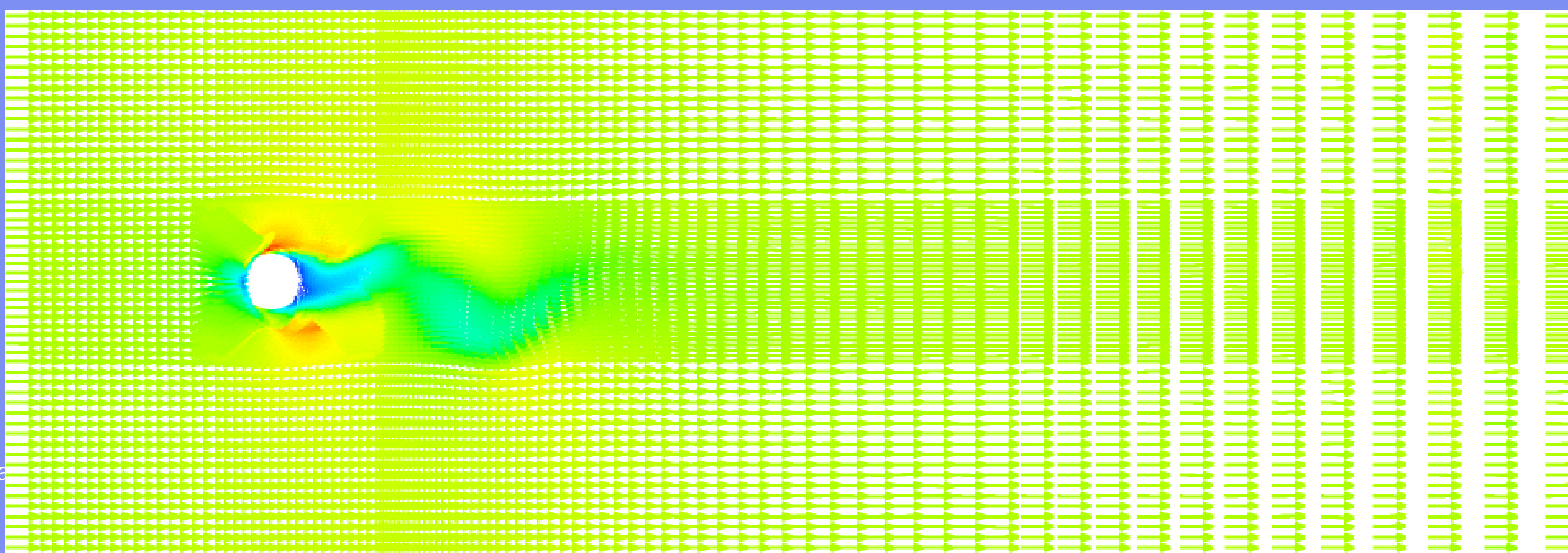


The two colours represent the different density. The heavier one sinks and runs under the lighter. Kelvin-Helmholtz billows are created at the interface due to shear.

Flow past a cylinder

(Re = 100, Minimum mesh size= 0.04, Maximum mesh size =1) forward (top), POD (bottom) (20 basis functions, 41 snapshots)





Aims and Objectives

----Develop a reduced order POD controller for a novel advanced mesh adaptive finite element model which includes many recent developments in ocean modelling.

In particular, our aim is to develop a new goal-based approach to:

- guide the mesh adaptivity and inversion;
- estimate error and optimise the POD bases.

What is Proper Orthogonal Decomposition (POD)?

POD- A numerical procedure that can be used to extract a basis for a model decomposition from an ensemble of signals.

-- originally proposed by Kosamibi, Loeve and Karhunen;
-- also known as Principal Components analysis (PCA) in statistics;
Empirical Orthogonal Function (EOF) in oceanography and meteorology.

The variables can be expressed as an expansion in Φ_k

$$U^{POD}(t,x,y,z)=\overline{U}(x,y,z)+\sum_{m=1}^{M}\alpha_m(t)\Phi_m(x,y,z)$$

The original PDE → The reduced order ODE

POD-based reduced order model

Define the mean of variables $U = (u, v, w, p)$: $\overline{U} = \frac{1}{K}\sum_{i=1}^{K} U^i$

The spatial correlation matrix $\mathrm{H}_{i,j} = \int_{\Omega} (U^i - \overline{U})(U^j - \overline{U}) d\Omega;\ 1 \le i, j \le K$

Solve the eigenvalue problem: $Hv = \lambda v$

The POD basis functions:

$$\Phi_m = \sum_{i=1}^{K} (v_m)_i (U^i - \overline{U}),\quad m = 1, \ldots, M$$

Energy: $I(M) = \frac{\sum_{i=1}^{M} \lambda_i}{\sum_{i=1}^{K} \lambda_i}$

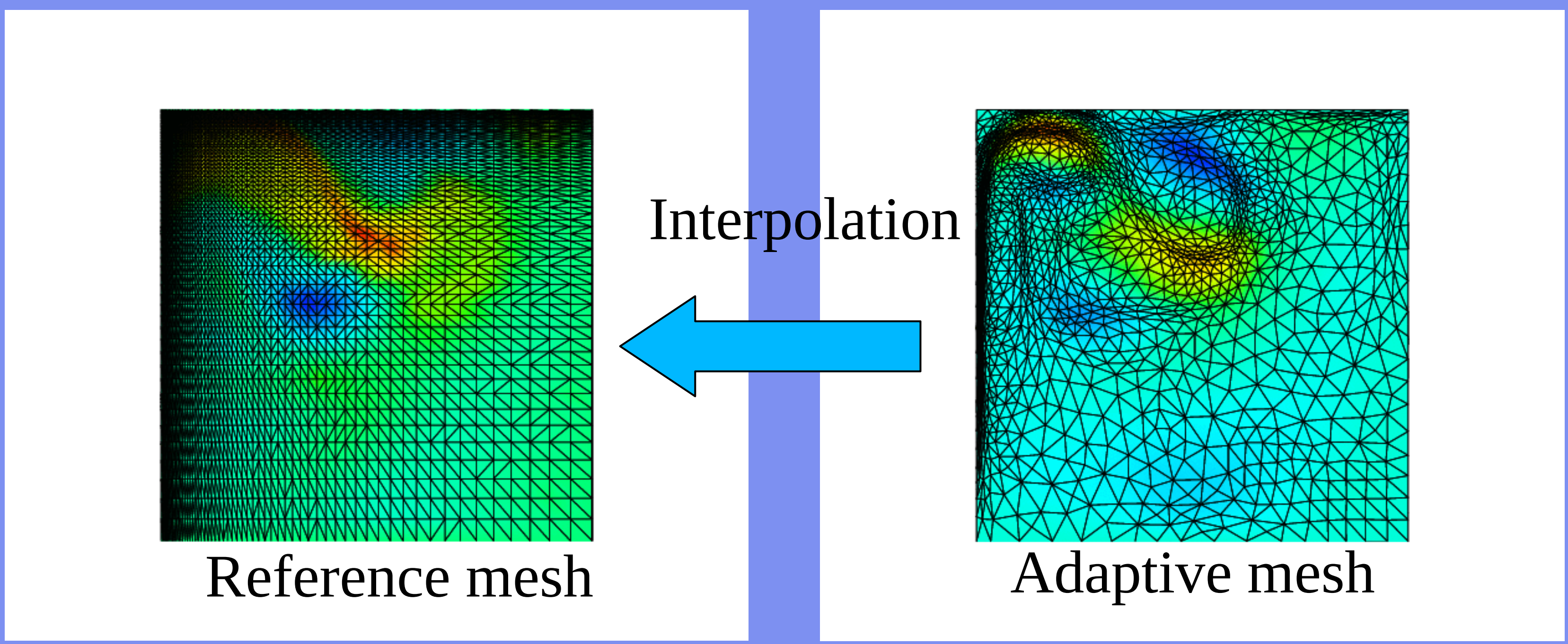
The variables can be expressed as an expansion in Φ_k

$$U^{POD}(t, x, y, z) = \overline{U}(x, y, z) + \sum_{m=1}^{M} \alpha_m(t) \Phi_m(x, y, z)$$

Mesh adaptivity in the POD model

Challenge:

The snapshots can be of different length at different time levels



A functional or goal

A functional can be defined as the model reduction error or solution which is of interest in a target region.

$$\Im(\psi) = \int_{\Omega} f(\psi)\, dV$$

The functional is used to

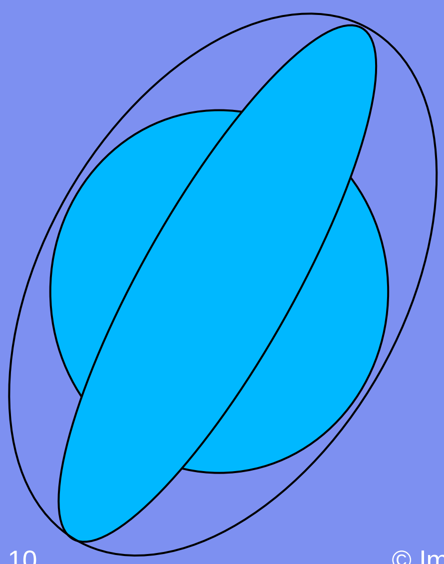
- optimise uncertainties (inversion problem) in models;
- determine the error measure for mesh adaptivity;
- optimise the POD bases.

The goal-based error measure approach for mesh adaptivity

$$\bar{\mathbf{M}}_i = \frac{\gamma}{|\bar{\epsilon}_i|}\left|\bar{\mathbf{H}}_i\right| .$$

$$\bar{\mathbf{M}}_i^* = \frac{\gamma}{|\bar{\epsilon}_i^*|}\left|\bar{\mathbf{H}}_i^*\right|$$

$$\bar{\bar{\mathbf{M}}}_i^{\mathcal{G}_I} = \mathcal{G}_I\left(\bar{\mathbf{M}}_i , \bar{\mathbf{M}}_i^*\right) .$$



To satisfy the goal, the minimal ellipsoid is obtained by superscribing both ellipses and used for mesh adaptivity

Dual-weight POD approach

A dual weighted method is developed to analyse the error of models and find an optimal POD basis.

To maximise the accuracy of the functional, a weighted diagonal Matrix $\tilde{M} \in R^{\mathcal{N} \times \mathcal{N}}$ is introduced to the snapshots

$$V_k = V_{k,i},\ 1 \le i \le \mathcal{N}$$

$$\tilde{A} = \left(\tilde{\omega}_1^{\frac{1}{2}} V_1, \ldots, \tilde{\omega}_k \tilde{M}^{\frac{1}{2}} V_k, \ldots, \tilde{\omega}_K \tilde{M}^{\frac{1}{2}} V_K \right)$$

$$\tilde{\omega}_i^l = (\delta F_i^l)^{\frac{1}{2}}$$

$$(\delta F_i^l)^{forward} \approx \frac{1}{NL} \sum_{il=1}^{NL} v_{il}^T \left[\sum_{l=1}^{\mathcal{M}} \left| \lambda_i^{*l,n} \right| \left| \mathbf{H}_i^{l,n} \right| \right] v_{il}$$

$$(\delta F_i^l)^{adjoint} \approx \frac{1}{NL} \sum_{il=1}^{NL} v_{il}^T \left[\sum_{l=1}^{\mathcal{M}} \left| r_i^{l,n} \right| \left| \mathbf{H}_i^{*l,n} \right| \right] v_{il}$$

$$\tilde{\omega}_i^l = (\delta F_i^l)^{\frac{1}{2}} = \left(min\{ (\delta F_i^l)^{forward}, (\delta F_i^l)^{adjoint} \} \right)^{\frac{1}{2}}$$

Case: Gyre (Re = 400)

Computational domain: 1000 km x 1000 km

$\rho = 1000 \quad \beta = 1.8 \times 10^{-11} \quad \tau_0 = 0.1$

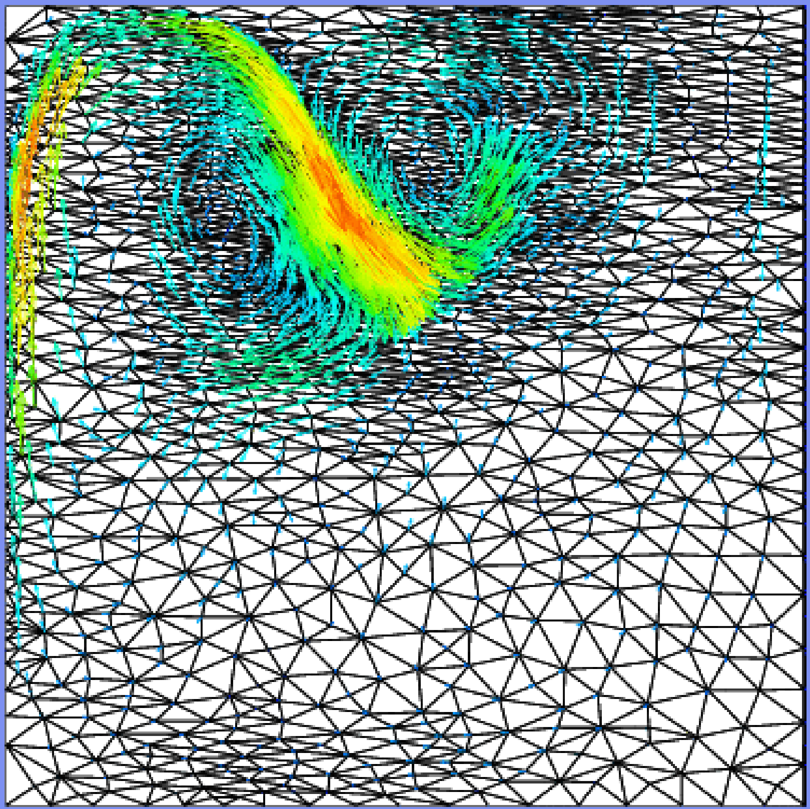
Assimilation period: 200 days   Time step: 3 hrs

Aim: to find an optimised mesh for the reduced forward and adjoint models

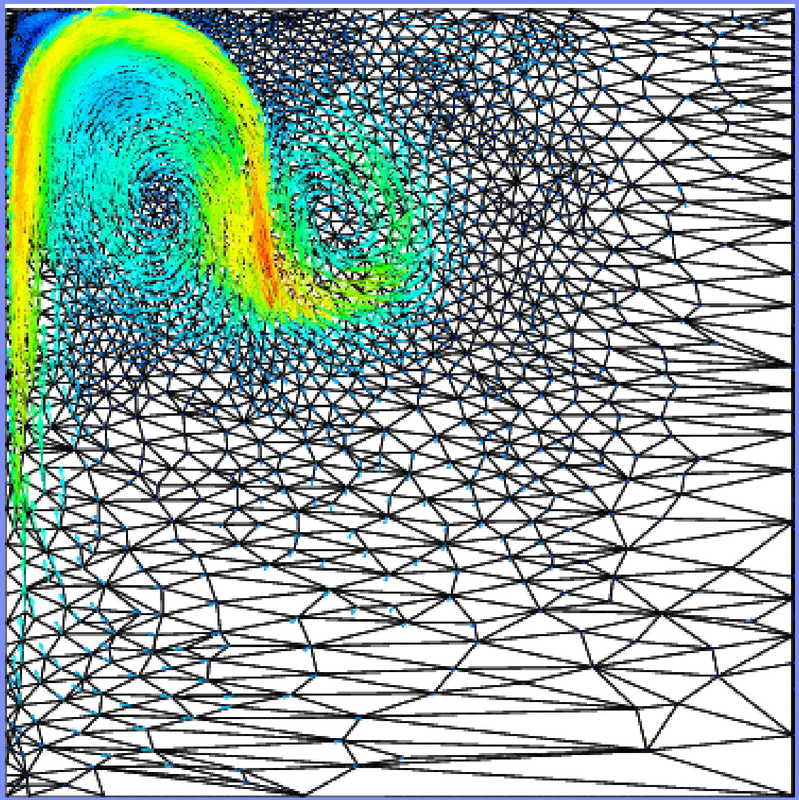
Run the reduced forward and adjoint models to find

$$\bar{\bar{\mathbf{M}}}_i^{\mathcal{G}_f} = \mathcal{G}_f\left(\bar{\mathbf{M}}_i, \bar{\mathbf{M}}_i^*\right).$$

Optimised adaptive mesh for the reduced order forward and adjoint models

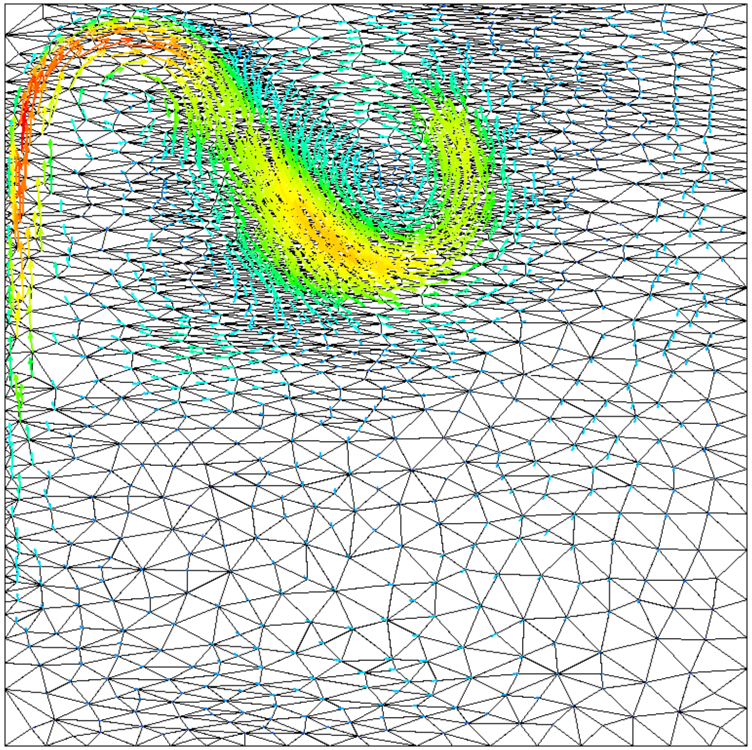


Inversion



Vorticity

Optimised adaptive mesh for the reduced order forward and adjoint models



Case 4: Gyre – inversion of initial conditions

Objective Function used in Inversion

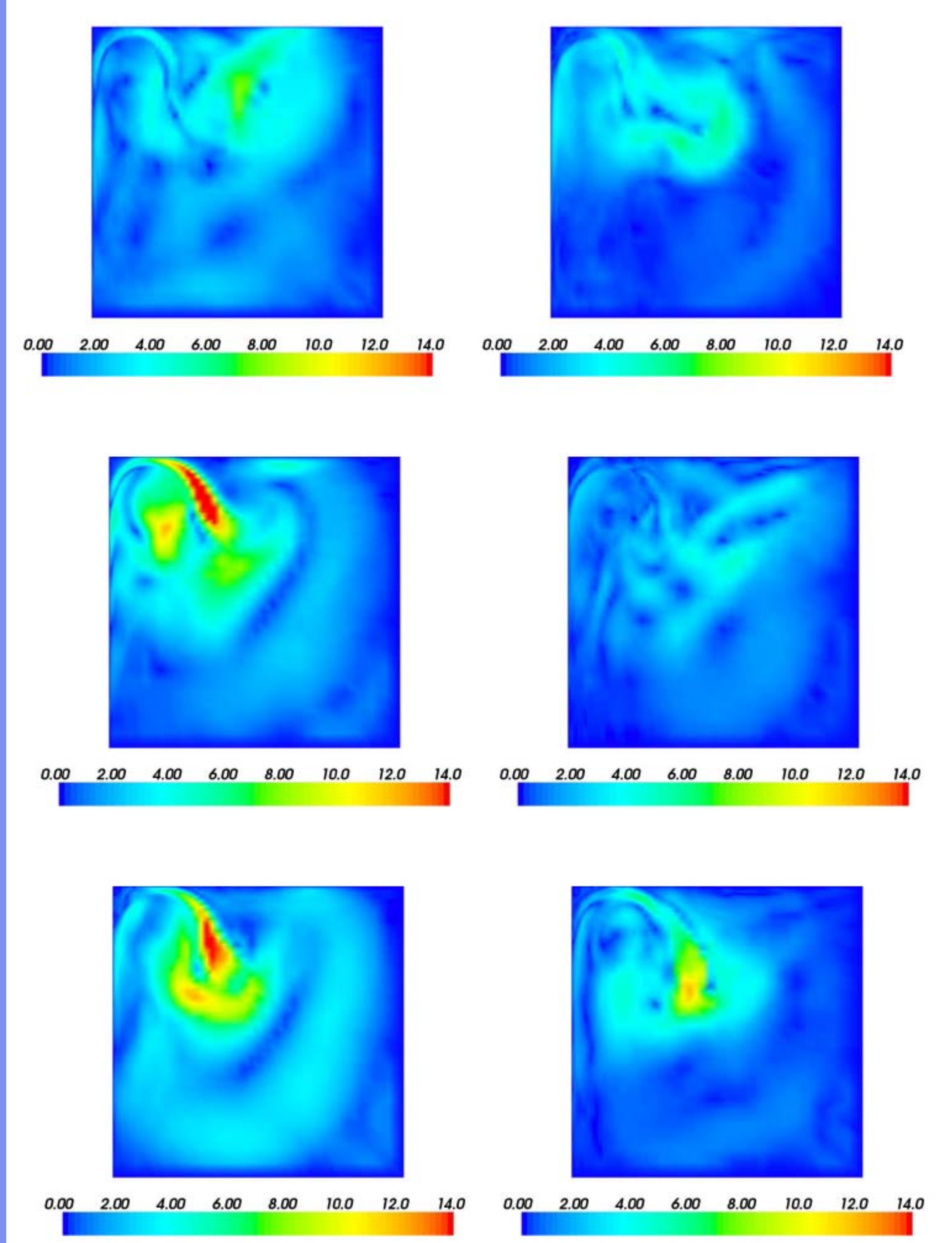
$$\Im(U^0) = \frac{1}{2}(U^0 - U_b)^T \mathbf{B}^{-1}(U^0 - U_b) + \frac{1}{2}\sum_{n=1}^{NT}(\mathbf{H}U^n - y_o^n)^T W_o(\mathbf{H}U^n - y_o^n)$$

Spin-up period: 200days

Simulation period: [200, 400] day

Time step: 6hrs

Pseudo-observations are u and v, which are available at t= 300 and 350 days.



250 day

300 day

350 day

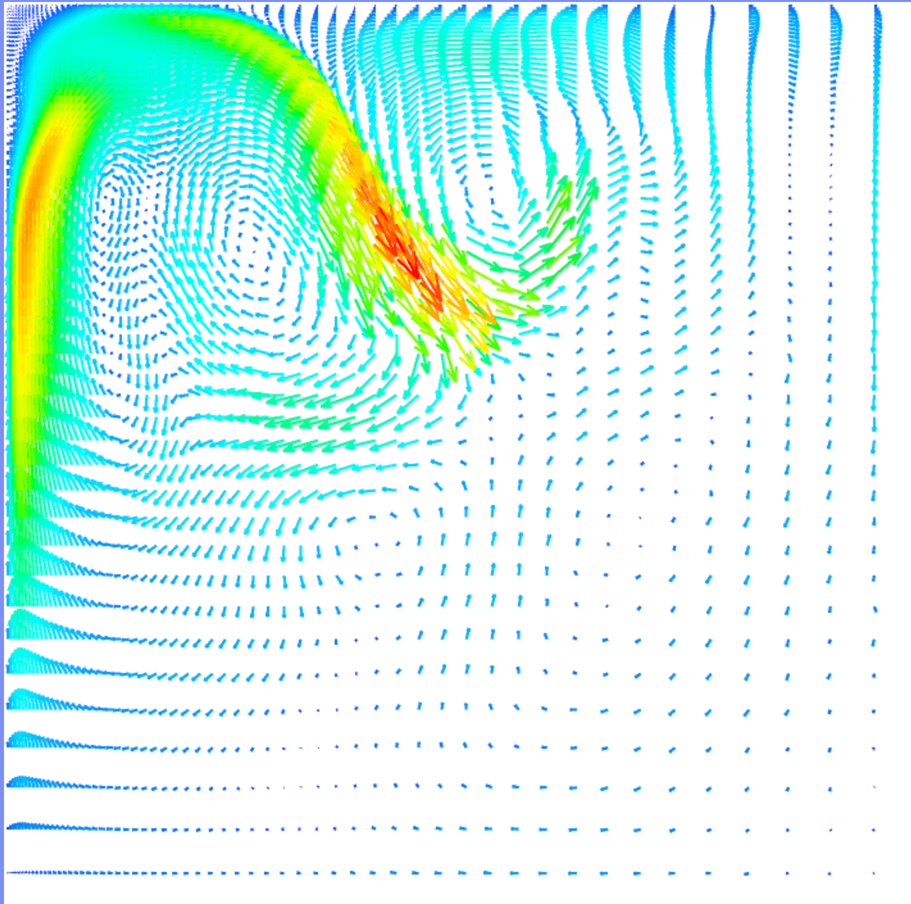
Conclusion

The advantages of the POD model developed here over existing POD approaches are the ability to:

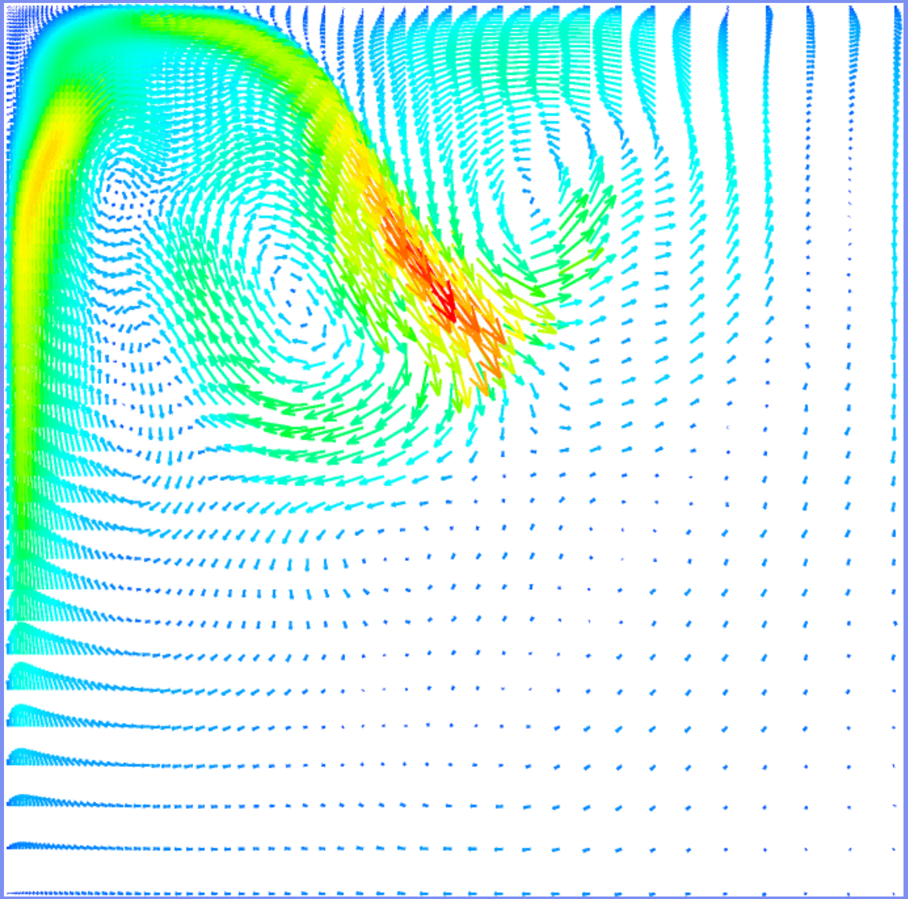
- **Introduce adaptive meshes into the POD model;**
- **The goal-based error measure approach developed here can be used to**
  (1) guide the mesh adaptivity and inversion;
  (2) optimise the POD bases (weight the snapshots).


THANK YOU!

Case: Gyre – inversion of initial conditions



Full model



Inversion model

The goal-based error measure approach for mesh adaptivity

$$\mathbf{A}\Psi - \mathrm{b} = 0$$

$$A^T \Psi^*_{exact} - \frac{\partial \Im}{\partial \Psi} = 0.$$

$$\bar{\mathbf{M}}_i = \frac{\gamma}{|\bar{\epsilon}_i|} \left|\bar{\mathbf{H}}_i\right|.$$

$$\bar{\mathbf{M}}^*_i = \frac{\gamma}{|\bar{\epsilon}^*_i|} \left|\bar{\mathbf{H}}^*_i\right|$$

$$\bar{\mathbf{H}}_i = \frac{1}{\sum_{n=1}^{NT}\sum_{l=1}^{\mathcal{M}} \left|\lambda_i^{l,n}\right|} \sum_{n=1}^{NT}\sum_{l=1}^{\mathcal{M}} \left|\lambda_i^{l,n}\right| \left|\mathbf{H}_i^{l,n}\right|$$

$$\bar{\mathbf{H}}^*_i = \frac{1}{\sum_{n=1}^{NT}\sum_{l=1}^{\mathcal{M}} \left|\lambda^{*l,n}_i\right|} \sum_{n=1}^{NT}\sum_{l=1}^{\mathcal{M}} \left|\lambda^{*l,n}_i\right| \left|\mathbf{H}^{*l,n}_i\right|.$$

$$|\hat{r}_i| = \left|\sum_{j\neq i} A_{i,j}\Psi_j + A_{i,i}\hat{\Psi}_i - b_i\right|$$

$$\left|\widehat{\mathbf{A}^T\Psi^*_{exact}}\right| = \left|\sum_{j\neq i} A^T_{i,j}\Psi^*_j + A^T_{i,i}\hat{\Psi}^*_i\right|$$

$$\bar{\epsilon}_i = \frac{\widetilde{\delta\Im}}{\sum_{n=1}^{NT}\sum_{l=1}^{\mathcal{M}} \left|\lambda_i^{l,n}\right|}$$

$$\bar{\epsilon}^*_i = \frac{\widetilde{\delta\Im}}{\sum_{n=1}^{NT}\sum_{l=1}^{\mathcal{M}} \left|\lambda^{*l,n}_i\right|}$$