And now, although the business of the angles has been worked out and there is left to add to these things now established a research into the positions of the principal cities of each province according to their longitudes and latitudes for the calculations of the appearance observed in them, we shall do this exposition as a special geographical treatise by itself; and we shall follow the writings of those who have especially worked in this kind of thing, finding out how many degrees each city is from the equator along the meridian drawn through that city, and how many degrees east or west of the meridian of Alexandria each city is. For we arrange the times of the other places with reference to this meridian.

Now we have thought it pertinent to add this much about these positions: whenever we wish to know, at a given hour in one of these places, what hour it is in some other if their meridians are different, we must take the number of degrees they differ from each other along the equator, and, according as the one sought is east or west of the one given, increase or decrease the given hour by that number of degrees to get the hour defined at the same time in the place sought.

BOOK THREE

Now that we have methodically gone through, in all that has been put together up until now, those things which have first to be completely grasped mathematically concerning the heavens and the earth, and also concerning the obliquity of the sun's path through the middle of the zodiac [along the ecliptic] and its particular incidents in the right sphere and the oblique sphere for each latitude, we consider it proper after all this to treat of the sun and moon and to take account of the incidents concerning their movements, since without a prior understanding of them none of the appearances having to do with the stars can be discovered. And we find the treatise on the sun's movement advanced first, for again, without this, matters concerning the moon could not be grasped in detail.

1. On the Year's Magnitude

Since finding the year's time-length is the first of all the things demonstrated concerning the sun, we shall first learn from the treatises of the ancients the disagreements and difficulties concerning their statement on this, and especially from that of Hipparchus, a diligent and truth-loving man. For he is brought to a difficulty of this kind especially by the fact that, for the apparent returns of the sun with respect to the tropics and equinoxes, the length of the year is found to be less than 365¼ days, but for its returns observed with respect to the fixed stars it is found to be more. And from that he conjectures that the sphere of the fixed stars also has a very slow movement, and like that of the planets is in the direction contrary to that of the prime movement which revolves the circle that passes through the poles of the equator and the ecliptic. And we shall show this is so and how it comes about, in the chapters on the fixed stars. For matters concerning them could not be seen in their entirety without a prior understanding of the sun and moon.

But for the present research we believe that we must consider the length of the year looking only to the sun's return with respect to itself—that is, with respect to the oblique circle made by it [the ecliptic]—and that we must define the length of the year as the time in which the sun proceeds continuously from some fixed point of this circle back to the same point, supposing as we do that the only proper principles of this return are the points of this circle determined by the tropics and equinoxes. For, if we tell the story mathematically, we shall not find a more proper return than that which carries the sun through the same configuration both in space and time, whether it be considered with respect to the horizons or the meridian or the magnitude of the solar days; and we shall find no other principles of the ecliptic except those accidentally defined by the tropic and equinoctial points. And if one examines the subject more physically, he will not find a more reasonable return than that which brings the sun from a like to a like weather-condition and from the same season to the same season; nor will he find other principles than those by which the seasons are the most completely distinguished. And the return of the sun considered with respect to the fixed stars seems quite inept for this reason particularly that their sphere is observed to make an ordered movement contrary to that of the heavens. For, with things this way, nothing will keep one from saying that the length of the sun's year is the time it takes the sun to overtake Saturn, for instance, or some other star. And so there would be many different years.

And so we think it proper to consider such a period of time the sun's year which is found, by as many observations as possible taken over a rather long interval, from one tropic or equinox back to the same.

But since a suspected inequality in the periods of even this return, suspected through continuous and successive observations, more or less worried Hipparchus, we shall try briefly to show how this is not at all disturbing, since we are sure by the continuous instrumental observations we have made of the tropics and equinoxes that these periods are not unequal. For we find them differing by no appreciable amount from the additional quarter day, but at times by about as much as could be attributed to the error due to the construction or position of the instruments. For we guess from what Hipparchus reports that the error with regards to inequality belongs rather to the observations. For after he has first set out, in his treatise *On the Precession of the Tropic and Equinoctial Points*, the summer and winter tropics seeming to him to have been accurately observed and in order, he himself agrees that there is not such a difference between them as to recognize for this reason an inequality in the year. For he adds this: "Then it is clear from these observations that the differences in the years have been altogether small. But in the case of the tropics I do not despair of Archimedes' and my having made an error of as much as a quarter day both in observation and calculation. But the irregularity of the years can be accurately perceived from observations made on the bronze ring situated in what is called the Square Hall in Alexandria which is supposed to indicate the equinoctial day as that on which its concave surface begins to be lighted up from the opposite side."[1]

Then he lists, first, the dates of those autumn equinoxes which have been very accurately observed. One fell in the year 17 of the Third Callippic Period, Mesore 30, at the setting of the sun; and another three years after, year 20, on the first of the intercalated days in the morning, which should have been noon, so that there was a disagreement of a quarter day. And another a year after, year 21, at the sixth hour, which agreed with the preceding observation. And another eleven years after, year 32, on the third of the intercalated days at midnight before the fourth. And it should have been in the morning, so that there was again a disagreement of a quarter day. And another a year after, year 33, on the fourth of the intercalated days in the morning, which agreed with the preceding observation. And another three years after, year 36, on the fourth of the intercalated days in the evening. And it should have been at midnight so that there was again a disagreement of a quarter day.

And next he lists those spring equinoxes likewise accurately observed. One fell in the year 32 of the Third Callippic Period, Mechir 27, in the morning. But he adds: "The ring in Alexandria was also lighted up equally on both sides at the fifth hour, so that the same equinox differently observed differed by nearly five hours." And he says the equinoxes following, up to the year 37, agreed with the addition of a quarter day. And eleven years after, year 43, Mechir 29-30, just after midnight, he says, there was a spring equinox, which also agreed with the observation in the year 32; and also, he says, with the observations in the following years up to the year 50. For in that year it fell on Phamenoth 1 at sunset, within very nearly $1\frac{3}{4}$ days of that of the year 43, which is also proportional to the 7 intervening years. And so in these observations there was no perceptible difference although it is possible for there to be an error of as much as a quarter day, not only as regards the tropic observations, but also the equinoctial. For even if the position or discrimination of the instruments is inaccurate by only $\frac{1}{3600}$ of the circle through the poles of the equator, at the equinoctial intersections the sun makes up for this advance in latitude by shifting $\frac{1}{4}°$ in longitude along the ecliptic, so that there could be an inconsistency of very nearly a quarter day. And there could be a greater error still in the case of instruments not placed permanently and not then corrected for each observation, but which have been attached for some time to the pavement with a view to keeping a steady position for a good while, where yet some long unnoticed shift has been made in them. And anyone can see an example of this in the bronze rings in the palestra of our city, which are supposed to be in the plane of the equator. For in making observations we find such a distortion in their placement, and especially in the case of the largest and oldest, that at times their concave surfaces twice suffer a shift in lighting at the same equinoxes.

But certainly from such things Hipparchus himself does not think there is anything solid to support a suspicion of inequality in the lengths of years. But he says he finds by calculating from certain eclipses of the moon that the irregularity of the years, on the average, does not embrace a difference greater than three quarters of a day. And this would merit some attention if it were so and not evidently belied by the reasons he offers. For he calculates, by the lunar eclipses observed near certain fixed stars, by how much in each case the star Spica precedes the autumn equinox. And in this way he thinks he finds that once it was at its greatest distance of $6\frac{1}{2}°$, for the time he observed, and once at its least distance of $5\frac{1}{4}°$. And he infers from this fact that, since it is not possible for Spica to move so far in such a short time, it is likely that the sun, from which Hipparchus examines the positions of the fixed stars, does not always make its return in an equal time.

But he has overlooked the point that, since this calculation cannot proceed without laying down the sun's position at the eclipse, he, taking for this purpose

[1]The inequality of the year is here being judged in terms of the number of solar days. This might seem arbitrary, since the days might be unequal. And indeed it will be seen later on in this Book that the solar days are considered to be unequal. But the theory of their irregularity will be such that their inequality is exactly symmetrical within each year. But, of course, to judge that the inequality is symmetrical, it is necessary to fall back on something else supposed equal. The Greeks usually supposed it to be the stellar day, the time it takes a fixed star to go from a meridian back to the same meridian again. This is practically true for Ptolemy, but not absolutely so, because of the precession of the equinoxes. For, as we shall see in Book VII, the fixed stars move from west to east about the poles of the ecliptic nearly a half a degree in a hundred years. This introduces an irregularity in the length of the stellar day exactly parallel to one of the irregularities in the length of the solar day described in Chapter 9 of this Book, but one so small in magnitude that it could not be perceived from day to day, or even from year to year.

in each case the tropics and equinoxes accurately observed by himself in those very years, thereby immediately makes clear that in comparing the years there is no difference beyond the addition of the quarter day.

For example, from the observation of the eclipse in the year 32 of the Third Callippic Period he thinks he finds Spica preceding the autumn equinox by $6\frac{1}{2}°$; but from the eclipse in the year 43 of the same period, by $5\frac{1}{4}°$. And likewise setting beside these calculations the spring equinoxes accurately observed in those same years—so that by means of these he may get the sun's positions at the middle of the eclipses, and from these the moon's positions, and from the moon's those of the stars—he says that the spring equinox of the year 32 fell on Mechir 27 in the morning, and that of the year 43 on Mechir 29-30 after midnight, nearly $2\frac{3}{4}$ days later than that of the year 32, which is just equal to the quarter day added for each of the 11 intervening years. If, then, the sun has made its return to these equinoxes in neither more nor less time than the additional quarter day, and if it is not possible for the star Spica to have moved $1\frac{1}{4}°$ in so few years, how could it be otherwise than absurd to take the results calculated from the principles assumed as an accusation against the very principles combined to produce them, as if one were unable to saddle anything else with the cause of this excessive movement of Spica except the equinoxes assumed at the same time to have been accurately and inaccurately observed, although there were many things which could have introduced such an error?[1] For it would seem much more possible that the distances of the moon at the eclipses with respect to the nearest fixed stars had been estimated rather roughly; or that the calculations either of the moon's parallaxes for the sighting of its apparent positions or of the sun's movement from the equinoxes to the middle of the eclipses had been effected neither truly nor accurately.

And I think Hipparchus himself recognized there is nothing convincing in such things as far as imposing a second anomaly on the sun, but I think he only wishes for the love of the truth not to keep back anything which could in anyway bring one to suspect. And so he himself used hypotheses concerning the sun and moon with just one anomaly belonging to the sun, an anomaly which is redeemed in the year considered with respect to the tropics and equinoxes. And in supposing these revolutions to be equal in time we do not observe the appearances at the eclipses differing in any perceptible way from the calculations based on these hypotheses. For it would be quite perceptible if a correction for the inequality of the year were not made at the same time, even if it were only a difference of one degree or very nearly two standard hours.

From all these things, and from the times of the returns which we ourselves have gotten from the consecutive passages of the sun observed by us, we do not find the magnitude of the year unequal if it is considered with respect to some one thing and not one time with respect to the tropic and equinoctial points and another with respect to the fixed stars. Nor do we find any more proper period of return than that which carries the sun from one tropic or equinoctial point, or any other point on the ecliptic, back to that same point. And we do think it entirely proper to explain the appearances by the simplest hypotheses

[1]Hipparchus' logic, contrary to what Ptolemy says, seems impeccable insofar as he is saying that assumptions which lead to their contradictories are false. But the reasoning must be right, and Ptolemy is also suggesting that there were many steps in between which might have been false.

possible, so long as nothing perceptible appears contrary to this deduction.

And therefore it has become clear to us from what Hipparchus has shown that the length of the year observed with respect to the tropics and equinoxes is less than $365\frac{1}{4}$ days, but it would not be possible to find out with very great certainty by how much it is less since the increase of a quarter day remains perceptibly unchanged for many years because of the very small difference. And so the extra amount can be perceived only when it is found added up together over a longer period of time. And it must be divided among the intervening years of the interval and it must be observed for a greater and a smaller number of years than this same interval. And the period of return will be gotten the more accurately the longer the time between the observations compared. And this is the case not only with this period of return, but with all of them. For the error resulting from the weakness of the observations themselves, even if they are managed accurately, is small and very nearly the same as far as the senses are concerned both for appearances considered over a long time and for those considered over a short time. And this error of observation, when it is distributed over fewer years, makes the error in the length of the year greater and also in multiples of it over a longer period of time; and it makes the error in the length of the year smaller when distributed over a greater number of years.

And therefore it is properly thought sufficient if, when we consider how much the time between us and the old yet accurate observations can help in the approximation of the supposed periods of revolution, we try to introduce them with the others and do not willingly forego the proper verification, and if we suppose the establishing of dates for a whole long age or for some great multiple of time between observations is the work for another's love of wisdom and truth. Because of their age, then, the summer tropics observed by the pupils of Meton and Euctemon and those observed after them by the pupils of Aristarchus should be compared with those observed by us. But because the observations of the tropics are generally hard to determine and, moreover, because the observations handed down by these people were taken more or less in the rough, as Hipparchus also seems to have thought, we pass them over. We have used for this comparison the observation of the equinoxes, and, because of their accuracy, especially those given Hipparchus' approval as having been most certainly taken by him, and those most carefully observed by ourselves with the instruments for such purposes described at the beginning of this treatise. And from these we find that, in very nearly 300 years, the tropics and equinoxes fall one day sooner than the quarter-day addition to 365 days allows.

For in the year 32 of the Third Callippic Period Hipparchus singles out especially the autumn equinox as most accurately observed, and he says he calculates it to have fallen at midnight between the third and fourth of the intercalated days. And this is the year 178 after the death of Alexander. And 285 years after in the year 3 of Antonine (which is 463 years after the death of Alexander) we observed, again most correctly, the autumn equinox as having fallen on Athyr 9 about one hour after sunrise. Therefore the return added on in all the 285 Egyptian years, that is those of 365 days each, all told $70+\frac{1}{4}+\frac{1}{20}$ days instead of the $71\frac{1}{4}$ days due these years by the regular quarter-day addition. And so the return fell sooner by very nearly 1 day less $\frac{1}{20}$ than the regular quarter-day addition allowed.

And likewise Hipparchus again says the spring equinox in the same year 32 of the Third Callippic Period was very accurately observed to have fallen on Mechir 27 in the morning. And this is 178 years after the death of Alexander. And likewise 285 years later (463 years after the death of Alexander) we find the spring equinox has fallen on Pactom 7 very nearly one hour after noon, so that the period reached the aforesaid $70+\frac{1}{4}+\frac{1}{20}$ days very nearly, instead of the regular quarter-day addition to the 285 years of $71\frac{1}{4}$ days. Therefore the return of the spring equinox fell sooner by 1 day less $\frac{1}{20}$ than the regular quarter-day addition allowed. And so, since 300 years to 285 years, and 1 day to 1 day less $\frac{1}{20}$, are the same ratio, it is inferred that in very nearly 300 years the sun's return with respect to the equinoctial points is sooner by 1 day than the regular quarter-day addition allows.

Even if, because of its antiquity, we compare the summer tropic more or less roughly recorded by the pupils of Meton and Euctemon with that calculated by us, we shall find the same thing. For it is recorded to have taken place Athenianwise in the Magistracy of Apseudes, Egyptianwise Phamenoth 21 in the morning, and we, in the same 463rd year after the death of Alexander, very carefully calculated it to have fallen on Mesore 11-12 two hours after midnight. And from the summer tropic recorded under Apseudes to that observed by the pupils of Aristarchus in the year 50 of the First Callippic Period, as Hipparchus also says, is 152 years. And from this year 50 (which was the 44th year after the death of Alexander) to the 463rd year [after the death of Alexander], the year of our observation, is 419 years. Therefore, in the intervening 571 years of the whole interval, if the summer tropic observed by the pupils of Euctemon fell at the beginning of Phamenoth 21, very nearly $140+\frac{1}{2}+\frac{1}{3}$ days have been added to the complete Egyptian years instead of the regular quarter-day addition to the 571 years of $142+\frac{1}{2}+\frac{1}{4}$ days; so that this return fell sooner by 2 days less $\frac{1}{12}$ than the regular quarter-day addition allowed. It is therefore evident that in 600 complete Egyptian years, the lengths of the solar years anticipate the regular quarter-day addition by nearly 2 whole days.

And with many other observations we find this same thing happening, and we see Hipparchus several times agreeing to this. For in his treatise *On the Length of the Year*, when he compares the summer tropic observed by Aristarchus at the end of the year 50 of the First Callippic Period with the one taken again very accurately by himself at the end of the year 43 of the Third Callippic Period, he says as follows: "It is evident, therefore, that in 145 years the tropic has fallen sooner by half a day and night together than the regular quarter-day addition allows." And again in his treatise *On Intercalated Months and Days*, saying first that the year, for the pupils of Meton and Euctemon, contains $365+\frac{1}{4}+\frac{1}{76}$ days and according to Callippus only $365\frac{1}{4}$ days, he adds this: "And we have found as many whole months in the 19 years as they, but we have found that the year comes to $\frac{1}{300}$ day less than the regularly added quarter day, so that in 300 Egyptian years the sum of the solar years is 5 days less than Meton's and only 1 day less than Callippus'." And also in summarizing his views by citing his own works, he says as follows: "And I have also treated the question of the length of the year in a book in which I show that the solar year (that is, the time in which the sun goes from a tropic back to the same tropic or from an equinox back to the same equinox) contains 365 days and less than $\frac{1}{4}$ day by $\frac{1}{300}$ of a day and night, and not as the mathematicians think $365\frac{1}{4}$ days."

I think, then, it has been made clear that the appearances observed up to this time concerning the magnitude of the year agree with the size just assigned the return of the tropic and equinoctial points by a concurrence of present appearances with earlier ones. And since all this is so, if we distribute the one day over the 300 years, there falls to each year 12″ of a day. And if we subtract this from the 365 days 15′ where the quarter day has been added, we shall have the length of the year we are looking for—that is, 365 days 14′ 48″. And this number of days can be taken by us as the nearest approximation possible from the observations we have at present.

And as regards the scrutiny of the movements of the sun and the other planets in their particularities which is best furnished ready to hand and all set out by the orderly construction of tables, we believe it is the necessary purpose and aim of the mathematician to show forth all the appearances of the heavens as products of regular and circular motions. And it is incumbent upon him to construct such tables as, proper and consequent upon this purpose, separate out the particular regular[1] motions from the anomaly which seems to result from the hypotheses of circles, and show forth their apparent movements as a combination and union of all together. In order, then, that we may get this sort of thing in more serviceable form for the demonstration under consideration, we shall set out the regular movements of the sun in their particularities in this way.

For since a return has been proved to be 365 days 14′48″, if we divide these into the 360° of one circle, we shall have the sun's mean daily movement along the ecliptic as approximately $0°59^{i}8^{ii}17^{iii}13^{iv}12^{v}31^{vi}$;[2] for it will suffice to carry out the fractions to this power of sixtieths. And again, taking $\frac{1}{24}$ of the daily movement along the ecliptic, we shall have for the hourly movement approximately $0°2^{i}27^{ii}50^{iii}43^{iv}3^{v}1^{vi}$. Likewise multiplying the daily movement by the 30 days of a month, we shall have the mean monthly movement of $29°34^{i}8^{ii}36^{iii}36^{iv}15^{v}30^{vi}$; and multiplying by the 365 days of an Egyptian year we shall have the mean yearly movement of $359°45^{i}24^{ii}45^{iii}21^{iv}8^{v}35^{vi}$. Again multiplying the mean yearly movement by 18 years, because of the symmetry which will appear in the construction of the tables, and subtracting the whole circles, we shall have the surplus for the 18-year period, that is $355°37^{i}25^{ii}36^{iii}20^{iv}34^{v}30^{vi}$.

We have accordingly drawn up three tables of the regular movement of the sun, one in forty-five rows and the others in two parts. The first table contains the mean movements for the 18-year periods; the second table contains first the movements for the Egyptian years, and then for hours; the third, first the movements for months and, under that, for days. The numbers designating the time are set out in the first columns, and in the next columns the degrees, minutes, etc., are put beside them according to the proper combinations of each. And the tables are as follows:

[1]The word "regular" is here used as a translation of the Greek word ὁμαλός. On the other hand ἀνομαλία, its privative, is translated by the technical "anomaly" instead of by the more obvious "irregularity." There will be times, however, when "irregularity" is used. The Greek word ὁμαλός has three meanings, all significant in an astronomical context: (1) regular, (2) uniform, (3) mean or average. It is evident that "regular" and "uniform" are here synonymous. But also the regular movement of the sun is computed as the average or mean movement of the sun for the interval of a solar year.

[2]The superscripts indicate the powers of the sixtieths in the denominator. Thus in ordinary fractions this would be written $\frac{59}{60} + \frac{8}{60^2} + \frac{17}{60^3} + \frac{13}{60^4}$, etc.

2. Table of the Sun's Regular Movement—Continued

Distance from the apogee 265°15′; mean epoch 0°45′ within the Fishes

Egyptian mo'ths	Degrees	I	II	III	IV	V	VI	Egyptian mo'ths	Degrees	I	II	III	IV	V	VI
30	29	34	8	36	36	15	30	210	206	59	0	16	13	48	30
60	59	8	17	13	12	31	0	240	236	33	8	52	50	4	0
90	88	42	25	49	48	46	30	270	266	7	17	29	26	19	30
120	118	16	34	26	25	2	0	300	295	41	26	6	2	35	0
150	147	50	43	3	1	17	30	330	325	15	34	42	38	50	30
180	177	24	51	39	37	33	0	360	354	49	43	19	15	6	0
Days	**Degrees**	**I**	**II**	**III**	**IV**	**V**	**VI**	**Days**	**Degrees**	**I**	**II**	**III**	**IV**	**V**	**VI**
1	0	59	8	17	13	12	31	16	15	46	12	35	31	20	16
2	1	58	16	34	26	25	2	17	16	45	20	52	44	32	47
3	2	57	24	51	39	37	33	18	17	44	29	9	57	45	18
4	3	56	33	8	52	50	4	19	18	43	37	27	10	57	49
5	4	55	41	26	6	2	35	20	19	42	45	44	24	10	20
6	5	54	49	43	19	15	6	21	20	41	54	1	37	22	51
7	6	53	58	0	32	27	37	22	21	41	2	18	50	35	22
8	7	53	6	17	45	40	8	23	22	40	10	35	3	47	53
9	8	52	14	34	58	52	39	24	23	39	18	53	17	0	24
10	9	51	22	52	12	5	10	25	24	38	27	10	30	12	55
11	10	50	31	9	25	17	41	26	25	37	35	27	43	25	26
12	11	49	39	26	38	30	12	27	26	36	43	44	56	37	57
13	12	48	47	43	51	42	43	28	27	35	52	2	9	50	28
14	13	47	56	1	4	55	14	29	28	35	0	19	23	2	59
15	14	47	4	18	18	7	45	30	29	34	8	36	36	15	30

3. On the Hypotheses concerning Regular and Circular Movement

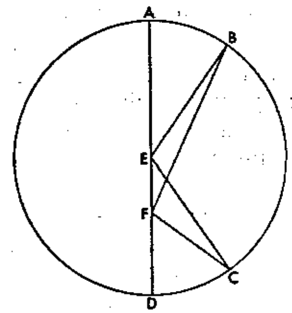
Since the next thing is to explain the apparent irregularity of the sun, it is first necessary to assume in general that the motions of the planets in the direction contrary to the movement of the heavens are all regular and circular by nature, like the movement of the universe in the other direction. That is, the straight lines, conceived as revolving the stars or their circles, cut off in equal times on absolutely all circumferences equal angles at the centres of each;[1] and their apparent irregularities result from the positions and arrangements of the circles on their spheres through which they produce these movements, but no departure from their unchangeableness has really occurred in their nature in regard to the supposed disorder of their appearances.

But the cause of this irregular appearance can be accounted for by as many as two primary simple hypotheses. For if their movement is considered with respect to a circle in the plane of the ecliptic concentric with the cosmos so that our eye is the centre, then it is necessary to suppose that they make their regular

[1]This principle of celestial mechanics will be considerably broadened in the case of the moon and the other five planets as treated in Books V, IX, and X.

movements either along circles not concentric with the cosmos, or along concentric circles; not with these simply, but with other circles borne upon them called epicycles. For according to either hypothesis it will appear possible for the planets seemingly to pass, in equal periods of time, through unequal arcs of the ecliptic circle which is concentric with the cosmos.

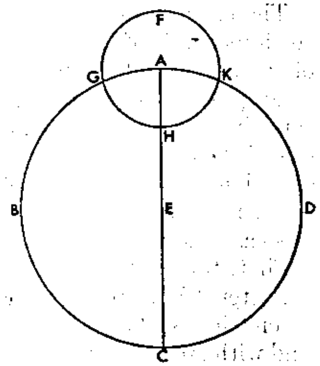
For if, in the case of the hypothesis of eccentricity, we conceive the eccentric circle $ABCD$ on which the star moves regularly, with E as center and with diameter AED, and the point F on it as your eye so that the point A becomes the apogee and the point D the perigee; and if, cutting off equal arcs AB and DC, we join BE, BF, CE, and CF, then it will be evident that the star moving through each of the arcs AB and CD in an equal period of time will seem to have passed through unequal arcs on the circle described around F as a centre. For since



$$\text{angle } BEA = \text{angle } CED,$$

therefore angle BFA is less than either of them, and angle CFD greater [Eucl. I, 16].

And if in the hypothesis of the epicycle we conceive the circle $ABCD$ concentric with the ecliptic with centre E and diameter AEC, and the epicycle $FGHK$ borne on it on which the star moves, with its centre at A, then it will be immediately evident also that as the epicycle passes regularly along the circle $ABCD$, from A to B for example, and the star along the epicycle, the star will appear indifferently to be at A the centre of the epicycle when it is at F or H; but when it is at other points, it will not. But having come to G, for instance, it will seem to have produced a movement greater than the regular movement by the arc AG; and having come to K, likewise less by the arc AK.



Then with the hypothesis of eccentricity it is always the case that the least movement belongs to the apogee and the greatest movement to the perigee, since angle AFB is always less than angle DFC. But both cases can come about with the hypothesis of the epicycle. For when the epicycle moves contrary to the heavens [from west to east], for example from A to B, if the star so moves on the epicycle that it goes from the apogee again contrary to the heavens (that is, from F in the direction of G), there will result at the apogee the greatest advance, because the epicycle and the star are moving the same way. But if the movement of the star on the epicycle is in the direction of that of the heavens [from east to west], that is, from F towards K, conversely the least advance will be effected at the apogee because the star is then moving contrary to the movement of the epicycle.

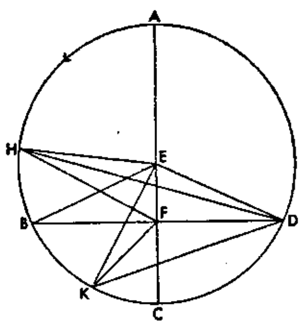
With these things established, it must next be understood that, in the case of those planets which effect two anomalies, it is possible to combine both of these hypotheses, as we shall show in the chapters concerning them. But, in the case of those planets subject to only one anomaly, one of the hypotheses will suffice. And it must be understood that all the appearances can be cared for interchangeably according to either hypothesis, when the same ratios are involved in each. In other words, the hypotheses are interchangeable when, in the case of the hypothesis of the epicycle, the ratio of the epicycle's radius to the radius of the circle carrying it[1] is the same as, in the case of the hypothesis of eccentricity, the ratio of the line between the centres (that is, between the eye and the centre of the eccentric circle), to the eccentric circle's radius; with the added conditions that the star move on the epicycle from the apogee in the direction of the movement of the heavens with the same angular velocity as the epicycle moves on the circle concentric with the eye in the direction opposite to that of the heavens, and that the star move regularly on the eccentric circle with the same angular velocity also and in the direction opposite to the movement of the heavens.

And we shall briefly show in a systematic way, first by reasoning and secondly by the numbers discovered in the appearances of the sun's anomaly, that with the above assumptions the same appearances agree with either hypothesis.

I say first, then, that on either hypothesis the greatest difference between the regular movement and the apparent irregular movement (difference by which the mean passage of the stars is apprehended) occurs when the apparent angular distance cuts off a quadrant from the apogee; and that the time from the apogee to this mean passage is greater than from this mean passage to the perigee.

Therefore it results—always on the hypothesis of the eccentric circles, and on the hypothesis of the epicycle whenever their movements occur in the direction of the movement of the heavens—that the time from the least passage to the mean passage is greater than that from the mean to the greatest passage, because in each case the least progress is effected at the apogee. But on the hypothesis of the epicycles which supposes the revolutions of the stars on them in the direction contrary to that of the heavens, conversely the time from the greatest to the mean passage is greater than that from the mean to the least, because in this case the greatest progress is effected at the apogee.

First, then, let there be the star's eccentric circle $ABCD$ with centre E and diameter AEC. And let the centre of the ecliptic be taken (that is, the point at the eye), and let it be F. And with BFD drawn through F at right angles to AEC, let the star be supposed at points B and D, so that the apparent angular distance on either side from the apogee A is clearly a quadrant.



It must be proved that the greatest difference between the regular and irregular movements occurs at the points B and D.

For let EB and ED be joined. Then it is immediately evident that the arc of the anomalistic difference has to the whole circle the same ratio that angle EBF has to 4 right angles, since the angle AEB subtends the arc of the regular movement and angle AFB that of the apparent irregular movement, and angle EBF is the difference between them.

[1]The circle carrying the epicycle is often called the deferent.

I say, then, that no other angle can be constructed on the circumference of the circle $ABCD$ and on the straight line EF greater than these two at B and D.

For let the angles EHF and EKF be constructed at the points H and K, and let HD and KD be joined. Since then in every triangle the greater side subtends the greater angle, and

$HF > FD$, [Eucl. III, 7,3]

also angle $HDF >$ angle DHF

But angle $EDH =$ angle EHD

since $EH = ED$.

And therefore angle $EDF >$ angle EHF,

angle $EBD >$ angle EHF.

Again since $DF > KF$,

angle $FKD >$ angle FDK.

But angle $EKD =$ angle EDK,

since again $EK = ED$.

And therefore, by subtraction,

angle $EDF >$ angle EKF,

angle $EBF >$ angle EKF.

Therefore it is not possible to construct other angles in the way we have described greater than those at points B and D.[1]

And at the same time it is proved that arc AB, which embraces the time from the least to the mean movement, is greater than arc BC which embraces the time from the mean to the greatest movement, by twice the arc containing the anomalistic difference. For

angle $AEB =$ angle $EFB +$ angle EBF,

$=$ rt. angle $+$ angle EBF,

and

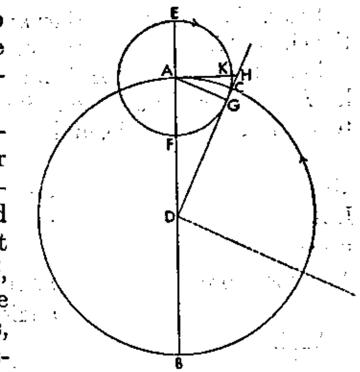
angle $BEC +$ angle $EBF =$ rt. angle.

Again, to prove the same thing occurs in the other hypothesis, let there be the circle ABC concentric with the cosmos, with center D and diameter ABD; and let there be in the same plane the epicycle EFG with centre A, carried on it. And let the star be supposed at G where it appears to be a quadrant's distance from the apogee point. And let AG and DGC be joined.

[1]The Greeks, in general, avoided the notion of a body's speed at a given point, and Ptolemy here handles the problem in the classic way, in terms of boundary points. Thus by proving that the greatest difference between the angle of the regular movement and that of the apparent irregular movement is at a point an apparent quadrant's distance from the apogee, it then follows that this point is a boundary point such that for all arcs between it and the apogee the star will appear to move more slowly than its regular or average movement, and for all arcs between it and the perigee the star will appear to move faster than its regular or average movement. Ptolemy therefore calls the point itself the point of the star's mean passage. It is not very different from saying in modern terms that the speed of the star at this point is its regular speed.

That the point of greatest anomalistic difference is such a boundary point is simply stated by Ptolemy and not explained. The explanation is this: From the apogee to the point of greatest difference the apparent angular speed of the star is always slower than its regular speed, for otherwise the difference between the two angles traveled would not be getting greater and greater. Likewise from the point of greatest difference to the perigee the apparent angular speed is greater than the regular speed, for otherwise the difference between the two angles traveled would not be getting less and less.

I say that the straight line DGC is tangent to the epicycle. For at that time there occurs the greatest difference between the regular and irregular movements.



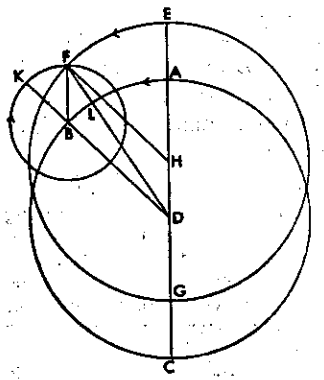
For since the regular movement from the apogee is contained by the angle EAG (for the star traverses the epicycle, and the epicycle the circle ABC, with the same angular velocity), and the difference between the regular and apparent movements is contained by the angle ADG, therefore it is evident that also the difference between angle EAG and angle ADG (that is, angle AGD) contains the apparent angular distance of the star from the apogee. And so, since it is assumed to be the angle of a quadrant, angle AGD will also be a right angle, and therefore the straight line DGC will be tangent to the epicycle EFG [Eucl. III, 16, Por.]. Therefore the arc AC between the centre A and the tangent is the greatest anomalistic difference.

And in the same way arc EG, which, according to the motion assumed for the epicycle, embraces the time from the least to the mean movement, is greater than arc GF, which embraces the time from the mean to the greatest movement. And it is greater by twice arc AC. For if we produce the straight line DGH and draw AKH at right angles to EF, then

angle KAG = angle ADC, [Eucl. VI, 8]

and arc KG is similar to arc AC. Therefore arc EKG is greater than a quadrant by arc AC, and arc FG is less than a quadrant by arc AC. Which it was required to prove.[1]

And next it will be clearly seen that, even in the other particular movements, in the case of both hypotheses, for equal times, all the same things will occur with respect to the regular and apparent movements and the differences between them—that is, the anomalistic difference.



For let there be the circle ABC with centre D, concentric with the ecliptic; and the eccentric EFG with center H, equal to the concentric circle ABC; and the diameter $EAHD$ common to both, through the centres D and H and the apogee E. And with arc AB taken at random length on the concentric circle, let the epicycle KF with centre B and radius DH be described, and let KBD be joined.

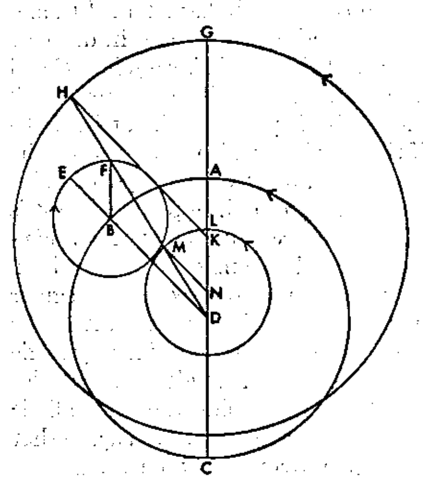
I say that the star will be borne by either movement to F, the intersection of the eccentric circle and the epicycle, in the same amount of time. That is, the three arcs, EF on the eccentric, AB on the concentric, and KF on the epicycle are similar to each other; and the difference between the regular and irregular movements, and the apparent passage of the star, will be similar and the same under either hypothesis.

For let FH, BF, and DF be joined. Since the opposite sides of the quadrilateral $BDHF$ are equal to each other, FH to BD, and BF to DH, the quadrilateral $BDFH$ is a parallelogram. Therefore the three angles EHF, ADB, and FBK are equal. And so, since they are angles at the centres, the arcs subtended by them—EF on the eccentric, AB on the concentric, and KF on the epicycle—are similar to one another. Therefore by either motion, the star will be brought to the same point F in an equal period of time, and will appear to have passed from the apogee along the same arc of the ecliptic, AL. And accordingly the anomalistic difference will be the same according to either hypothesis, since we have already proved that the difference contained by angle DFH on the hypothesis of eccentricity is of the same kind as that contained by angle BDF on the hypothesis of the epicycle, and since these angles are here also alternate and equal, with FH proved parallel to BD.

And it is clear that for all distances these same results will follow, $HDFB$ being always a parallelogram and the eccentric circle being described by the movement of the star on the epicycle whenever the relations under either hypothesis are both similar and equal.

But it will also become clear in the following way that, even if they are only similar but unequal in magnitude, the same appearances will again result. For in the same way, let there be the circle ABC with centre D, concentric with the cosmos; and its diameter ADC passing through the star's apogee and perigee; and the epicycle about B at the random distance of arc AB from the apogee A. And let the star have moved through arc EF similar to AB since the returns of the circles take place in the same time. And let the straight lines DBE, BF, and DF be joined.

It is immediately clear that, on this hypothesis, angle ADE and angle FBE are always equal, and that the star will appear on the straight line DF.



I say also that, on the hypothesis of eccentricity, both if the eccentric circle is greater than the concentric circle and if it is less, with only the similarity of the relations and the isochronism of the returns assumed, the star will again appear on the same straight line DF.

For let the eccentric circle GH be drawn greater, as we said, with its centre at K on AC; and likewise LM less, with centre N. And producing $DMFH$ and $DLAG$, let HK and MN be joined. Since

$$DB : BF :: HK : KD :: MN : ND,$$

and since

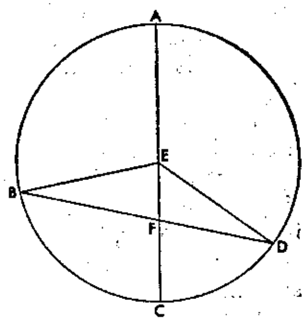
angle BFD = angle MDN

[1] In the case where the star moves on the epicycle in the same direction that the epicycle moves on the concentric circle, the mean passage and greatest anomalistic difference do not occur an apparent quadrant's distance from the apogee—that is, if the angular velocity of the star on the epicycle is the same as that of the epicycle's centre on the concentric circle. But it is greater than a quadrant's distance from the apogee. This is immediately evident if we refer to the previous figure and suppose DGH drawn tangent on the opposite side of the epicycle.

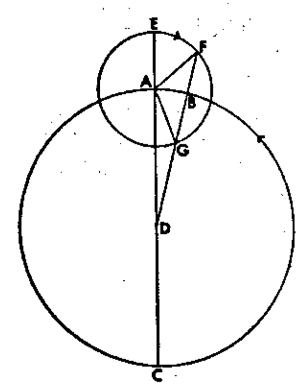
because of the parallels DA and BF, therefore the three triangles are equiangular [Eucl. VI, 7], and the angles BDF, DHK, and DMN, subtending the corresponding sides, are equal. Therefore the straight lines BD, HK, and MN are parallel, so that also angles ADB, AKH, and ANM are equal. And since they are angles at the centres of the circles, therefore the arcs subtended by them, AB, GH, and LM, will be similar. Therefore, in the same length of time, not only has the epicycle traversed arc AB, and the star arc EF, but also on the eccentric circles the star will have traversed arcs GH and LM; and in each case, therefore, it will be observed on the same straight line $DMFH$, being at the point F in the case of the epicycle, at H for the greater eccentric, at M for the smaller eccentric, and likewise for all positions.

And furthermore it results that, when the star appears to have traversed equal arcs both from the apogee and the perigee, the anomalistic difference in either position will be equal.

For, on the hypothesis of eccentricity, if we describe the eccentric circle $ABCD$ about centre E with diameter AEC through the apogee A, with the eye supposed on it at F, and if, drawing through F the straight line BFD at random, we join EB and ED, then the apparent courses will be equal and opposite; that is, angle AFB of the course from the apogee and angle CFD of that from the perigee. And the anomalistic difference will be the same because BE is equal to DE and angle EBF to angle EDF. And so the arc from the apogee A and the arc from the perigee C (that is, the arcs contained by angles AFB and CFD, respectively) are, the one greater and the other less, than the regular movement by the same difference of the apparent arc; because AEB is greater than angle AFB and angle CED less than angle CFD.



And, on the hypothesis of the epicycle, if we describe likewise the concentric circle around centre D and with diameter ADC, and the epicycle EFG around centre A, and if, drawing at random the straight line $DGBF$, we join AF and AG, then the arc of the anomalistic difference AB will be the same for both positions. That is, if the star is at F or if the star is at G, it will appear on the ecliptic at the same distance from the apogee point when it is at F as from the perigee point when it is at G, since the apparent arc from the apogee is contained by angle DFA. For angle DFA has been shown to be the difference between the regular movement and the anomalistic difference. And the apparent arc from the perigee is contained by angle FGA. For it is also equal to the angle of the regular movement from the perigee plus the anomalistic difference. And so it is thereupon inferred again that the mean movement is greater than the apparent movement about the apogee (that is, angle EAF than angle AFD) and the mean movement is less than the apparent about the perigee (that is, angle GAD than angle AGF), both by the same difference, by angle ADG. Which was to be proved.



4. On the Apparent Irregularity or Anomaly of the Sun

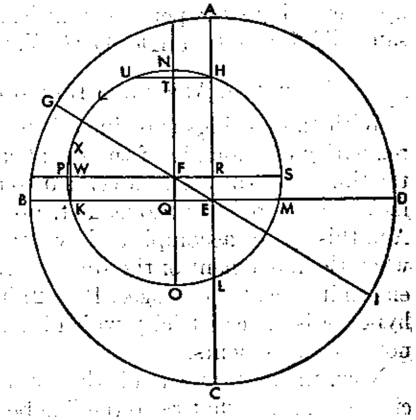
With these things explained, it is now necessary to take up the apparent irregularity or anomaly of the sun; because there is one only, and it is such that the time from the least movement to the mean is greater than the time from the mean to the greatest movement. For we find this agrees with the appearances. And this can be accomplished by either hypothesis:—(1) by that of the epicycle when the movement of the sun is in the direction of the movement of the heavens on its arc at the apogee. But (2) it would be more reasonable to stick to the hypothesis of eccentricity which is simpler and completely effected by one and not two movements.

Now, the first question is that of finding the ratio of eccentricity of the sun's circle—that is, what ratio the line between the eccentric circle's centre and the ecliptic's centre at the eye has to the radius of the eccentric circle; and next at what section of the ecliptic the apogee of the eccentric circle is to be found. And these things have been shown in a serious way by Hipparchus. For having supposed the time from the spring equinox to the summer tropic to be 94½ days, and the time from the summer tropic to the autumn equinox to be 92½ days, he proves from these appearances alone that the straight line between the aforesaid centres is very nearly $\frac{1}{24}$ the radius of the eccentric circle; and that its apogee precedes the summer tropic by very nearly 24½° of the ecliptic's 360°.

And we too find that the time-periods of these quarters and these ratios are very nearly the same even now, so that in this way it is clear to us that the sun's eccentric circle always preserves the same position with respect to the tropic and equinoctial points. And not to establish this position on hearsay only, but to expound the theory systematically with our own numbers, we shall prove these things ourselves, using these same appearances as regards the eccentric circle—that is, as we said, 94½ days from the spring equinox to the summer tropic and 92½ days from the summer tropic to the autumn equinox. For with the very accurate observations made by us in the year 463 after the death of Alexander we find a complete agreement in the number of days between the summer tropic and the equinoxes. For as we said [pp. 81-82], the autumn equinox fell on Athyr 9 after sunrise, the spring equinox on Pachom 7 after midday, which makes an interval of 178¼ days, and the summer tropic on Mesore 11-12 after midnight, which makes the interval from the spring equinox to the summer tropic 94½ days, and leaves very nearly 92½ days for the interval from the summer tropic to the following autumn equinox.

Then let there be the ecliptic circle $ABCD$ with centre E, and let the two diameters AC and BD be drawn in it perpendicular to each other through the tropic and equinoctial points. And let A be supposed the spring point, B the summer, and the rest accordingly.

Now, that the centre of the eccentric circle will fall between the straight lines EA and EB, is clear on the one hand from the fact that the semicircle ABC embraces more time than half a year and therefore cuts off a section of the eccentric greater than a semicircle, and on the other hand from the fact that the quadrant AB itself also embraces more time and cuts off a greater arc of the eccentric than quadrant BC.



This being so, let the point F be supposed the centre of the eccentric circle, and let the diameter EFG be drawn through both centres and the apogee. And with centre F and any radius, let the eccentric circle of the sun $HKLM$ be drawn, and through F the line NQO parallel to AC and the line PRS parallel to BD. And again let HTU, the perpendicular from H to NQO, and KWX, the perpendicular from K to PRS, be drawn.

Since, then, the sun, moving regularly on the circle $HKLM$, traverses arc HK in 94½ days and arc KL in 92½ days, and since it covers regularly in 94½ days very nearly 93°9′, and in 92½ days 91°11′ [Chap. II, Table of Sun's Regular Movement], therefore

$$\text{arc } HKL = 184°20',$$

and

$$\text{arc } NH + \text{arc } LO = 4°20'$$

by subtraction of the semicircle NPO. And

$$\text{arc } HNU = 2 \text{ arc } HN = 4°20'.$$

And so

$$\text{chord } HU = 4^p32'$$

where

$$\text{ecc. diam.} = 120^p$$

And, the half of chord HU,

$$HT = EQ = 2^p16'.$$

Again, since

$$\text{arc } HNPK = 93°9'$$

and

$$\text{arc } HN = 2°10'$$

and

$$\text{quadrant } NP = 90°,$$

therefore, by subtraction,

$$\text{arc } PK = 0°59'$$

and

$$\text{arc } KPX = 2 \text{ arc } PK = 1°58'.$$

And so

$$\text{chord } KWX = 2^p4'$$

and, the half of it,

$$KW = FQ = 1^p2'.$$

But, it was proved

$$EQ = 2^p16'.$$

And since

$$\text{sq. } FQ + \text{sq. } EQ = \text{sq. } EF,$$

therefore, in length,

$$EF = 2^p29'30''$$

where

$$\text{rad. ecc.} = 60^p.$$

Therefore the radius of the eccentric circle is very nearly twenty-four times the line between its centre and that of the ecliptic.

Again since

$$FQ = 1^p2'$$

where it was proved

$$EF = 2^p29'30'',$$

therefore

$$FQ = 49^p46'$$

where

$$\text{hypt. } EF = 120^p,$$

and, on the circle about right triangle EFQ,

$$\text{arc } FQ = 49°.$$

And therefore

$$\text{angle } FEQ = 49° \text{ to 2 rt.}$$
$$= 24°30'.$$

And so, since the angle is at the centre of the ecliptic, arc BG by which the apogee G precedes the summer tropic point B, is also 24°30′.

Finally, since the quadrants OS and SN are each 90°, and

$$\text{arc } OL = \text{arc } HN = 2°10',$$

and

$$\text{arc } MS = 0°59',$$

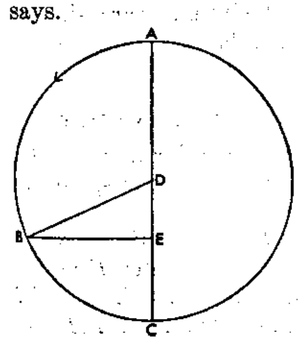
therefore

$$\text{arc } LM = 86°51'$$

and

$$\text{arc } MH = 88°49'.$$

But the sun moves regularly through 86°51′ in 88⅛ days, and through 88° 49′ in very nearly 90⅛ days [Chap. 2, Table of Sun's Regular Movement]. And so the sun will appear to traverse arc CD, which is the arc from the autumn equinox to the winter tropic in 88⅛ days; and arc DA, which is the arc from the winter tropic to the spring equinox, in very nearly 90⅛ days. And these things have been found by us in accord with what Hipparchus says.



Now with these quantities, let us find out first how much is the greatest difference between the regular and irregular movements, and at what point this occurs.

Then let there be the eccentric circle ABC with centre D and diameter ADC through the apogee A; and on it let there be the centre of the ecliptic E. And let EB be drawn perpendicular to AC, and DB be joined. Since

$$\text{l. betw. c. } DE = 2^p30'$$

where

$$\text{rad. } BD = 60^p$$

according to the ratio of 1 to 24, therefore

$$DE = 5^p$$

where

$$\text{hypt. } BD = 120^p,$$

and, on the circle about right triangle BDE,

$$\text{arc } DE \doteq 4°46'.$$

And so angle DBE which contains the greatest anomalistic difference will be 4°46′ to 2 right angles' and 2°23′ to 4 right angles' 360°. And

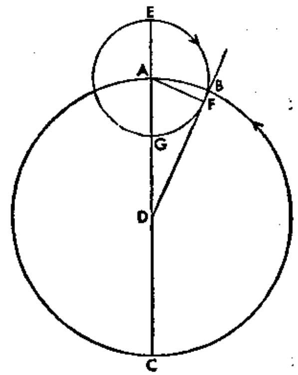
$$\text{rt. angle } BED = 90°$$

and

$$\text{angle } BDA = \text{angle } BED + \text{angle } DBE = 92°23'.$$

And since angle BDA is at the centre of the eccentric circle and angle BED of the ecliptic, we shall have the greatest anomalistic difference as 2°23′. And as for the arcs at which this occurs, that of the eccentric which is regular is 92°23′ from the apogee, and that of the ecliptic which is apparent and irregular is a quadrant or 90°, as we have already proved. And it is clear from what has been set out that in the opposite section the apparent mean passage and the greatest anomalistic difference will be at 270°, and the regular mean passage at 267°37′ on the eccentric.

In order to show with numbers also that the same quantities can be inferred on the hypothesis of the epicycle when there are the same ratios in the way we described, let there be the circle ABC with centre D and diameter ADC, concentric with the ecliptic, and the epicycle EFG with centre A. And let the straight line DFB be drawn from D tangent to the epicycle, and let AF be joined. Then likewise



$$AD = 24\ AF,$$

so that again also

$$AF = 5^p$$

where

$$\text{hypt. } AD = 120^p,$$

and, on the circle about right triangle ADF,

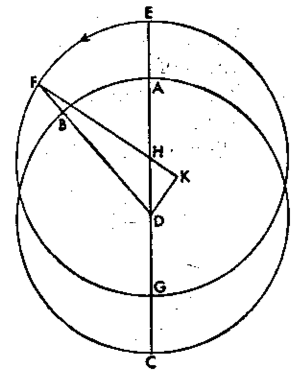
$$\text{arc } AF \doteq 4°46' \text{ to 2 rt.}$$
$$= 2°23'$$

Therefore, the greatest anomalistic difference (that is, arc AB) is thereupon found rightly to be 2°23′; and the irregular arc, since it is contained by the right angle AFD, to be 90° and the regular arc, contained by angle EAF, again to be 92°23′.

5. On the Examination of Particular Sections of the Anomaly

And to be able to distinguish at any time particular irregular movements, we shall again show for either hypothesis how, given one of these arcs, we can get the others also.

Then first let there be the circle ABC with centre D, concentric with the ecliptic; and the eccentric circle with centre H; and the diameter $EAHDG$ through both centres and the apogee E. And with arc EF cut off, let FD and FH be joined. And first let arc EF be given, for instance, as 30°; and on FH produced let fall the perpendicular DK from D.



Since then it is assumed

$$\text{arc } EF = 30°,$$

therefore

$$\text{angle } EHF = \text{angle } DHK = 30°$$
$$= 60° \text{ to 2 rt.}$$

And therefore, on the circle about right triangle DHK,

$$\text{arc } DK = 60°,$$

and, the rest of the semicircle,

$$\text{arc } HK = 120°. \quad \text{[Eucl. III, 31]}$$

And therefore

$$DK = 60^p$$

and

$$KH = 103^p55'$$

where

$$\text{hypt. } DH = 120^p.$$

And so

$$DK = 1^p15',$$
$$HK = 2^p10',$$
$$KHF = 62^p10'$$

where

$$DH = 2^p30',$$
$$\text{rad. } FH = 60^p.$$

And since

$$\text{sq. } DK + \text{sq. } FHK = \text{sq. } FD,$$
$$\text{hypt. } FD \doteq 62^p11'.$$

And therefore

$$DK = 2^p25'$$

where

$$FD = 120^p,$$

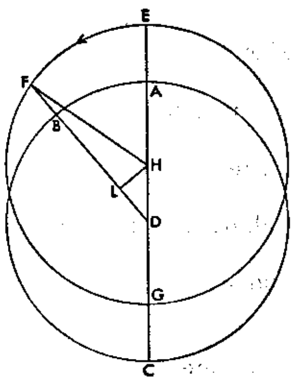
and, on the circle about right triangle FDK,

$$\text{arc } DK = 2°18'.$$

And so

$$\text{angle } DFK = 2°18' \text{ to 2 rt.}$$
$$= 1°9'.$$

Therefore the anomalistic difference at that time is 1°9′. But angle EHF was 30°, and therefore the remaining angle ADB (that is, arc AB on the ecliptic) is 28°51′.



And with the same construction, if HL is dropped from H perpendicular to FD, it will be immediately clear that also, if any other angle is given, the rest are given. For if we suppose the arc AB on the ecliptic given (that is, angle HDL), then the ratio of DH to HL is given. And if the ratio DH to HF is given, the ratio of HF to HL is also given; and therefore we shall have angle HFL given (that is, the anomalistic difference) and angle EHF (that is, arc EF of the eccentric circle).

apogees into 15 sections, so that the comparison is at intervals of 6°; and the quadrants at the perigees into 30 sections so that the comparison in this case is at intervals of 3°. This is done because the differences in excess at the perigees are greater than those corresponding to equal sections at the apogee.

And so we shall arrange the table of the sun's anomaly into forty-five rows and three columns. And of these columns the first two contain the numbers of the 360° of regular movement, while the first fifteen rows embrace the two quadrants at the apogee, and the other thirty those at the perigee. And the third column contains the degrees of anomalistic difference to be added or subtracted, corresponding to each of the regular numbers. And here is the table:

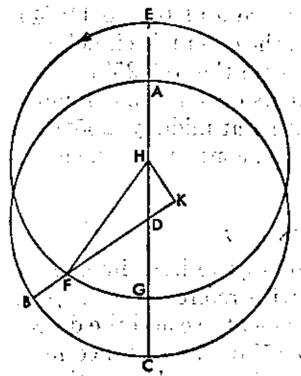
6. Table of the Sun's Anomaly

1. Common numbers (Degrees of Regular Movement)	2.	3. Additive—subtractive differences		1. Common numbers (Degrees of Regular Movement)	2.	3. Additive—subtractive differences	
6°	354°	0°	14′	120°	240°	2°	6′
12°	348°	0°	28′	123°	237°	2°	2′
18°	342°	0°	42′	126°	234°	1°	58′
24°	336°	0°	56′	129°	231°	1°	54′
30°	330°	1°	9′	132°	228°	1°	49′
36°	324°	1°	21′	135°	225°	1°	44′
42°	318°	1°	32′	138°	222°	1°	39′
48°	312°	1°	43′	141°	219°	1°	33′
54°	306°	1°	53′	144°	216°	1°	27′
60°	300°	2°	1′	147°	213°	1°	21′
66°	294°	2°	8′	150°	210°	1°	14′
72°	288°	2°	14′	153°	207°	1°	7′
78°	282°	2°	18′	156°	204°	1°	0′
84°	276°	2°	21′	159°	201°	0°	53′
90°	270°	2°	23′	162°	198°	0°	46′
93°	267°	2°	23′	165°	195°	0°	39′
96°	264°	2°	23′	168°	192°	0°	32′
99°	261°	2°	22′	171°	189°	0°	24′
102°	258°	2°	21′	174°	186°	0°	16′
105°	255°	2°	20′	177°	183°	0°	8′
108°	252°	2°	18′	180°	180°	0°	0′
111°	249°	2°	16′				
114°	246°	2°	13′				
117°	243°	2°	10′				

7. On the Epoch of the Sun's Mean Course

Since there remains to be established the epoch of the sun's regular movement for finding out its particular course at any time, we shall make the following exposition, using again in general, both in the case of the sun and of the other planets, the passages most accurately observed by ourselves, and taking the establishing of the epochs back to the beginning of the reign of Nabonassar by means of the mean movements already demonstrated. For we have ancient observations completely preserved from that period to the present.

Then let there be the circle with centre D concentric with the ecliptic; and the eccentric circle of the sun EFG with centre H; and the diameter $EAGC$ through both centres and the apogee E. And let B be supposed the fall equinox point on the ecliptic; and let BFD and FH be joined; and let HK be dropped from H perpendicular to FD.



Since the point B, the autumn equinox, is at the beginning of the sign of the Balance, and the perigee C 5½° within the sign of the Archer, therefore

$$\text{arc } BC = 65°30'.$$

And therefore

$$\text{angle } BDC = \text{angle } HDK = 65°30'$$
$$= 131° \text{ to 2 rt.}$$

And so, on the circle about right triangle DHK,

$$\text{arc } HK = 131°,$$

and

$$\text{chord } HK = 109^p12'$$

where

$$\text{diam. } DH = 120^p.$$

Therefore

$$HK = 4^p33'$$

where

$$DH = 5^p,$$
$$\text{hypt. } FH = 120^p.$$

And, on the circle about right triangle FHK,

$$\text{arc } HK = 4°20'.$$

And so

$$\text{angle } HFK = 4°20' \text{ to 2 rt.}$$
$$= 2°10'.$$

But it was proved

$$\text{angle } BDC = 65°30',$$

and therefore the remaining angle FHG (that is, arc FG on the eccentric circle) is 63°20′. Therefore, when the sun is at the autumn equinox, it precedes the perigee (that is, the point 5½° within the Archer) by 63°20′ mean movement; and it is 116°40′ in mean movement from the apogee (that is, 5½° within the Twins) in the direction contrary to the movement of the heavens.

Now that this is understood—since, of the first equinoxes observed by us, one of the most accurate occured as the autumn equinox in the year 17 of Hadrian, Egyptianwise Athyr 7, very nearly 2 hours after midday—it is clear that at that time the sun was 116°40′ in mean movement on the eccentric circle, away from the apogee in the direction contrary to the movement of the heavens. But from the reign of Nabonassar to the death of Alexander amounts to 424 Egyptian years; and from the death of Alexander to the reign of Augustus, 294 years; and from the year 1 of Augustus, Egyptianwise Thoth 1, midday (for we establish the epochs from midday) to the year 17 of Hadrian, Athyr 7, 2 hours after midday, amounts to 161 years, 66 days, and 2 equatorial hours. And therefore from the year 1 of Nabonassar, Egyptianwise Thoth 1, midday, to the time of the autumn equinox just mentioned amounts to 879 Egyptian years, 66 days, and 2 equatorial hours.

But in that amount of time the sun makes a mean movement of very nearly 211°25′ over and above the complete circles. If then we add to the 116°40′, representing the distance of this autumn equinox from the eccentric circle's apogee, the 360° of a whole circle and subtract from the sum the 311°25′ left over from the time between, then we shall have the sun at its epoch of mean movement in the year 1 of Nabonassar, Egyptianwise Thoth 1 at midday, 265°15′ of mean movement distant from the apogee in the direction contrary to the movement of the heavens, and 0°45′ within the Fishes.

8. On Calculating the Sun

Whenever we wish to know the course of the sun for any desired time, taking the total time from the epoch to the proposed date with reference to the hour in Alexandria and taking it to the tables of mean movement, we add the degrees corresponding to the particular numbers to the 265°15′ of the distance found above; and striking the complete circles out of the result, we subtract the rest from the 5°30′ within the Twins backwards in the order of the signs [from west to east]. And wherever the number falls, there we find the mean course of the sun. Next we take the same number (that is, the number of degrees from the apogee to the mean course) to the Table of Anomaly. And, if the number falls in the first column (that is, if it is not greater than 180°), then we subtract the corresponding degrees in the third column from the position of the mean course; but, if it falls in the second column (that is, if it is greater than 180°), then we add it to the mean course. And thus we find the true and apparent sun.

9. On the Inequality of Solar Days

Now this is pretty nearly all of the theory of the sun considered by itself. But it would be well to add briefly to this something concerning the inequality of the solar days, a matter which ought to be cleared up before what follows. For each of the simple mean movements we have given receives a uniform increase, as if the solar days were equal in time; but this is contrary to true theory. Now, given that the revolution of the universe is effected regularly about the poles of the equator, if the cyclical return is taken either with respect to the horizon or the meridian (whichever point is the more easily distinguishable), it is clear that one complete turn of the cosmos is the return of the same equatorial point from some section either of the horizon or the meridian back to that same section, and the solar day considered simply is the return of the sun from some section either of the horizon or of the meridian back again to the same section. The regular [or mean] solar day is, therefore, that embracing a course of 360° equatorial time proper to one revolution of the equator plus very nearly 59′ equatorial time for the sun's contrary mean movement along the ecliptic. And the irregular solar day is that embracing a course of 360° equatorial time proper to one revolution of the equator plus the extra arc, either at the horizon or at the meridian, corresponding to the sun's contrary irregular movement.

And so this section of the equator which is traversed over and above the 360° equatorial time must be an unequal one because of the sun's apparent irregularity, and because equal sections of the ecliptic do not traverse the horizon or the meridian in equal periods of time. Each of these extra sections makes the difference between the regular and irregular return, in the case of one solar day, a difference indistinguishable by the senses; but a total of many solar days makes a very sensible difference.

Now, the sun's greatest anomalistic difference occurs at intervals of one mean movement of the sun to the other. For in this way the total of solar days will differ from the total of regular [or mean] solar days by very nearly 4¾° equatorial time; and the total in one interval will differ from the total in the other by twice as much, that is by 9½° equatorial time. And this is so because the apparent course of the sun is less, in the semicircle of the apogee, than the regular course by 4¾°, but greater, in the semicircle of the perigee, by the same amount.[1]

And the greatest difference of irregularity in the corresponding risings or settings occurs in the semicircles bounded by the tropic points. For then the ascensions in each of these semicircles will differ from the 180° equatorial time of those considered regularly by the difference between the shortest or longest day and the equinoctial day; and they will differ from each other by the difference between the longest day or night and the shortest day or night. But the greatest difference of inequality in the case of simultaneous culminations occurs in the intervals containing two signs, one sign on each side of the tropic or equinoctial point. For those at the tropics taken together will differ from the regular by very nearly 4½° equatorial time, but from those at the equinoctial points taken together by 9° equatorial time. For the latter are less than the mean movement, and the former are greater by nearly an equal amount.[2] And thence we establish the beginnings of the solar days at the positions of the sun as it passes the meridian, and not from the sun's risings or settings; because the difference considered with respect to horizons can amount to many hours and is not the same everywhere, but changes with the excess of the longest over the shortest day in each latitude of the sphere. But the difference at the meridian is the same everywhere and does not exceed the total time of the sun's anomalistic difference.

And the difference in the intervals to be added or subtracted is constructed out of the combination of both of the aforesaid differences: that of the sun's

[1]For the greatest anomalistic differences, it has already been proved, occur at the apparent distance of a quadrant from the apogee, which are also the points of mean movement. Now, it has been shown that the difference between the angles of the regular and irregular movements from the mean to the apogee is 2°23′ and from the apogee to the other mean again 2°23′, the regular exceeding in both cases. The same thing occurs from the mean through the perigee to the other mean, the irregular exceeding in both cases.

[2]This difference is demonstrated in the Table of Ascensions for the Right Sphere, Book II. The difference in the co-ascensions of the ecliptic and equator is additive for the ecliptic in the Twins and the Crab, and the Archer and the Goat. It is subtractive for the ecliptic in the Fishes and the Ram, and the Virgin and the Balance. The other signs are all mixed. The equator here, of course, represents the regular movement of the sun for the mean solar days; and the ecliptic represents the irregular movement of the sun for the irregular true solar days.

anomaly and that of the culminations. And the interval from the middle of the Water Bearer to the Balance is subtractive for either difference; and the interval from the Scorpion to the middle of the Water Bearer is additive. For each of these sections either adds or subtracts with respect to the solar anomaly at most very nearly 3⅔° equatorial time, and with respect to the culminations very nearly 4⅔° equatorial time. And so, at most, the difference in the solar day, gotten from the combination of the two, amounts, for either interval, to 8⅓° equatorial time over and above the regular difference—that is, ½+⅟18 [equatorial] hour. And with respect to each other it amounts to 16⅔° equatorial time —that is, 1⅑ [equatorial] hours. Now, if this difference is neglected in the case of the sun, it perhaps would not hurt the study of its appearances to any appreciable extent; but in the case of the moon, because of the rapidity of its movement, it would produce a considerable difference even for 3⅕°.

In order to reduce the solar days given over any interval of time (and by solar days I mean from midday or midnight to midday or midnight) to regular solar days, we find out for the beginning and end of the given interval at what part of the ecliptic the sun is to be found both in its regular and irregular movements. Then, taking the surplus over and above complete revolutions accumulated in the irregular interval (that is, from apparent position to apparent position) to the Table of Ascensions in the Right Sphere, we find out with how many degrees equatorial time the degrees of the irregular interval, as we said, culminate. Then we take the difference between the amount of time thus found and the degrees of the regular interval and calculate the part of the corresponding equatorial hour. And if the amount of time found in the Table of Ascensions in the Right Sphere is greater than that of the regular interval, we add this difference to the given number of solar days; if less, we subtract. And finally, in this way, we have the time expressed in mean solar days. We shall use this correction especially in the successive additions of the mean movements in the moon's tables. And it is immediately seen that, given the regular [or mean] solar days, the simply considered seasonal solar days are gotten by an addition or subtraction in the converse order.[1]

[1]If the sun moved regularly along the equator instead of moving irregularly along the ecliptic, the solar days would all be regular. Their irregularity, then, is due to two things: (1) the irregularity of the sun's movement, and (2) the fact that the sun moves on a circle oblique to the equator.

Now, as far as the solar days are concerned, it is not the arc of the ecliptic that is directly involved, but the corresponding or co-ascending arc of the equator. For the culminations which define solar days take place at the meridian, and the meridian is through the pole of the equator. Therefore we first find the arc on the equator co-ascending with the apparent arc traversed by the sun along the ecliptic.

Now if the arc of the sun's regular movement is less than the equatorial arc co-ascending with the apparent arc, then the number of actual solar days is less than the number of regular ones. But if the arc of the sun's regular movement is greater, then the number of actual solar days is greater than the number of regular ones. For when the sun moves faster than the regular, then the actual solar day is longer than the mean solar day and there are fewer of them in a given time; but, when it moves slower, then the actual solar day is shorter.

And in each case the difference between the number of actual solar days and the number of corresponding regular solar days is expressed by the difference between the equatorial arc corresponding to the apparent arc on the ecliptic and the arc of regular movement. For (1) if the regular arc is less than the equator's arc, then the apparent arc is reaching the meridian that many time-degrees later; and so just so many time-degrees have been lost to the number of regular solar days. For the same number of regular solar days as actual solar days would only take the sun the length of the arc of regular movement, if we consider it as moving on the

Now, according to the epoch we have chosen (that is, the year 1 of Nabonassar, Egyptianwise Thoth 1 at midday), the sun's mean position, as we have just shown [p. 104], was 0°45′ within the Fishes. But, considered with its anomaly, its position was very nearly 3°8′ within the Fishes.

equator. It would perhaps be clearer if we imagined that the regular arc were less by a revolution than the equator's arc representing the apparent arc on the ecliptic. Then the number of actual solar days would be one day less than the number of regular solar days. For the sun would have moved back one whole circle, and thus would have come to the meridian one time less over the period assigned than if it had moved back according to the regular movement. And by the same argument (2) if the regular arc is greater, just so many time-degrees have been added to the number of regular solar days. Therefore, in the first case, we add the difference to the number of actual solar days to get the number of regular solar days; and, in the second case, we subtract.