Polynomial solutions on ellipses

In Volume 61, Number 2, of the "Notices of the AMS" (available on the AMS web site), Dmitry Khavinson and Erik Lundberg mention the following theorem.

Theorem Consider the ellipsoid

$$\Omega = \left\{ x \in \mathbb{R}^n : \sum_{j=1}^{n} \frac{x_j^2}{a_j^2} - 1 \leq 0 \right\},$$

where $a_1 \geq a_2 \geq \cdots \geq a_n > 0$. The solution u to the Dirichlet problem

$$\Delta u = 0 \text{ in } \Omega$$
$$u = p \text{ on } \Omega$$

where p is a polynomial of n variables, is a harmonic polynomial. Moreover, $\deg u \leq \deg p$.

They relate a well-known proof of this theorem that is short and beautiful.

For this project, you will write a program that takes as input

1. The major and minor axis lengths of an ellipse in $\mathbb{R}^2$, and
2. The coefficients of a polynomial (as in (1) below) of degree ≤ 5 in two variables.

Given this information you will solve Laplace's equation on the ellipse centered at the origin with given major and minor axes, and with Dirichlet boundary conditions equal to the given polynomial. Since this solution is actually a polynomial, you will compute its coefficients when written as

$$u(x,y) = c_{00} + c_{10}x + c_{01}y + c_{20}x^2 + c_{11}xy + c_{02}y^2 + \ldots + c_{n0}x^n + c_{0n}y^n \quad (1)$$

for $n \leq 5$. Compute the error in your solution as $\int_{\partial\Omega}(u(x,y) - p(x,y))^2\,dxdy$, where p is the given boundary value polynomial and u is your solution, *computed from (1)*.

Remark: For linear polynomials, the solution $u(x,y)$ is equal to the given boundary value polynomial. For $n \geq 2$, the given boundary value polynomial need not be harmonic, so the solution will not necessarily agree with the given boundary value polynomial in the interior of the ellipse.

Using your program, investigate how accurate your solution is as a function of n for several aspect ratios (ratio of major to minor axis) 1.0 and larger.